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ON THE CHARACTERISTIC WORD OF THE INHOMOGENEOUS BEATTY SEQUENCE

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We detail the sequence (f_n) where $f_n = [(n+1)\theta + \phi] - [n\theta + \phi] - [\theta]$. This description of the inhomogeneous Beatty sequence generalises earlier work dealing with special cases in which ϕ is restricted to rational values.

1. INTRODUCTION

The *Beatty* sequence $([n\theta + \phi])$ of integer parts of $n\theta + \phi$, $n = 1, 2, \ldots$ has been studied extensively. In that context it is natural to consider the sequence of differences $f_1, f_2, f_3, \ldots$, where

$$f_n = [(n+1)\theta + \phi] - [n\theta + \phi] - [\theta].$$

Plainly each f_n is equal to either 0 or 1. The word $f_1 f_2 f_3 \cdots$ is called the *characteristic word*, or just *word* of the sequence.

Stolarsky [9] constructs this characteristic word in the case when $\phi = 0$ by using *shift operators*; and this is generalised by Fraenkel, Muskin and Tassa [3]. On the other hand, van Ravenstein, Winley and Tognetti [8] obtain the word in a special case from the *three gap theorem*. Danilov [1] had a similar result by a different method. Recently, Nishioka, Shiokawa and Tamura [7] get the word for an irrational θ and a rational ϕ from *Mahler functions*. However, their result does not match the facts when $\phi \neq 0$ (The correct version is described in [5]). Two other papers dealing with the inhomogeneous case are [4] and [6].

Venkov [10, 65–68] rewrites Markov's method to obtain the characteristic word. We apply Venkov's method with the goal of obtaining the word $f_1 f_2 f_3 \cdots$ in the inhomogeneous case.

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2. The word of $f(n;\theta,\phi)$

We write

$$f_n = f(n;\theta,\phi) = [(n+1)\theta+\phi] - [n\theta+\phi] - [\theta].$$

As always, $[\psi]$ denotes the integer part (floor) of ψ, and $\{\psi\} = \psi - \lfloor\psi\rfloor$ is its fractional part. We shall assume that $0 < \theta < 1$.

Venkov details the characteristic of $f(n;\theta,0)$ and $f(n;\theta,1/2)$. We obtain a more general result, that is, with θ and ϕ arbitrary real numbers.

We may take $0 < \theta < 1$ without loss of generality, because the result is trivial if θ is an integer. We also let $0 \leqslant \phi < 1$. We begin by supposing that $\theta + \phi < 1$.

We start from the continued fraction expansion $\theta = [0, a_1, a_2, \dots] = [0, a_1 + \theta_1]$, where $\theta_{n-1} = [0, a_n, a_{n+1}, \dots]$. And we also introduce the expansion of ϕ in terms of the sequence $\{\theta_0, \theta_1, \dots\}$, that is, by using the notation $\lceil\psi\rceil$, the ceiling of ψ

$$\phi = b_0 - \phi_0, \qquad b_0 = \lceil\phi\rceil$$
$$\frac{\phi_{n-1}}{\theta_{n-1}} = b_n - \phi_n, \qquad b_n = \left\lceil\frac{\phi_{n-1}}{\theta_{n-1}}\right\rceil \quad (n \geqslant 1).$$

If for $m = 1, 2, \dots$ we put $\alpha_m = ma_1 + [m\theta_1]$, then $[\alpha_m\theta + \phi] = m + [\phi - \{m\theta_1\}\theta]$ and $[(\alpha_m+1)\theta+\phi] = m + [\phi + (1 - \{m\theta_1\})\theta]$. So for any non-negative integer λ_m we have $[(\alpha_m - \lambda_m)\theta + \phi] = m + [\phi - (\lambda_m + \{m\theta_1\})\theta]$.

Noting that $0 \leqslant \{m\theta_1\}\theta < 1$ and that $(1 - \{m\theta_1\})\theta < 1 - \phi$, it follows that

$$[\alpha_m\theta+\phi] = \begin{cases} m-1, & \text{if } \{m\theta_1\}\theta > \phi, \\ m, & \text{otherwise}, \end{cases}$$
$$[(\alpha_m+1)\theta+\phi] = m,$$
$$[(\alpha_m-\lambda_m)\theta+\phi] = \begin{cases} m-1, & \text{if } (\lambda_m + \{m\theta_1\})\theta > \phi, \\ m, & \text{otherwise}. \end{cases}$$

If $\{(m+1)\theta_1\}\theta > \phi$, then

$$f_{\alpha_m+1} + f_{\alpha_m+2} + \cdots + f_{\alpha_{m+1}-1} = [\alpha_{m+1}\theta+\phi] - [(\alpha_m+1)\theta+\phi] = m - m = 0,$$
$$f_{\alpha_{m+1}} = [(\alpha_{m+1}+1)\theta+\phi] - [\alpha_{m+1}\theta+\phi] = m+1-m = 1.$$

Thus, $f_{\alpha_m+1}f_{\alpha_m+2}\cdots f_{\alpha_{m+1}} = \underbrace{00\cdots\cdots\cdots0}_{\alpha_{m+1}-\alpha_m-1}1.$

And if $\{(m+1)\theta_1\}\theta \leqslant \phi$,

$$[\alpha_{m+1}\theta+\phi] - [(\alpha_m+1)\theta+\phi] = m+1-m = 1,$$
$$[(\alpha_{m+1}+1)\theta+\phi] - [\alpha_{m+1}\theta+\phi] = m+1-(m+1) = 0.$$

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Thus in this case

$$\begin{aligned} f_{\alpha_m+1} + f_{\alpha_m+2} + \cdots + f_{\alpha_{m+1}-\lambda_{m+1}-1} &= [(\alpha_{m+1} - \lambda_{m+1})\theta + \phi] - [(\alpha_m + 1)\theta + \phi] \\ &= \begin{cases} m - m = 0, & \text{if } (\lambda_{m+1} + \{(m+1)\theta_1\})\theta > \phi, \\ m + 1 - m = 1, & \text{otherwise} . \end{cases} \end{aligned}$$

So, if λ_{m+1} is the least non-negative integer satisfying $(\lambda_{m+1} + \{(m+1)\theta_1\})\theta > \phi$,

$$f_{\alpha_m+1} f_{\alpha_m+2} \cdots f_{\alpha_{m+1}} = \underbrace{00 \cdots\cdots\cdots\cdots 0}_{\alpha_{m+1}-\alpha_m-\lambda_{m+1}-1} 1 \underbrace{0 \cdots 0}_{\lambda_{m+1}} .$$

Next, consider $f_1 f_2 \cdots f_{a_1}$, noting that $\alpha_1 = a_1$. Much as above, we have the following:

$$[a_1\theta + \phi] = 1 + [\phi - \theta\theta_1] = \begin{cases} 0, & \text{if } \theta\theta_1 > \phi, \\ 1, & \text{otherwise} . \end{cases}$$

$$[(a_1 + 1)\theta + \phi] = 1 + [\phi + (1 - \theta_1)\theta] = 1.$$

For a non-negative integer λ_1,

$$[(a_1 - \lambda_1)\theta + \phi] = 1 + [\phi - (\lambda_1 - \theta_1)\theta] = \begin{cases} 0, & \text{if } (\lambda_1 + \theta_1)\theta > \phi, \\ 1, & \text{otherwise.} \end{cases}$$

If $\theta\theta_1 > \phi$,

$$\begin{aligned} f_1 + f_2 + \cdots + f_{a_1-1} &= [a_1\theta + \phi] - [\theta + \phi] = 0, \\ f_1 + f_2 + \cdots + f_{a_1} &= [(a_1 + 1)\theta + \phi] - [\theta + \phi] = 1. \end{aligned}$$

Thus,

$$f_1 f_2 \cdots f_{a_1} = \underbrace{0 \cdots 0}_{a_1-1} 1.$$

If $\theta\theta_1 \leqslant \phi$, $f_1 + f_2 + \cdots + f_{a_1-1} = 1$ and $f_1 + f_2 + \cdots + f_{a_1} = 0$. So in this case

$$\begin{aligned} f_1 + f_2 + \cdots + f_{a_1-\lambda_1-1} &= [(a_1 - \lambda_1)\theta + \phi] - [\theta + \phi] \\ &= \begin{cases} 0, & \text{if } (\lambda_1 + \theta_1)\theta > \phi, \\ 1, & \text{otherwise.} \end{cases} \end{aligned}$$

So, if λ_1 is the least non-negative integer satisfying $(\lambda_1 + \theta_1)\theta > \phi$,

$$f_1 f_2 \cdots f_{a_1} = \underbrace{00 \cdots\cdots 0}_{a_1-\lambda_1-1} 1 \underbrace{0 \cdots 0}_{\lambda_1} .$$

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But $\alpha_{m+1} - \alpha_m = a_1 + h_m$. Thus if we put $h_m = [(m+1)\theta_1] - [m\theta_1]$, then for $m = 1, 2, \ldots$ and $h_0 = 0$,

$$f_1 f_2 f_3 \cdots\cdots = I_1 I_2 \cdots\cdots ,$$

where for $m = 1, 2, \ldots$

$$I_m = f_{\alpha_{m-1}+1} f_{\alpha_{m-1}+2} \cdots f_{\alpha_m} = \underbrace{00\cdots\cdots\cdots\cdots0}_{a_1-1-\lambda_m+h_{m-1}} 1 \underbrace{0\cdots0}_{\lambda_m}.$$

Moreover,

$$\lambda_1 = [\phi/\theta - \theta_1] + 1 = [a_1 - b_1 + \phi_1] + 1 = a_1 - b_1 + 1 .$$

And if we put $g_m = h_m - \lambda_{m+1} + \lambda_m$, from $\lambda_m = [\phi/\theta - \{m\theta_1\}] + 1$ we have

$$\begin{aligned} g_m &= -[(a_1 + \theta_1)\phi - (m+1)\theta_1] + [(a_1 + \theta_1)\phi - m\theta_1] \\ &= -[\theta_1 + \phi_1 - (m+1)\theta_1] + [\theta_1 + \phi_1 - m\theta_1] \\ &= [(m+1)\theta_1 + (1 - \theta_1 - \phi_1)] - [m\theta_1 + (1 - \theta_1 - \phi_1)] , \end{aligned}$$

under the assumption $m_1\theta + \phi \neq m_2$ for any $m_1, m_2 \in \mathbb{Z}$. From now on this is always assumed. Thus, we have the following theorem:

Theorem 1. *Let $f_n = f(n;\theta,\phi) = [(n+1)\theta + \phi] - [n\theta + \phi] - [\theta]$ and suppose the continued fraction expansion of θ is given by $\theta = [0, a_1, a_2, \ldots] = [0, a_1 + \theta_1]$. For $m = 1, 2, \ldots$ let $g_m = [(m+1)\theta_1 + (1 - \theta_1 - \phi_1)] - [m\theta_1 + (1 - \theta_1 - \phi_1)]$.*

If $0 < \{\theta\} + \{\phi\} < 1$, then the characteristic word is

$$f_1 f_2 f_3 \cdots = J_0 J_1 J_2 J_3 \cdots ,$$

where

$$J_0 = \underbrace{0\cdots0}_{b_1-2} 1, \qquad J_m = \underbrace{00\cdots\cdots0}_{a_1-1+g_m} 1 \quad (m \geqslant 1).$$

Remark. Our discussion is quite general. When $\theta + \phi > 1$, $n = 1, 2, \ldots$ the characteristic word of $f = f(n;\theta,\phi)$ coincides with the characteristic word of $f' = f(n; 1-\theta, 1-\phi)$ with $(1-\theta) + (1-\phi) < 1$, if 0 and 1 are interchanged. When $\theta + \phi = 1$, we can reduce to the homogeneous case, that is,

$$f_1 f_2 f_3 \cdots = h_0 h_1 h_2 \cdots .$$

When $\theta + \phi < 1$, $b_1 - \phi_1 = \phi_0/\theta_0 = (1-\phi)/\theta > 1$, so $b_1 \geqslant 2$.

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We wish to rewrite this in a different way. Therefore we consider the word $g_1g_2g_3\cdots$ and according as g_m is 0 or 1 we replace each g_m by the rule

$$g_m = \begin{cases} 0 \longrightarrow \underbrace{0\cdots0}_{a_1-1}1 = w_1, \text{ say,} \\ 1 \longrightarrow \underbrace{0\cdots0}_{a_1}1 = w_0w_1\,. \end{cases}$$

Here we have set $w_0 = 0$.

Writing $\psi(0,1) = f_1f_2f_3\cdots$ and $\psi_1(0,1) = g_1g_2g_3\cdots$, we have that

$$\psi(0,1) = J_0\psi_1(w_1, w_0w_1).$$

If $\theta_1 + \phi_1 < 1$, we can continue with a similar step. This time we consider θ as θ_1, ϕ as $1-\theta_1-\phi_1$. Then, $\theta_1 + (1-\theta_1-\phi_1) = 1-\phi_1 < 1$. If we put $\theta^{(1)} = \theta_1$ and $\phi^{(1)} = 1-\theta_1-\phi_1$, this $\lambda_m \left(= \lambda_m^{(2)}, \text{ say}\right)$ is the least non-negative integer satisfying $\left(\lambda_m^{(2)} + \{m\theta_2\}\right)\theta^{(1)} > \phi^{(1)}$. Hence,

$$\lambda_m^{(2)} = \left[\frac{\phi^{(1)}}{\theta^{(1)}} - \{m\theta_2\}\right] + 1 = a_2 - b_2 + [\theta_2 + \phi_2 - \{m\theta_2\}]\,.$$

Writing $h_m^{(2)} = [(m+1)\theta_2] - [m\theta_2]$ and $g_m^{(2)} = h_m^{(2)} - \lambda_{m+1}^{(2)} + \lambda_m^{(2)}$, we obtain

$$g_m^{(2)} = [(m+1)\theta_2 + (1-\theta_2-\phi_2)] - [m\theta_2 + (1-\theta_2-\phi_2)]$$

and

$$a_2 - \lambda_1^{(2)} - 1 = b_2 - 1\,.$$

Thus,

$$g_1g_2g_3\cdots = J_0^{(2)}J_1^{(2)}J_2^{(2)}J_3^{(2)}\cdots\,,$$

where

$$J_0^{(2)} = \underbrace{0\cdots0}_{b_2-1}1, \qquad J_m^{(2)} = \underbrace{00\cdots\cdots0}_{a_2-1+g_m^{(2)}}1 \quad (m \geqslant 1)\,.$$

Therefore,

$$f_1f_2f_3\cdots = J_0\psi_1(w_1, w_0w_1) = J_0K_0^{(2)}K_1^{(2)}K_2^{(2)}K_3^{(2)}\cdots\,,$$

where

$$K_0^{(2)} = \underbrace{w_1\cdots w_1}_{b_2-1}w_0w_1, \qquad K_m^{(2)} = \underbrace{w_1w_1\cdots\cdots w_1}_{a_2-1+g_m^{(2)}}w_0w_1 \quad (m \geqslant 1)\,.$$

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If $\theta_2 + \phi_2 < 1$ again, we consider the word $g_1^{(2)} g_2^{(2)} g_3^{(2)} \cdots$ and the rule

$$g_m^{(2)} = \begin{cases} 0 \longrightarrow \underbrace{0 \cdots 0}_{a_2 - 1} 1, \\ 1 \longrightarrow \underbrace{0 \cdots 0}_{a_2} 1. \end{cases}$$

Writing $\psi_2(0,1) = g_1^{(2)} g_2^{(2)} g_3^{(2)} \cdots$, we have

$$\psi(0,1) = J_0 K_0^{(2)} \psi_2(w_2, w_1 w_2),$$

where $w_2 = \underbrace{w_1 \cdots w_1}_{a_2 - 1} w_0 w_1$. Ultimately we obtain that for $k = 1, 2, \ldots$,

$$f_1 f_2 f_3 \cdots = J_0 K_0^{(2)} K_0^{(3)} \cdots K_0^{(k)} \psi_k(w_k, w_{k-1} w_k),$$

where $w_k = \underbrace{w_{k-1} \cdots w_{k-1}}_{a_k - 1} w_{k-2} w_{k-1}$ and $K_0^{(k)} = \underbrace{w_{k-1} \cdots w_{k-1}}_{b_k - 1} w_{k-2} w_{k-1}$. We therefore have the following theorem:

THEOREM 2. *Let $f_n = f(n; \theta, \phi) = [(n+1)\theta + \phi] - [n\theta + \phi] - [\theta]$ and suppose the continued fraction expansion of θ is given by $\theta = [0, a_1, a_2, \ldots]$. Let $w_0 = 0$, $w_1 = \underbrace{0 \cdots 0}_{a_1 - 1} 1$, and $w_k = \underbrace{w_{k-1} \cdots w_{k-1}}_{a_k - 1} w_{k-2} w_{k-1}$ for $k \geqslant 2$.*

If $0 < \{\theta\} + \{\phi\} < 1$ and $\theta_k + \phi_k < 1$ for every $k \geqslant 1$, then the characteristic word is

$$f_1 f_2 f_3 \cdots = \lim_{k \to \infty} J_0 K_0^{(2)} K_0^{(3)} \cdots K_0^{(k)},$$

where $J_0 = \underbrace{0 \cdots 0}_{b_1 - 2} 1$ and $K_0^{(k)} = \underbrace{w_{k-1} \cdots w_{k-1}}_{b_k - 1} w_{k-2} w_{k-1}$.

REMARK. This result matches the Main theorem of [5]. $\theta_k + \phi_k < 1$ leads to $a_k \geqslant b_k$ because $1 > (1 - \phi_{k-1})/\theta_{k-1} - (a_k - b_k)$ and $0 \leqslant b_k \leqslant a_k + 1$. And if we write $W_0 = w_0$, $W_1 = w_1$, $W_2 = W_1^{b_2 - 1} W_0 W_1^{a_2 - b_2 + 1}$ and $W_k = W_{k-1}^{b_k} W_{k-2} W_{k-1}^{a_k - b_k}$ for $k \geqslant 3$, then it is easily seen that for $i = 1, 2, \ldots$

$$w_2^i = W_1^{a_2 - b_2} W_2^i W_1^{b_2 - a_2}$$

and

$$w_k^i = W_1^{a_2 - b_2} W_2^{a_3 - b_3 - 1} \cdots W_{k-1}^{a_k - b_k - 1} W_k^i W_{k-1}^{b_k - a_k + 1} \cdots W_2^{b_3 - a_3 + 1} W_1^{b_2 - a_2} \quad (k \geqslant 3).$$

Therefore, we get $K_0^{(2)} = w_1^{b_2-1}w_0w_1 = W_2W_1^{b_2-a_2}$ and

$$\begin{aligned} K_0^{(2)}K_0^{(3)} &= W_2W_1^{b_2-a_2}w_2^{b_3-1}w_1w_2 \\ &= W_2W_1^{b_2-a_2}W_1^{a_2-b_2}W_2^{b_3-1}W_1^{b_2-a_2}W_1W_1^{a_2-b_2}W_2W_1^{b_2-a_2} \\ &= W_2^{b_3}W_1W_2W_1^{b_2-a_2}. \end{aligned}$$

Generally for $k \geqslant 4$, from

$$\begin{aligned} K_0^{(k)} &= w_{k-1}^{b_k-1}w_{k-2}w_{k-1} \\ &= W_1^{a_2-b_2}W_2^{a_3-b_3-1}\cdots W_{k-2}^{a_{k-1}-b_{k-1}-1}W_{k-1}^{b_k-1}W_{k-2}W_{k-1} \\ &\qquad W_{k-2}^{b_{k-1}-a_{k-1}+1}\cdots W_2^{b_3-a_3+1}W_1^{b_2-a_2}, \end{aligned}$$

we obtain

$$K_0^{(2)}\cdots K_0^{(k)} = W_{k-1}^{b_k}W_{k-2}W_{k-1}W_{k-2}^{b_{k-1}-a_{k-1}+1}\cdots W_2^{b_3-a_3+1}W_1^{b_2-a_2}.$$

This W_k is the same as the w_k in that Main Theorem.

3. The general cases

We can see that $g_m^{(i)}$ always has the same form if $\theta_n + \phi_n < 1$ for all n. In this section we shall show that the form of $g_m^{(i)}$ is always the same, even if $\theta_k + \phi_k > 1$ for some k. We introduce $\theta_{k,l}$ and $\phi_{k,l}$ for convenience, satisfying

$$\frac{1}{\theta_k} + \frac{1}{\theta_{k,l}} = l \qquad \text{and} \qquad \frac{\phi_k}{\theta_k} + \frac{\phi_{k,l}}{\theta_{k,l}} = l - 1$$

for an non-negative integer l. That is,

$$\theta_{k,l} = -\frac{\theta_k}{1 - l\theta_k} \qquad \text{and} \qquad \phi_{k,l} = 1 + \frac{\theta_k + \phi_k - 1}{1 - l\theta_k}.$$

The case $l = 1$ is known to be the necessary and sufficient condition in order that $\{[n\theta_k + \phi_k]\}_{n=1}^{\infty}$ and $\{[n\theta_{k,1} + \phi_{k,1}]\}_{n=1}^{\infty}$ partition the positive integers (for example, [2]).

Our notation allows us to write that for some integers k and l

$$g_m^{(i)} = -[(m+1)\theta_{k,l} + \phi_{k,l}] + [m\theta_{k,l} + \phi_{k,l}].$$

3.1 Case 1. Suppose $l = 0$ and $\theta_k + \phi_k < 1$. Then,

$$g_m^{(i)} = [(m+1)\theta_k + (1 - \theta_k - \phi_k)] - [m\theta_k + (1 - \theta_k - \phi_k)]$$

for some positive integers i and k. By setting $\theta^{(i)} = \theta_k = -\theta_{k,0}$, $\phi^{(i)} = 1 - \theta_k - \phi_k = 1 - \phi_{k,0}$ and $\alpha_m^{(i+1)} = ma_{k+1} + [m\theta_{k+1}]$, we have

$$\begin{aligned}\lambda_m^{(i+1)} &= [\phi^{(i)}/\theta^{(i)} - \{m\theta_{k+1}\}] + 1 \\ &= a_{k+1} - b_{k+1} + [\theta_{k+1} + \phi_{k+1} - m\theta_{k+1}] + [m\theta_{k+1}], \\ g_m^{(i+1)} &= [(m+1)\theta_{k+1} + (1 - \theta_{k+1} - \phi_{k+1})] - [m\theta_{k+1} + (1 - \theta_{k+1} - \phi_{k+1})]\end{aligned}$$

and $a_{k+1} - \lambda_1^{(i+1)} - 1 = b_{k+1} - 1$. Therefore,

$$g_1^{(i)} g_2^{(i)} \cdots = J_0^{(i+1)} J_1^{(i+1)} J_2^{(i+1)} \cdots ,$$

where

$$J_0^{(i+1)} = \underbrace{0\cdots0}_{b_{k+1}-1}1 \quad \text{and} \quad J_m^{(i+1)} = \underbrace{00\cdots\cdots\cdots0}_{a_{k+1}-1+g_m^{(i+1)}}1 \quad (m \geqslant 1).$$

3.2 Case 2(1). Let

$$g_m^{(i)} = -[(m+1)\theta_{k,l-1} + \phi_{k,l-1}] + [m\theta_{k,l-1} + \phi_{k,l-1}]$$

for some positive integers i, k and l. Suppose $\theta_k + \phi_k > 1$ and $\theta_k < 1/l$. Put

$$\theta^{(i)} = \theta_{k,l-1} + 1 = [\ 0,\ 1,\ a_{k+1} - l,\ a_{k+2},\ a_{k+3},\ \dots\ ] \quad \text{and} \quad \phi^{(i)} = \phi_{k,l-1} - 1$$

so that

$$g_m^{(i)} = 1 - [(m+1)\theta^{(i)} + \phi^{(i)}] + [m\theta^{(i)} + \phi^{(i)}]$$

with

$$0 < \theta^{(i)},\ \phi^{(i)},\ \theta^{(i)} + \phi^{(i)} < 1.$$

If $a_{k+1} \geqslant l + 1$, from $\theta^{(i)} = [\ 0,\ 1,\ a_{k+1} - l + \theta_{k+1}\ ]$ we set $\alpha_m^{(i+1)} = m + [m(-\theta_{k,l})]$. Hence, we have

$$\begin{aligned}\lambda_m^{(i+1)} &= [\phi_{k,l} - 1 - m(-\theta_{k,l})] + [m(-\theta_{k,l})] + 1, \\ g_m^{(i+1)} &= -\ [\phi_{k,l} - 1 - (m+1)(-\theta_{k,l})] + [\phi_{k,l} - 1 - m(-\theta_{k,l})] \\ &= -\ [(m+1)\theta_{k,l} + \phi_{k,l}] + [m\theta_{k,l} + \phi_{k,l}]\end{aligned}$$

and

$$1 - \lambda_1^{(i+1)} - 1 = -[\theta_{k,l} + \phi_{k,l} - 1] - 1 = [1 - \theta_{k,l} - \phi_{k,l}] = 0.$$

Noticing that $\theta_k < 1/(l+1)$ and $\theta_k + \phi_k > 1$, we remark that

$$0 < \theta_{k,l} + 1,\ \phi_{k,l} - 1,\ \theta_{k,l} + \phi_{k,l} < 1.$$

Therefore,

$$g_1^{(i)}g_2^{(i)}\cdots = J_0^{(i+1)}J_1^{(i+1)}J_2^{(i+1)}\cdots ,$$

where

$$J_0^{(i+1)} = 0 \quad \text{and} \quad J_m^{(i+1)} = \underbrace{1\cdots1}_{g_m^{(i+1)}}0 \quad (m \geqslant 1).$$

3.3. CASE 2(2). Let $g_m^{(i)}$, $\theta^{(i)}$ and $\phi^{(i)}$ be the same as those in the case 2(1). Again suppose $\theta_k + \phi_k > 1$ and $\theta_k < 1/l$.

If $a_{k+1} = l$, from $\theta^{(i)} = [\,0,\ a_{k+2}+1+\theta_{k+2}\,]$ we set $\alpha_m^{(i+1)} = (a_{k+2}+1)m + [m\theta_{k+2}]$. Then we have

$$\begin{aligned}
\lambda_m^{(i+1)} &= [\phi_{k,l} - 1 - m\theta_{k+2}] + [m\theta_{k+2}] + 1,\\
g_m^{(i+1)} &= -\,[\phi_{k,l} - 1 - (m+1)\theta_{k+2}] + [\phi_{k,l} - 1 - m\theta_{k+2}]\\
&= \big[(m+1)\theta_{k+2} + \big(1 - (b_{k+1}+1-l)\theta_{k+2} - \phi_{k+2}\big)\big]\\
&\qquad - \big[m\theta_{k+2} + \big(1 - (b_{k+1}+1-l)\theta_{k+2} - \phi_{k+2}\big)\big]
\end{aligned}$$

and

$$a_{k+2} + 1 - \lambda_1^{(i+1)} - 1 = a_{k+2} - [\phi_{k,l} - 1 - \theta_{k+2}] - 1 = [1 - \theta_{k,l} - \phi_{k,l}].$$

Note that

$$0 < \theta_{k+2} = -\theta_{k,l} - a_{k+2} < 1$$

and

$$-\phi_{k,l} = -(b_{k+1}+1-l)\theta_{k+2} - \phi_{k+2} - a_{k+2}(b_{k+1}+1-l) + b_{k+2},$$

where $b_{k+1} = l$ or $l+1$ because $\theta_k + \phi_k > 1$. Therefore,

$$g_1^{(i)}g_2^{(i)}\cdots = J_0^{(i+1)}J_1^{(i+1)}J_2^{(i+1)}\cdots ,$$

where

$$J_0^{(i+1)} = \underbrace{11\cdots\cdots\cdots1}_{[1-\theta_{k,l}-\phi_{k,l}]}0 \quad \text{and} \quad J_m^{(i+1)} = \underbrace{11\cdots\cdots1}_{a_{k+2}+g_m^{(i+1)}}0 \quad (m \geqslant 1).$$

3.4. CASE 1′. Let

$$g_m^{(i)} = [(m+1)\theta_k + \big(1 - (b_{k-1}+1-l)\theta_k - \phi_k\big)] - [m\theta_k + \big(1 - (b_{k-1}+1-l)\theta_k - \phi_k\big)]$$

for some integers i, k and l with $k \geqslant 3$ and $l \geqslant 1$. From the fact in the case 2(2), $b_{k-1} = l$ or $l+1$. If $b_{k-1} = l$, we are back to case 1. If $b_{k-1} = l+1$,

$$g_m^{(i)} = [m\theta_k + (1 - \theta_k - \phi_k)] - [(m-1)\theta_k + (1 - \theta_k - \phi_k)]$$

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with

$$g_1^{(i)} = \begin{cases} 0, & \text{if } \theta_k + \phi_k < 1; \\ 1, & \text{if } \theta_k + \phi_k > 1. \end{cases}$$

Combining these remarks, we obtain

$$g_1^{(i)} g_2^{(i)} \cdots = J_{-1}^{(i+1)} J_0^{(i+1)} J_1^{(i+1)} J_2^{(i+1)} \cdots ,$$

where $J_{-1}^{(i+1)} = \underbrace{00\cdots\cdots 0}_{b_{k-1}-l}$ if $\theta_k + \phi_k < 1$, $\underbrace{11\cdots\cdots 1}_{b_{k-1}-l}$ if $\theta_k + \phi_k > 1$. The others are the same as those in the previous cases.

4. Together

Suppose that the characteristic word of the sequence $f_n = f(n;\theta,\phi)$ is given by

$$f_1 f_2 f_3 \cdots = J_0 K_0^{(2)} K_0^{(3)} \cdots K_0^{(i-1)} K_1^{(i-1)} K_2^{(i-1)} \cdots ,$$

where $K_m^{(i-1)} = u^{c+g_m^{(i-1)}} v$ for $m \geqslant 1$ and

$$g_m^{(i-1)} = [(m+1)\theta_k + 1 - (b_{k-1} + 1 - l)\theta_k - \phi_k] - [m\theta_k + 1 - (b_{k-1} + 1 - l)\theta_k - \phi_k]$$

for some positive integer k.

If $\theta_k + \phi_k < 1$, then from the argument in the previous section

$$g_1^{(i-1)} g_2^{(i-1)} \cdots = J_{-1}^{(i)} J_0^{(i)} J_1^{(i)} J_2^{(i)} \cdots ,$$

where

$$J_{-1}^{(i)} = \underbrace{00\cdots\cdots 0}_{b_{k-1}-a_{k-1}}, \qquad J_0^{(i)} = \underbrace{0\cdots 0}_{b_{k+1}-1} 1, \qquad J_m^{(i)} = \underbrace{00\cdots\cdots 0}_{a_{k+1}-1+g_m^{(i)}} 1 \quad (m \geqslant 1),$$

and

$$g_m^{(i)} = [(m+1)\theta_{k+1} + (1 - \theta_{k+1} - \phi_{k+1})] - [m\theta_{k+1} + (1 - \theta_{k+1} - \phi_{k+1})].$$

Therefore, we have

$$f_1 f_2 f_3 \cdots = J_0 K_0^{(2)} K_0^{(3)} \cdots K_0^{(i-1)} K_{-1}^{(i)} K_0^{(i)} K_1^{(i)} K_2^{(i)} \cdots ,$$

where

$$K_{-1}^{(i)} = (u^c v)^{b_{k-1}-a_{k-1}}, \quad K_0^{(i)} = (u^c v)^{b_{k+1}-1} u^{c+1} v$$

and for $m \geqslant 1$

$$K_m^{(i)} = (u^c v)^{a_{k+1}-1+g_m^{(i)}} u^{c+1} v .$$

In the case when $a_{k-1} = b_{k-1}$, $J_{-1}^{(i)}$ (so, $K_{-1}^{(i)}$) is omitted.

We can go to the next step by applying the substitutions

$$u \longrightarrow u^c v, \quad v \longrightarrow u^{c+1} v \quad \text{and} \quad c \longrightarrow a_{k+1} - 1 .$$

Next, let $\theta_k + \phi_k > 1$. We put $a_{k+1} = a (= l)$ and $d_k = [1 - \theta_{k,l} - \phi_{k,l}]$. We use the argument of the previous section repeatedly.

If $a_{k+1} \geqslant 2$,

$$g_1^{(i-1)} g_2^{(i-1)} \cdots = J_{-1}^{(i)} J_0^{(i)} J_1^{(i)} J_2^{(i)} \cdots ,$$

where

$$J_{-1}^{(i)} = \underbrace{11 \cdots\cdots 1}_{b_{k-1}-a_{k-1}}, \qquad J_0^{(i)} = 0, \qquad J_m^{(i)} = \underbrace{1 \cdots 1}_{g_m^{(i)}} 0 \quad (m \geqslant 1),$$

and

$$g_m^{(i)} = -[(m+1)\theta_{k,1} + \phi_{k,1}] + [m\theta_{k,1} + \phi_{k,1}] .$$

Thus, we have

$$f_1 f_2 f_3 \cdots = J_0 K_0^{(2)} K_0^{(3)} \cdots K_0^{(i-1)} K_{-1}^{(i)} K_0^{(i)} K_1^{(i)} K_2^{(i)} \cdots ,$$

where

$$K_{-1}^{(i)} = \left(u^{c+1} v\right)^{b_{k-1}-a_{k-1}}, \qquad K_0^{(i)} = u^c v \quad \text{and} \quad K_m^{(i)} = \left(u^{c+1} v\right)^{g_m^{(i)}} u^c v \quad (m \geqslant 1).$$

In the case when $a_{k-1} = b_{k-1}$, $J_{-1}^{(i)}$ (so, $K_{-1}^{(i)}$) is omitted.

Similarly, if $a_{k+1} \geqslant a$, then

$$g_1^{(i+a-3)} g_2^{(i+a-3)} \cdots = J_0^{(i+a-2)} J_1^{(i+a-2)} J_2^{(i+a-2)} \cdots ,$$

where

$$J_0^{(i+a-2)} = 0, \qquad J_m^{(i+a-2)} = \underbrace{1 \cdots 1}_{g_m^{(i+a-2)}} 0 \quad (m \geqslant 1),$$

and

$$g_m^{(i+a-2)} = -[(m+1)\theta_{k,a-1} + \phi_{k,a-1}] + [m\theta_{k,a-1} + \phi_{k,a-1}] .$$

Thus, we have

$$f_1 f_2 f_3 \cdots = J_0 K_0^{(2)} K_0^{(3)} \cdots K_0^{(i-1)} K_{-1}^{(i)} K_0^{(i)} \cdots K_0^{(i+a-2)} K_1^{(i+a-2)} K_2^{(i+a-2)} \cdots ,$$

where

$$K_0^{(i+a-2)} = u^c v \quad \text{and} \quad K_m^{(i+a-2)} = \left(u(u^c v)^{a-1}\right)^{g_m^{(i+a-2)}} u^c v \quad (m \geqslant 1).$$

Finally, if $a_{k+1} \not> a+1$ and $a_{k+1} = a$, then

$$g_1^{(i+a-2)} g_2^{(i+a-2)} \cdots = J_0^{(i+a-1)} J_1^{(i+a-1)} J_2^{(i+a-1)} \cdots ,$$

where

$$J_0^{(i+a-1)} = \underbrace{1 \cdots 1}_{d_k} 0, \qquad J_m^{(i+a-1)} = \underbrace{11 \cdots\cdots\cdots 1}_{a_{k+2}+g_m^{(i+a-1)}} 0 \quad (m \geqslant 1),$$

and

$$g_m^{(i+a-1)} = [(m+1)\theta_{k+2} + 1 - (b_{k+1} - a_{k+1} + 1)\theta_{k+2} - \phi_{k+2}] \\ - [m\theta_{k+2} + 1 - (b_{k+1} - a_{k+1} + 1)\theta_{k+2} - \phi_{k+2}]$$

where $b_{k+1} = a_{k+1}$ or $a_{k+1} + 1$. Thus, we have

$$f_1 f_2 f_3 \cdots = J_0 K_0^{(2)} K_0^{(3)} \cdots K_0^{(i-1)} K_{-1}^{(i)} K_0^{(i)} \cdots K_0^{(i+a-1)} K_1^{(i+a-1)} K_2^{(i+a-1)} \cdots ,$$

where

$$K_0^{(i+a-1)} = \left(u(u^c v)^a\right)^{d_k} u^c v, \quad K_m^{(i+a-1)} = \left(u(u^c v)^a\right)^{a_{k+2}+g_m^{(i+a-1)}} u^c v \quad (m \geqslant 1).$$

Therefore, we summarise the case when $\theta_k + \phi_k > 1$, including $a_{k+1} = 1$, and we have

$$f_1 f_2 f_3 \cdots = J_0 K_0^{(2)} K_0^{(3)} \cdots K_0^{(i-1)} K_{-1}^{(i)} K_0^{(i)} K_1^{(i)} K_2^{(i)} \cdots ,$$

where

$$K_{-1}^{(i)} = \left(u^{c+1} v\right)^{b_{k-1} - a_{k-1}}, \quad K_0^{(i)} = (u^c v)^{a_{k+1}-1} \left(u(u^c v)^{a_{k+1}}\right)^{d_k} u^c v .$$

For $m \geqslant 1$

$$K_m^{(i)} = \left(u(u^c v)^{a_{k+1}}\right)^{a_{k+2}+g_m^{(i)}} u^c v$$

and

$$g_m^{(i)} = [(m+1)\theta_{k+2} + 1 - (b_{k+1} - a_{k+1} + 1)\theta_{k+2} - \phi_{k+2}] \\ - [m\theta_{k+2} + 1 - (b_{k+1} - a_{k+1} + 1)\theta_{k+2} - \phi_{k+2}],$$

where $b_{k+1} = a_{k+1}$ or $a_{k+1} + 1$.

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In the case when $a_{k-1} = b_{k-1}$, $J_{-1}^{(i)}$ (so, $K_{-1}^{(i)}$) is omitted.

We can go to the next step by applying the substitutions

$$u \longrightarrow u(u^c v)^{a_{k+1}}, \quad v \longrightarrow u^c v \quad \text{and} \quad c \longrightarrow a_{k+2}.$$

Furthermore, $\theta_k + \phi_k > 1$ implies $a_{k+1} \leqslant b_{k+1}$. When $a_{k+1} = b_{k+1}$, we have $d_k = b_{k+2}$ since $1 - \theta_{k,a_{k+1}} - \phi_{k,a_{k+1}} = b_{k+2} + 1 - \phi_{k+2}$. When $a_{k+1} + 1 = b_{k+1}$, we have $d_k = 0$ since $1 - \theta_{k,a_{k+1}} - \phi_{k,a_{k+1}} = 1 - (1 - \phi_{k+1})/\theta_{k+1}$ and $\theta_k < 1/a_{k+1}$.

We obtain the following general theorem which incorporates Theorem 2:

Theorem 3. *Let θ be irrational and ϕ be real, satisfying $0 < \theta, \phi, \theta + \phi < 1$ and $m_1\theta + \phi \neq m_2$ for any m_1, $m_2 \in \mathbb{Z}$. Then the characteristic word of the sequence $f_n = f(n; \theta, \phi)$ is given by*

$$f_1 f_2 f_3 \cdots = \lim_{n\to\infty} J_0 J_{k_1} J_{k_2} J_{k_3} \cdots J_{k_n}.$$

Here, k_1, k_2, k_3, ..., are determined by

$$k_1 = 1, \qquad k_{n+1} = \begin{cases} k_n + 1, & \text{if } \theta_{k_n} + \phi_{k_n} < 1; \\ k_n + 2, & \text{if } \theta_{k_n} + \phi_{k_n} > 1 \end{cases} \quad (n \geqslant 1);$$

and $J_0 = \underbrace{0 \cdots 0}_{b_1 - 2} 1$. For $k = k_1, k_2, k_3, \ldots$, if $\theta_k + \phi_k < 1$,

$$J_k = \underline{u_{k+1}^{b_{k-1} - a_{k-1}}} u_{k+1}^{b_{k+1} - 1} v_{k+1},$$

where $u_1 = 0$, $v_1 = 1$, and for $k = k_n \geqslant 1$

$$u_{k+1} = \begin{cases} u_k^{a_k - 1} v_k, & \text{if } k_{n-1} = k - 1; \\ u_k^{a_k} v_k, & \text{if } k_{n-1} = k - 2, \end{cases} \qquad v_{k+1} = u_k u_{k+1}.$$

If $\theta_k + \phi_k > 1$,

$$J_k = \underline{(u_k v_{k+2})^{b_{k-1} - a_{k-1}}} v_{k+2}^{a_{k+1} - 1} u_{k+2}^{d_k} v_{k+2},$$

where $u_1 = 0$, $v_1 = 1$, and for $k = k_n \geqslant 1$

$$u_{k+2} = u_k v_{k+2}^{a_{k+1}}, \qquad v_{k+2} = \begin{cases} u_k^{a_k - 1} v_k, & \text{if } k_{n-1} = k - 1; \\ u_k^{a_k} v_k, & \text{if } k_{n-1} = k - 2. \end{cases}$$

$$d_k = \begin{cases} 0, & \text{if } a_{k+1} + 1 = b_{k+1}; \\ b_{k+2}, & \text{if } a_{k+1} = b_{k+1}. \end{cases}$$

The underlined parts occur as stated when $k_n = k (\geqslant 3)$ and $k_{n-1} = k - 2$. Otherwise, they are empty.

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5. Example

Let $\theta = \sqrt{3} - 1$ and $\phi = \sqrt{5} - 2$. Then

$$a_1 = 1,\ a_2 = 2,\ a_3 = 1,\ a_4 = 2,\ a_5 = 1,\ a_6 = 2,\ a_7 = 1,\ a_8 = 2,\ a_9 = 1,\ \ldots$$
$$b_1 = 2,\ b_2 = 3,\ b_3 = 1,\ b_4 = 2,\ b_5 = 1,\ b_6 = 1,\ b_7 = 2,\ b_8 = 3,\ b_9 = 2,\ \ldots$$

and

$$\theta + \phi,\ \theta_3 + \phi_3,\ \theta_5 + \phi_5,\ \theta_{10} + \phi_{10},\ \ldots < 1,$$
$$\theta_1 + \phi_1,\ \theta_2 + \phi_2,\ \theta_4 + \phi_4,\ \theta_6 + \phi_6,\ \theta_7 + \phi_7,\ \theta_8 + \phi_8,\ \theta_9 + \phi_9,\ \ldots > 1.$$

So, $d_1 = 0,\ d_2 = 2,\ d_4 = 1,\ d_6 = 0,\ d_7 = 0,\ d_8 = 0,\ \ldots.$

Therefore,

$$f_1 f_2 f_3 \cdots = J_0 J_1 J_3 J_4 J_6 \cdots ,$$

where $J_0 = 1$, $J_1 = v_3^{a_2 - 1} u_3^{d_1} v_3 = 11$ from $v_3 = u_1^{a_1 - 1} v_1 = 1$ and $u_3 = u_1 v_3^{a_2} = 011$, $J_3 = u_4^{b_2 - a_2} u_4^{b_4 - 1} v_4 = 011101110110111$ from $u_4 = u_3^{a_3} v_3 = 0111$ and $v_4 = u_3 u_4 = 0110111$,

$$J_4 = v_6^{a_5 - 1} u_6^{d_4} v_6 = 0111011101101110110110111,$$
$$J_6 = v_8^{a_7 - 1} u_8^{d_6} v_8 = 0111011101101110111011101101110110110111,$$

because $v_6 = u_4^{a_4 - 1} v_4 = 01110110111$ and $u_6 = u_4 v_6 = 011101110110111$, $v_8 = u_6^{a_6} v_6 = 0111011101101110111011101101110110110111$ and $u_8 = u_6 v_8^{a_7}$.

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