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PERSISTENT PROPERTIES FROM THE GROMOV–HAUSDORFF VIEWPOINT

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Abstract In this paper, we introduce topologically IGH-stable, IGH-persistent, average IGH-persistent and pointwise weakly topologically IGH-stable homeomorphisms of compact metric spaces. We prove that every topologically IGH-stable homeomorphism is topologically stable and every expansive topologically stable homeomorphism of a compact manifold is topologically IGH-stable. We further prove that every equicontinuous pointwise weakly topologically IGH-stable homeomorphism is IGH-persistent and every pointwise minimally expansive IGH-persistent homeomorphism is pointwise weakly topologically IGH-stable. Finally, we prove that every mean equicontinuous pointwise weakly topologically IGH-stable homeomorphism is average IGH-persistent.

Keywords: Gromov–Hausdorff; persistent; topologically stable

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1. Introduction

In the stability theory of dynamical systems, we aim to obtain the sufficient conditions under which the stability of a trajectory under small enough perturbations can be achieved. In [15], Walters has addressed this problem for homeomorphisms and has proved that every expansive homeomorphism of a compact metric space with the shadowing property is topologically stable. In the current literature, this result is known as Walters' stability theorem. Many variants of this result have also been proved under different dynamical settings including in [2, 8–10].

Topological stability of a homeomorphism $f : X \to X$ of a compact metric space X says that in the class of all homeomorphisms of X equipped with the C^0-metric, there exists a neighbourhood N of f in which f can be seen with prescribed error via continuous image of h for every homeomorphism $h \in N$. In [5], the author has studied Gromov–Hausdorff metric to measure the distance between two metric spaces, which has motivated authors of

[2] to combine C^0-distance with the Gromov–Hausdorff distance to measure the distance between two homeomorphisms of possibly two distinct metric spaces. They have used the resultant C^0-Gromov–Hausdorff distance to study the stability of a homeomorphism of a compact metric space with respect to the class of all homeomorphisms of arbitrary compact metric spaces. This notion is known as topological GH stability, where "GH" denotes the dependence of the notion on the Gromov–Hausdorff distance. They have proved that every expansive homeomorphism of a compact metric space with the shadowing property is topologically GH-stable. Moreover, every topologically GH-stable circle homeomorphism is topologically stable, but every topologically stable homeomorphism need not be topologically GH-stable. However they have not answered that whether every topologically GH-stable homeomorphism is topologically stable. To address this problem, we get motivated to introduce a stronger form of topologically stable and topologically GH-stable homeomorphisms, namely, topologically IGH-stable homeomorphisms, where I denotes the dependence of the notion on δ-isometries. In Example 3.7, we give an example to show that topologically stable homeomorphism need not be topologically IGH-stable in order to address the converse of the Theorem 3.5.

In [9], the authors have introduced topologically stable points and have proved that every shadowable point of an expansive homeomorphism of a compact metric space is topologically stable. In [8], the authors have introduced minimally expansive points and have proved that every minimally expansive shadowable point of a homeomorphism of a compact metric space is topologically stable. In [11], the author has introduced the persistent property which is a weaker notion than topological stability for homeomorphisms of compact manifolds. In [4, 7], the authors have proved that this relationship holds for equicontinuous homeomorphisms as well. Precisely, they have proved that every equicontinuous pointwise topologically stable homeomorphism of a compact metric space is persistent. In [6], the authors have addressed the converse of this result for pointwise weakly topologically stable homeomorphisms. The second motivation of this paper comes from these results. We introduce the persistent property by using the GromovHausdorff distance and then prove the analogue of the latter result in Theorem 3.12(1). We also address the converse of this result in Theorem 3.9(3).

This paper is distributed as follows. In § 2, we give the necessary preliminaries required for the remaining section. In § 3, we introduce topological IGH stability, IGH persistence, IGH persistent points, weakly topologically IGH-stable points and average IGH persistence for homeomorphisms of compact metric spaces. Then we prove Theorems 3.5, 3.9 and 3.12.

2. Preliminaries

Throughout this paper, $(X, d_X), (Y, d_Y)$ and (Z, d_Z) denotes compact metric spaces. If no confusion arises, then we use "d" for the metric on X. For a given $\epsilon > 0$ and for each $x \in X$, we define $B(x, \epsilon) = \{y \in X \mid \mathrm{d}(x, y) < \epsilon\}$. Let $f : X \to X$ be a homeomorphism. The orbit of a point $x \in X$ under f is the set $\mathcal{O}_f(x) = \{f^n(x) \mid n \in \mathbb{Z}\}$.

Let $f : X \to X$ be a continuous map. We say that f is mean equicontinuous if for each $\epsilon > 0$, there exists a $\delta > 0$ such that for every $x, y \in X$ with $\mathrm{d}(x, y) < \delta$, we have $\limsup\limits_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathrm{d}(f^i(x), f^i(y)) < \epsilon$ [12].

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Let $f : X \to X$ be a homeomorphism. We say that f is equicontinuous if for each $\epsilon > 0$, there exists a $\delta > 0$ such that for every $x, y \in X$ with $\mathrm{d}(x, y) < \delta$, we have $\mathrm{d}(f^n(x), f^n(y)) < \epsilon$, for each $n \in \mathbb{Z}$. Clearly, every equicontinuous homeomorphism is mean equicontinuous.

Let $f : X \to X$ be a homeomorphism. We say that f is expansive on a subset B of X if there exists a $\mathfrak{c} > 0$ (called the expansivity constant) such that for each pair of distinct points $x, y \in B$, there exists an $n \in \mathbb{Z}$ satisfying $\mathrm{d}(f^n(x), f^n(y)) > \mathfrak{c}$. We say that f is expansive if f is expansive on X [14]. We say that a point $x \in X$ is a minimally expansive point of f if there exists a $\mathfrak{c} > 0$ such that for each $y \in B(x, \mathfrak{c})$, f is expansive on $\overline{\mathcal{O}_f(y)}$ with an expansivity constant $\mathfrak{c}$. Such a constant $\mathfrak{c}$ is said to be an expansivity constant for the minimal expansivity of f at x. The set of all minimally expansive points of f is denoted by $M_f(X)$. We say that f is pointwise minimally expansive if $M_f(X) = X$. Recall that if f is expansive, then f is pointwise minimally expansive [8].

For a metric space (X, d_X) and $A, B \subseteq X$, we define

$$d^X(A, B) = \inf\{d_X(a, b) \mid (a, b) \in A \times B\}.$$

We replace A by "a" if $A = \{a\}$.

The Hausdorff distance between A and B is given by 33

$$d_H^X(A, B) = \max\left\{\sup_{a\in A} d^X(a, B), \sup_{b\in B} d^X(A, b)\right\}.$$

We say that an onto map $i : X \to Y$ is an isometry if $d_X(x, x') = d_Y(i(x), i(x'))$, for every $x, x' \in X$. For a given $\delta > 0$, we say that a map $j : X \to Y$ is a δ-isometry if

$$\max\left\{d_H^Y(j(X), Y), \sup_{x,x'\in X} |d_Y(j(x), j(x')) - d_X(x, x')|\right\} < \delta.$$

The C^0-distance between maps $f : X \to Y$ and $\overline{f} : X \to Y$ is given by

$$d_{C^0}^Y(f, \overline{f}) = \sup_{x\in X} d_Y(f(x), \overline{f}(x)).$$

The C^0-Gromov–Hausdorff distance [2] between continuous maps $h : X \to X$ and $g : Y \to Y$ is given by

$$\begin{aligned} d_{GH^0}(h, g) = \inf\{\delta > 0 \mid & \text{ there exist } \delta\text{-isometries } i : X \to Y \text{ and } j : Y \to X \text{ such that} \\ & d_{C^0}^Y(i \circ h, g \circ i) < \delta \text{ and } d_{C^0}^X(h \circ j, j \circ g) < \delta\}. \end{aligned}$$

For a given $\delta > 0$, we define $I_\delta(h, g) = \{(i, j) \mid i : X \to Y$ and $j : Y \to X$ are δ-isometries such that $d_{C^0}^Y(i \circ h, g \circ i) < \delta$ and $d_{C^0}^X(h \circ j, j \circ g) < \delta\}$ and $P(I_\delta(h, g)) = \{j : Y \to X \mid j$ is a δ-isometry and there exists a δ-isometry $i : X \to Y$ such that $(i, j) \in I_\delta(h, g)\}$. Recall that $d_{GH^0}(h, g) \leq d_{C^0}(h, g)$ [2, Theorem 1(1)].

Let $f : X \to X$ be a homeomorphism and $x \in X$. Then we say that

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(i) f is topologically stable if for each $\epsilon > 0$, there exists a $\delta > 0$ such that for each homeomorphism $g : X \to X$ satisfying $d_{C^0}(f, g) < \delta$, there exists a continuous map $h : X \to X$ such that $f \circ h = h \circ g$ and $d(h(x), x) < \epsilon$, for each $x \in X$ [15].

(ii) f is topologically GH-stable if for each $\epsilon > 0$, there exists a $\delta > 0$ such that for each homeomorphism $g : Y \to Y$ of a compact metric space Y satisfying $d_{GH^0}(f, g) < \delta$, there exists a continuous ϵ-isometry $h : Y \to X$ such that $f \circ h = h \circ g$ [2].

Let $f : X \to X$ be a homeomorphism. We say that f is persistent through a subset B of X if for each $\epsilon > 0$, there exists a $\delta > 0$ such that for each homeomorphism $g : X \to X$ satisfying $d_{C^0}(f, g) \leq \delta$ and for each $x \in B$, there exists a $y \in X$ such that $d(f^n(x), g^n(y)) < \epsilon$, for each $n \in \mathbb{Z}$. We say that f is persistent if f is persistent through X [11]. We say that a point $x \in X$ is a persistent point of f if f is persistent through x. The set of all persistent points of f is denoted by $P_f(X)$. We say that f is pointwise persistent if $P_f(X) = X$ [4].

Let $f : X \to X$ be a homeomorphism. Choose an $\eta > 0$ and a subset B of X. We say that a sequence $\rho = \{x_n\}_{n\in\mathbb{Z}}$ of elements of X is through B if $x_0 \in B$. We say that ρ is an η-pseudo orbit of f through B if ρ is through B and $d(f(x_n), x_{n+1}) < \eta$, for each $n \in \mathbb{Z}$. We say that ρ can be η-traced through f if there exists a $z \in X$ such that $d(f^n(z), x_n) < \eta$, for each $n \in \mathbb{Z}$. We say that f has the shadowing property if for each $\epsilon > 0$, there exists a $\delta > 0$ such that each δ-pseudo orbit of f through X can be ϵ-traced through f by some point of X [1]. We say that a point $x \in X$ is a shadowable point of f if for each $\epsilon > 0$, there exists a $\delta > 0$ such that each δ-pseudo orbit of f through x can be ϵ-traced through f by some point of X. The set of all shadowable points of f is denoted by $\mathrm{Sh}_f(X)$ [13].

3. Weakly topologically IGH-stable and IGH persistence

In this section, we define topologically IGH-stable, IGH-persistent, average IGH-persistent and pointwise weakly topologically IGH-stable homeomorphisms and study the relationship between these notions. Then we prove Theorems 3.5, 3.9 and 3.12.

Definition 3.1. *Let $f : X \to X$ be a homeomorphism. We say that f is topologically IGH-stable if for each $\epsilon > 0$, there exists a $\delta > 0$ such that if $g : Y \to Y$ is a homeomorphism satisfying $d_{GH^0}(f, g) < \delta$, then for each $j \in P(I_\delta(f, g))$, there exists a continuous map $h : Y \to X$ such that $f \circ h = h \circ g$ and $d_X(h(y), j(y)) < \epsilon$, for each $y \in Y$.*

Definition 3.2. *Let $f : X \to X$ be a homeomorphism. We say that a point $x \in X$ is a weakly topologically IGH-stable point of f if for each $\epsilon > 0$, there exists a $\delta > 0$ such that if $g : Y \to Y$ is a homeomorphism satisfying $d_{GH^0}(f, g) < \delta$, then for each $j \in P(I_\delta(f, g))$, there exists a $z \in B(x, \epsilon)$ such that for each $y \in j^{-1}(z)$, there exists a continuous map $h : \overline{\mathcal{O}_g(y)} \to X$ such that $f \circ h = h \circ g$ and $d_X(h(u), j(u)) < \epsilon$, for each $u \in \overline{\mathcal{O}_g(y)}$. The set of all weakly topologically IGH-stable points of f is denoted by $WGH_f(X)$. We say that f is pointwise weakly topologically IGH-stable if $WGH_f(X) = X$.*

Definition 3.3. *Let $f : X \to X$ be a homeomorphism. We say that f is IGH-persistent through a subset B of X if for each $\epsilon > 0$, there exists a $\delta > 0$ such that if $g : Y \to Y$ is*

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a homeomorphism satisfying $d_{GH^0}(f,g) < \delta$, then for each $j \in P(I_\delta(f,g))$ and for each $x \in B$, there exists a $z \in X$ such that if $y \in j^{-1}(z)$, then $d_X(f^n(x), j(g^n(y))) < \epsilon$, for each $n \in \mathbb{Z}$. We say that f is IGH-persistent if f is IGH-persistent through X. We say that a point $x \in X$ is an IGH-persistent point of f if f is IGH-persistent through x. The set of all IGH-persistent points of f is denoted by $GHP_f(X)$. We say that f is pointwise IGH-persistent if $GHP_f(X) = X$.

Definition 3.4. *Let $f : X \to X$ be a homeomorphism. We say that f is average IGH-persistent if for each $\epsilon > 0$, there exists a $\delta > 0$ such that if $g : Y \to Y$ is a homeomorphism satisfying $d_{GH^0}(f,g) < \delta$, then for each $j \in P(I_\delta(f,g))$ and for each $x \in X$, there exists a $z \in X$ such that if $y \in j^{-1}(z)$, then $\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d_X(f^i(x), j(g^i(y))) < \epsilon$.*

Theorem 3.5. *Let $f : X \to X$ be a homeomorphism of a compact metric space X. If f is topologically IGH-stable, then f is topologically stable and topologically GH-stable.*

Proof. Note that if j is a δ-isometry and $h : X \to Y$ is a continuous map satisfying $d_X(h(y), j(y)) < \epsilon$, for each $y \in Y$, then h is a continuous $(2\epsilon + \delta)$-isometry. Thus, for a given $\epsilon > 0$, we can choose an appropriate $\delta > 0$ and use the corresponding definitions to conclude that every topologically IGH-stable homeomorphism is topologically GH-stable as well as topologically stable. □

Theorem 3.6. *Let $f : X \to X$ be a homeomorphism and $x \in X$. Then the following statements are true:*

(1) *If f is topologically IGH-stable, then f is pointwise weakly topologically IGH-stable.*
(2) *If f is IGH-persistent, then f is persistent and pointwise IGH-persistent.*
(3) *If x is an IGH-persistent point of f, then x is a persistent point of f.*
(4) *If f is IGH-persistent, then f is average IGH-persistent.*

Proof. Proofs of the statements (1), (2) and (3) are similar. Proof of statement (4) follows from the corresponding definitions. Therefore we prove only statement (3). Let $x \in X$ be an IGH-persistent point of f and choose an $\epsilon > 0$. For this ϵ, choose a $\delta > 0$ by the definition of IGH-persistent point. Let $g : X \to X$ be a homeomorphism satisfying $d_{C^0}(f,g) < \delta$. Since $d_{GH^0}(f,g) \leq d_{C^0}(f,g) < \delta$ and $I_X \in P(I_\delta(f,g))$, where I_X denotes the identity map of X, we get that there exists a $z \in X$ such that if $y \in I_X^{-1}(z) = \{z\}$, then $d_X(f^n(x), I_X(g^n(y))) = d_X(f^n(x), g^n(y)) < \epsilon$, for each $n \in \mathbb{Z}$. Since $I_X^{-1}(u) \neq \phi$, for each $u \in X$, $I_X \in P(I_\alpha(f,g))$, for each $\alpha > 0$ and ϵ chosen arbitrarily, we get that x is a persistent point of f. □

Example 3.7. In [2, Theorem 2], the authors have proved that there exist a compact metric space (X, d) and a homeomorphism $f : X \to X$ such that f is topologically stable, but f is not topologically GH-stable. From Theorem 3.5, we get that every topologically IGH-stable homeomorphism is topologically GH-stable. Therefore, f is not topologically IGH-stable. Hence, every topologically stable homeomorphism need not be topologically IGH-stable. □

We do not know that whether every topologically GH-stable homeomorphism is topologically IGH-stable. However, if there exists a homeomorphism which is topologically

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GH-stable but not topologically stable, then we can follow the similar arguments as in the last example to answer this question in negative. Moreover, we are not presently aware about any fact which holds for topologically IGH-stable homeomorphisms but does not hold for topologically GH-stable homeomorphisms.

Now we recall the following Lemma from [8] which will be useful to prove the second main result of this paper, namely, Theorem 3.9.

Lemma 3.8. [8] *Let $f : X \to X$ be a homeomorphism and $x \in X$ be a minimally expansive point of f with an expansivity constant $\mathfrak{c}$. Then, for each $y \in B(x, \mathfrak{c})$ and for each $0 < \epsilon < \mathfrak{c}$, there exists an $N \in \mathbb{N}$ such that for each pair $u, v \in \mathcal{O}_f(y)$ with $d(f^n(u), f^n(v)) < \mathfrak{c}$, for all $-N \leq n \leq N$, we have $d(u, v) < \epsilon$.*

Theorem 3.9. *Let $f : X \to X$ be a homeomorphism of a compact metric space X and $x \in X$. Then the following statements are true:*

(1) *If f is an expansive homeomorphism with the shadowing property, then f is topologically IGH-stable.*
(2) *If x is a minimally expansive shadowable point of f, then x is a weakly topologically IGH-stable point of f.*
(3) *If x is a minimally expansive IGH-persistent point of f, then x is a weakly topologically IGH-stable point of f.*

Proof.

(1) Let f be an expansive homeomorphism with an expansivity constant $\mathfrak{c}$. We claim that if f has the shadowing property, then f is topologically IGH-stable. Let $\epsilon > 0$ be given. For $\eta = \frac{\min\{\epsilon, \mathfrak{c}\}}{8}$, choose $0 < \delta < \eta$ by the shadowing property. Let $g : Y \to Y$ be a homeomorphism satisfying $d_{GH^0}(f, g) < \delta$ and choose a $j \in P(I_\delta(f, g))$. Thus, we have $d^X_{C^0}(j \circ g, f \circ j) < \delta$ implying that $\overline{x}_n = \{j(g^n(\overline{x}))\}_{n \in \mathbb{Z}}$ is a δ-pseudo orbit of f, for each $\overline{x} \in Y$. Choose an $x \in X$ such that $d(f^n(x), jg^n(\overline{x})) < \eta$, for each $n \in \mathbb{Z}$. Note that if there exists another $z \in X$ such that $d(f^n(z), jg^n(\overline{x})) < \eta$, for each $n \in \mathbb{Z}$, then $d(f^n(x), f^n(z)) < \mathfrak{c}$, for each $n \in \mathbb{Z}$. Since f is expansive, we get that $x = z$. Thus, we can define $h : Y \to X$ by $h(\overline{x}) = x$, for each $\overline{x} \in Y$. In particular, for $n = 0$, we get that $d^X(h(u), j(u)) < \epsilon$, for each $u \in Y$. Moreover, $\mathrm{d}(f^n(h(g(u))), f^n(f(h(u)))) \leq \mathrm{d}(f^n(h(g(u))), j(g^n(g(u)))) + \mathrm{d}(j(g^n(g(u))), f^n(f(h(u)))) \leq \mathfrak{c}$, for each $n \in \mathbb{Z}$ and for each $u \in Y$. Since f is expansive, we get that $(f \circ h)(u) = (h \circ g)(u)$, for each $u \in Y$.

Now, we claim that h is continuous. For $0 < \epsilon < \mathfrak{c}$, choose an $N \in \mathbb{N}$ such that if $\mathrm{d}(f^n(x_1), f^n(x_2)) \leq \mathfrak{c}$, for all $-N \leq n \leq N$, then $d(x_1, x_2) < \epsilon$ [15]. From the uniform continuity of g, we can choose $0 < \gamma < \epsilon$ such that for every $u, v \in Y$ with $d_Y(u, v) < \gamma$, we have $d_Y(g^n(u), g^n(v)) < \frac{\mathfrak{c}}{2}$, for all $-N \leq n \leq N$. Therefore, for every $u, v \in Y$ with $d_Y(u, v) < \gamma$ and for all $-N \leq n \leq N$, we have

$$\begin{aligned} d_X(f^n(h(u)), f^n(h(v))) &= d_X(h(g^n(u)), h(g^n(v))) \\ &\leq d_X(h(g^n(u)), j(g^n(u))) + d_X(j(g^n(u)), j(g^n(v))) + d_X(h(g^n(v)), j(g^n(v))) \\ &\leq d_X(h(g^n(u)), j(g^n(u))) + \delta + d_Y(g^n(u), g^n(v)) + d_X(h(g^n(v)), j(g^n(v))) < \mathfrak{c}. \end{aligned}$$

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Therefore, $d_X(h(u), h(v)) < \epsilon$ implying that h is continuous which completes the proof.

(2) Let $x \in X$ be a minimally expansive point of f with an expansivity constant $\mathfrak{c}$. We claim that if x is a shadowable point of f, then x is a weakly topologically IGH-stable point of f. Let $\epsilon > 0$ be given. For $\eta = \frac{\min\{\epsilon,\mathfrak{c}\}}{5}$, choose $0 < \delta < \eta$ by the definition of shadowable point. Let $g : Y \to Y$ be a homeomorphism satisfying $d_{GH^0}(f, g) < \delta$ and choose a $j \in P(I_\delta(f, g))$. Thus, we have $d^X_{C^0}(j \circ g, f \circ j) < \delta$ implying that $\overline{x}_n = \{j(g^n(y))\}_{n\in\mathbb{Z}}$ is a δ-pseudo orbit of f through x, for each $y \in j^{-1}(x)$. Choose a $\overline{y} \in X$ such that $d_X(f^n(\overline{y}), j(g^n(y))) < \eta$, for each $n \in \mathbb{Z}$ and for each $y \in j^{-1}(x)$. Note that if $j^{-1}(x) = \phi$, then we are done. Therefore, fix a $y \in j^{-1}(x)$ and define $h : \mathcal{O}_g(y) \to X$ by $h(g^n(y)) = f^n(\overline{y})$, for each $n \in \mathbb{Z}$. To check that h is well defined, choose $k, m \in \mathbb{Z}$ such that $g^k(y) = g^m(y)$. Then $j(g^{n+k}(y)) = j(g^{n+m}(y))$, for each $n \in \mathbb{Z}$, and hence,

$$\begin{aligned} d_X(f^n(f^k(\overline{y})), f^n(f^m(\overline{y}))) &\le d_X(f^{n+k}(\overline{y}), j(g^{n+k}(y))) \\ &\quad + d_X(j(g^{n+k}(y)), j(g^{n+m}(y))) \\ &\quad + d_X(j(g^{n+m}(y)), f^{n+m}(\overline{y})) \\ &= d_X(f^{n+k}(\overline{y}), j(g^{n+k}(y))) + d_X(j(g^{n+m}(y)), f^{n+m}(\overline{y})) \\ &< 2\eta < \mathfrak{c}, \ \text{for each } n \in \mathbb{Z}. \end{aligned}$$

Since x is a minimally expansive point of f with the expansivity constant $\mathfrak{c}$, we get that f is expansive on $\overline{\mathcal{O}_f(\overline{y})}$ with the expansivity constant $\mathfrak{c}$, and hence, $f^k(\overline{y}) = f^m(\overline{y})$. Therefore, h is well defined. Moreover, for each $n \in \mathbb{Z}$, we get that

$$\begin{aligned} (f \circ h)(g^n(y)) &= f \circ (f^n(x)) = f^{n+1}(\overline{y}) = h(g^{n+1}(y)) = h(g(g^n(y))) \\ &= (h \circ g)(g^n(y)). \end{aligned}$$

Therefore, $(f \circ h)(u) = (h \circ g)(u)$, for each $u \in \mathcal{O}_g(y)$. Also, $d_X(h(g^n(y)), j(g^n(y))) < \eta$, for each $n \in \mathbb{Z}$ implying that $d_X(h(u), j(u)) < \eta$, for each $u \in \mathcal{O}_g(y)$. Now, we claim that h is uniformly continuous. For $\overline{y}$ as above and $0 < \epsilon < \mathfrak{c}$, choose an $N \in \mathbb{N}$ from Lemma 3.8. From the uniform continuity of g, we can choose $0 < \gamma < \epsilon$ such that for every $u, v \in Y$ with $d_Y(u, v) < \gamma$, we have $d_Y(g^n(u), g^n(v)) < \frac{\mathfrak{c}}{2}$, for all $-N \le n \le N$. Therefore, for every $u, v \in \mathcal{O}_g(y)$ with $d_Y(u, v) < \gamma$, we have

$$\begin{aligned} d_X(f^n(h(u)), f^n(h(v))) &= d_X(h(g^n(u)), h(g^n(v))) \\ &\le d_X(h(g^n(u)), j(g^n(u))) + d_X(j(g^n(u)), j(g^n(v))) \\ &\quad + d_X(h(g^n(v)), j(g^n(v))) \\ &\le d_X(h(g^n(u)), j(g^n(u))) + \delta + d_Y(g^n(u), g^n(v)) \\ &\quad + d_X(h(g^n(v)), j(g^n(v))) \\ &< 3\eta + \frac{\mathfrak{c}}{2} \\ &< \mathfrak{c}, \ \text{for all } -N \le n \le N. \end{aligned}$$

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Therefore, $d_X(h(u), h(v)) < \epsilon$, implying that h is uniformly continuous. Since Y is a compact metric space and $d_X(j(y_1), j(y_2)) < \delta + d_Y(y_1, y_2)$, for all $y_1, y_2 \in Y$, we can extend h continuously to the function $H : \overline{\mathcal{O}_g(y)} \to X$ such that $f \circ H = H \circ g$ and $d_X(H(u), j(u)) < \epsilon$, for each $u \in \overline{\mathcal{O}_g(y)}$. Since y and ϵ are chosen arbitrarily, we get that x is a weakly topologically *IGH*-stable point of *f*.

(3) Let $x \in X$ be a minimally expansive point of f with an expansivity constant $\mathfrak{c}$. We claim that if x is an *IGH*-persistent point of *f*, then x is a weakly topologically *IGH*-stable point of *f*. Let $\epsilon > 0$ be given. For $\eta = \frac{\min\{\epsilon, \mathfrak{c}\}}{5}$, choose $0 < \delta < \eta$ by the definition of *IGH*-persistent point. Let $g : Y \to Y$ be a homeomorphism satisfying $d_{GH^0}(f, g) < \delta$ and choose a $j \in P(I_\delta(f, g))$. Then, there exists a $z \in X$ such that if $y \in j^{-1}(z)$, then $d_X(f^n(x), j(g^n(y))) < \eta$, for each $n \in \mathbb{Z}$. Note that if $j^{-1}(z) = \phi$, then we are done. Therefore, fix a $y \in j^{-1}(z)$. Define $h : \mathcal{O}_g(y) \to X$ by $h(g^n(y)) = f^n(x)$, for each $n \in \mathbb{Z}$.

To check that h is well defined, choose $k, m \in \mathbb{Z}$ such that $g^k(y) = g^m(y)$. Then, $j(g^{n+k}(y)) = j(g^{n+m}(y))$, for each $n \in \mathbb{Z}$, and hence,

$$\begin{aligned} d_X(f^n(f^k(x)), f^n(f^m(x))) &\le d_X(f^{n+k}(x), j(g^{n+k}(y))) \\ &\quad + d_X(j(g^{n+k}(y)), j(g^{n+m}(y))) \\ &\quad + d_X(j(g^{n+m}(y)), f^{n+m}(x)) \\ &= d_X(f^{n+k}(x), j(g^{n+k}(y))) + d_X(j(g^{n+m}(y)), f^{n+m}(x)) \\ &< 2\eta < \mathfrak{c}, \text{ for each } n \in \mathbb{Z}. \end{aligned}$$

Since x is a minimally expansive point of f with the expansivity constant $\mathfrak{c}$, we get that f is expansive on $\overline{\mathcal{O}_f(x)}$ with the expansivity constant $\mathfrak{c}$, and hence, $f^k(x) = f^m(x)$. Therefore h is well defined. Moreover,

$$\begin{aligned} (f \circ h)(g^n(y)) &= f \circ (f^n(x)) = f^{n+1}(x) \\ &= h(g^{n+1}(y)) = h(g(g^n(y))) \\ &= (h \circ g)(g^n(y)), \text{ for each } n \in \mathbb{Z}. \end{aligned}$$

Therefore, $(f \circ h)(u) = (h \circ g)(u)$, for each $u \in \mathcal{O}_g(y)$. Also, $d_X(h(g^n(y)), j(g^n(y))) < \eta$, for each $n \in \mathbb{Z}$ implying that $d_X(h(u), j(u)) < \eta$, for each $u \in \mathcal{O}_g(y)$.

Now, we claim that h is uniformly continuous. For the x as above and $0 < \epsilon < \mathfrak{c}$, choose an $N \in \mathbb{N}$ from Lemma 3.8. From the uniform continuity of g, we can choose $0 < \gamma < \epsilon$ such that for every $u, v \in Y$ with $d_Y(u, v) < \gamma$, we have $d_Y(g^n(u), g^n(v)) < \frac{\mathfrak{c}}{2}$, for all $-N \le n \le N$. Therefore, for every $u, v \in \mathcal{O}_g(y)$ with $d_Y(u, v) < \gamma$ and for all $-N \le n \le N$, we have

$$\begin{aligned} d_X(f^n(h(u)), f^n(h(v))) &= d_X(h(g^n(u)), h(g^n(v))) \\ &\le d_X(h(g^n(u)), j(g^n(u))) + d_X(j(g^n(u)), j(g^n(v))) \\ &\quad + d_X(h(g^n(v)), j(g^n(v))) \\ &\le d_X(h(g^n(u)), j(g^n(u))) + \delta + d_Y(g^n(u), g^n(v)) \\ &\quad + d_X(h(g^n(v)), j(g^n(v))) < \mathfrak{c} \end{aligned}$$

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Therefore, $d_X(h(u), h(v)) < \epsilon$, implying that h is uniformly continuous. Since Y is a compact metric space and $d_X(j(y_1), j(y_2)) < \delta + d_Y(y_1, y_2)$, for all $y_1, y_2 \in Y$, we can extend h continuously to the function $H : \overline{\mathcal{O}_g(y)} \to X$ such that $f \circ H = H \circ g$ and $d_X(H(u), j(u)) < \epsilon$, for each $u \in \overline{\mathcal{O}_g(y)}$. Since y and ϵ are chosen arbitrarily, we get that x is a weakly topologically IGH-stable point of f.

□

Corollary 3.10. *Let $f : X \to X$ be an expansive homeomorphism of a compact manifold X. Then f has the shadowing property if and only if f is topologically stable if and only if f is topologically IGH-stable.*

Proof. Recall that if $f : X \to X$ is an expansive homeomorphism of a compact manifold, then f has the shadowing property if and only if f is topologically stable [15]. Now, we use Theorem 3.5 and Theorem 3.9(1) to complete the proof. □

In the next example, we give a homeomorphism on which one can apply Theorem 3.9(2) but cannot apply Theorem 3.9(1).

Example 3.11. Let $g : Y \to Y$ be an expansive homeomorphism with the shadowing property on an uncountable compact metric space (Y, d_0). Let p be a periodic point of g with prime period $t \geq 2$. Let $X = Y \cup E$, where E is an infinite enumerable set. Set $Q = \bigcup_{k\in\mathbb{N}} \{1, 2, 3\} \times \{k\} \times \{0, 1, 2, 3, \cdots, t-1\}$. Suppose that $r : \mathbb{N} \to E$ and $s : Q \to \mathbb{N}$ are bijections. Consider the bijection $q : Q \to E$ defined as $q(i, k, j) = r(s(i, k, j))$, for each $(i, k, j) \in Q$. Therefore, any point $x \in E$ has the form $x = q(i, k, j)$ for some $(i, k, j) \in Q$. Consider the function $d : X \times X \to \mathbb{R}^+$ defined by

$$d(a,b) = \begin{cases} 0 & \text{if } a = b, \\ d_0(a,b) & \text{if } a, b \in Y \\ \frac{1}{k} + d_0(g^j(p), b) & \text{if } a = q(i,k,j) \text{ and } b \in Y \\ \frac{1}{k} + d_0(a, g^j(p)) & \text{if } a \in Y \text{ and } b = q(i,k,j) \\ \frac{1}{k} & \text{if } a = q(i,k,j), b = q(l,k,j) \text{ and } i \neq l \\ \frac{1}{k} + \frac{1}{m} + d_0(g^j(p), g^r(p)) & \text{if } a = q(i,k,j), b = q(i,m,r) \text{ and } k \neq m \text{ or } j \neq r \end{cases}$$

and $f : X \to X$ defined by

$$f(x) = \begin{cases} g(x) & \text{if } x \in Y \\ q(i, k, (j+1)) \bmod t & \text{if } x = q(i,k,j). \end{cases}$$

Recall that (X, d) is a compact metric space and f is a pointwise minimally expansive homeomorphism with the shadowing property [3, 8]. Therefore, f is pointwise weakly topologically IGH-stable.

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Theorem 3.12. *Let $f : X \to X$ be a pointwise weakly topologically IGH-stable homeomorphism of a compact metric space X. Then, the following statements are true:*

(1) *If f is equicontinuous, then f is IGH-persistent.*
(2) *If f is mean equicontinuous, then f is average IGH-persistent.*

Proof. Let $f : X \to X$ be a pointwise weakly topologically IGH-stable homeomorphism.

(1) Suppose that f is equicontinuous. We first claim that $\mathrm{WGH}_f(X) \subseteq \mathrm{GHP}_f(X)$. Let $x \in \mathrm{WGH}_f(X)$ and $\epsilon > 0$ be given. For $\frac{\epsilon}{3}$, choose $0 < \alpha < \frac{\epsilon}{3}$ by the definition of equicontinuity. For this α, choose a $\delta > 0$ by the definition of weakly topologically IGH-stable point. Let $g : Y \to Y$ be a homeomorphism satisfying $d_{GH^0}(f, g) < \delta$ and choose a $j \in P(I_\delta(f, g))$. Then, there exists a $z \in B(x, \alpha)$ such that for each $y \in j^{-1}(z)$, there exists a continuous map $h : \overline{\mathcal{O}_g(y)} \to X$ such that $f \circ h = h \circ g$ and $d_X(h(u), j(u)) < \alpha$, for each $u \in \overline{\mathcal{O}_g(y)}$. Hence, $d_X(f^n(x), j(g^n(y))) \leq [d_X(f^n(x), f^n(z)) + d_X(f^n(j(y)), f^n(h(y))) + d_X(f^n(h(y)), j(g^n(y)))] \leq [\frac{\epsilon}{3} + \frac{\epsilon}{3} + \alpha] < \epsilon$, for each $n \in \mathbb{Z}$. Since y and ϵ are chosen arbitrarily, we get that $x \in GHP_f(X)$. Since f is pointwise weakly topologically IGH-stable, we get that f is pointwise IGH-persistent as well.

Now, we claim that f is IGH-persistent as well. Define $\mathrm{GHP}^*_f(X) = \{x \in X \mid$ for each $\epsilon > 0$, there exists a $\delta > 0$ such that if $g : Y \to Y$ is a homeomorphism satisfying $d_{GH^0}(f, g) < \delta$, then for each $j \in P(I_\delta(f, g))$ and for each $u \in B(x, \delta)$, there exists a $z \in X$ such that if $y \in j^{-1}(z)$, then $d_X(f^n(u), j(g^n(y))) < \epsilon$, for each $n \in \mathbb{Z}\}$. We first claim that $\mathrm{GHP}^*_f(X) = X$. Since f is pointwise IGH-persistent, it is enough to show that $\mathrm{GHP}_f(X) \subseteq \mathrm{GHP}^*_f(X)$. For $\frac{\epsilon}{3}$, choose $0 < \alpha < \frac{\epsilon}{3}$ by the definition of equicontinuity. For this α, choose a $\delta > 0$ by the definition of IGH-persistent point. Let $g : Y \to Y$ be a homeomorphism satisfying $d_{GH^0}(f, g) < \delta$ and choose a $j \in P(I_\delta(f, g))$. Then, there exists a $z \in X$ such that if $y \in j^{-1}(z)$, then $d_X(f^n(x), j(g^n(y))) < \alpha$, for each $n \in \mathbb{Z}$. Therefore, for each $u \in B(x, \delta)$ and for each $y \in j^{-1}(z)$, we have $d_X(f^n(u), j(g^n(y))) \leq [d_X(f^n(u), f^n(x)) + d_X(f^n(x), j(g^n(y)))] < [\frac{\epsilon}{3} + \alpha] < \epsilon$. Since y, u and ϵ are chosen arbitrarily, we get that $x \in \mathrm{GHP}^*_f(X)$. Hence, $\mathrm{GHP}^*_f(X) = X$. We now complete the proof by showing that f is IGH-persistent. Let $\epsilon > 0$ be given. For each $x \in X = \mathrm{GHP}^*_f(X)$, there exists a $\delta_x > 0$ depending on x and ϵ by the definition of elements of $\mathrm{GHP}^*_f(X)$. Since X is a compact metric space, we can choose finitely many elements $\{x_i\}_{i=1}^k$ of X such that $X = \bigcup_{i=1}^k B(x_i, \delta_{x_i})$. Set $\delta = \min_{1 \leq i \leq k} \{\delta_{x_i}\}$. Clearly if $g : Y \to Y$ is a homeomorphism satisfying $d_{GH^0}(f, g) < \delta$, then for each $j \in P(I_\delta(f, g))$ and for each $x \in X$, there exists a $z \in X$ such that if $y \in j^{-1}(z)$, then $d_X(f^n(x), j(g^n(y))) < \epsilon$, for each $n \in \mathbb{Z}$. Since ϵ is chosen arbitrarily, we get that f is IGH-persistent.

(2) Suppose that f is a mean equicontinuous homeomorphism. Define $\mathrm{AGHP}^*_f(X) = \{x \in X \mid$ for each $\epsilon > 0$, there exists a $\delta > 0$ such that if $g : Y \to Y$ is a

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homeomorphism satisfying $d_{GH^0}(f,g) < \delta$, then for each $j \in P(I_\delta(f,g))$ and for each $u \in B(x,\delta)$, there exists a $z \in X$ such that if $y \in j^{-1}(z)$, then $\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d_X(f^n(u), j(g^n(y))) < \epsilon$, for each $n \in \mathbb{Z}\}$. We can follow the similar steps as in the proof of (1) to first prove that $\mathrm{WGH}_f(X) = \mathrm{AGHP}^*_f(X) = X$ and then again following similar steps as in the proof of (1), we can conclude that f is average IGH-persistent.

□

Corollary 3.13. *Let $f : X \to X$ be an equicontinuous pointwise minimally expansive homeomorphism. Then, f is pointwise weakly topologically IGH-stable if and only if f is IGH-persistent.*

Proof. Proof follows from Theorems 3.12(1), 3.6(2) and 3.9(3). □

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