Glasgow Math. J. **46** (2004) 63–75.
DOI: 10.1017/S0017089503001538. Printed in the United Kingdom

ONE DIMENSIONAL COMBUSTION FREE BOUNDARY PROBLEM [1]

FAHUAI YI

Department of Mathematics, South China Normal University, Guangzhou 510631, P.R.China
e-mail: fhyi@scnu.edu.cn

(Received 26 August, 2002; accepted 13 June, 2003)

Abstract. A free boundary problem for a parabolic system arising from the mathematical theory of combustion will be considered in the one dimensional case. The existence and uniqueness of the classical solution locally in time will be obtained by the use of a fixed point theorem. Also the existence of the classical solution globally in time and a convergence result with respect to a parameter λ will be proved under some reasonable assumptions.

2000 *Mathematics Subject Classification.* 35R35.

1. Introduction. The aim of this paper is to investigate the existence and uniqueness of following free boundary problem

$$\partial_t\theta - \partial_{xx}\theta = 0, \quad x < g(t) \tag{1.1}$$
$$\partial_t S_1 - \partial_{xx} S_1 = -\lambda\partial_{xx}\theta, \quad x < g(t) \tag{1.2}$$
$$\partial_t S_2 - \partial_{xx} S_2 = 0, \quad x > g(t) \tag{1.3}$$
$$\theta = 1, \quad x = g(t) \tag{1.4}$$
$$\partial_x\theta = e^{S_1}, \quad x = g(t) \tag{1.5}$$
$$S_1 = S_2, \quad x = g(t) \tag{1.6}$$
$$\partial_x S_1 - \partial_x S_2 = \lambda\partial_x\theta, \quad x = g(t) \tag{1.7}$$
$$\theta(x,0) = \theta_0(x), \tag{1.8}$$
$$S_1(x,0) = S_{1,0}(x), \quad S_2(x,0) = S_{2,0}(x) \tag{1.9}$$
$$g(0) = 0, \tag{1.10}$$

where θ represents the renormalized temperature in combustion processes, S_1 and S_2 are reduced enthalpies, $\lambda = -l/2$, l is a constant representing the reduced Lewis number and $x = g(t)$ is an (unknown) curve. In this system $\theta(x,t)$, $S_1(x,t)$, $S_2(x,t)$ and $g(t)$ are unknown functions. For more physical background see [**1**], where the problem was originally derived, and [**2**], where instabilities of travelling wave solution of this problem were studied in the two dimensional case.

[1]The project supported by the National Natural Science Foundation of China (No. 10371045) and Guangdong Provincial Natural Science Foundation of China (No. 000671).

It is easy to check that system (1.1)–(1.7) admits a travelling wave solution, with velocity -1, defined by

$$\begin{aligned} &g(t) = -t, \\ &\theta(x,t) = e^{x+t}, \quad S(x,t) = \lambda(x+t)e^{x+t}, \quad \text{if } x \leq -t \\ &\theta(x,t) = 1, \quad S = 0, \quad \text{if } x \geq -t. \end{aligned}$$

The free boundary problem (1.1)–(1.10) is the one predicted by physicists in the near equi-diffusional limit of a parabolic system without a free boundary (see [**3**], [**4**]), and this prediction was proved mathematically in [**5**].

If $S_1 \equiv S_2 \equiv 0$, the system becomes

$$\begin{aligned} \partial_t\theta - \partial_{xx}\theta &= 0, \quad x < g(t) \\ \theta &= 1, \quad x = g(t) \\ \partial_x\theta &= 1, \quad x = g(t) \\ \theta(x,0) &= \theta_0(x), \\ g(0) &= 0. \end{aligned}$$

It is also called a combustion free boundary problem. Its global classical solution was obtained in [**6**]. In the multidimensional case this free boundary problem was thoroughly researched in the elliptic case (see [**7**]) and the parabolic case (see [**8**]).

In fact there is a special method for studying the problem (1.1)–(1.10) in the one dimensional case. That is we set

$$u(x,t) = \partial_x\theta(x,t), \tag{1.11}$$

noticing that (1.4) implies

$$\partial_t\theta(g(t),t) + \partial_x\theta(g(t),t)g'(t) = 0$$

and it follows that, by (1.1), (1.5) and (1.11),

$$g'(t) = -e^{-S_1}\partial_x u \quad \text{on} \quad x = g(t). \tag{1.12}$$

The advantage of this transformation is that we have an explicit presentation (1.12) for the free boundary just as in a Stefan problem. In this way we have the problem for $u(x,t)$, $S_1(x,t)$, $S_2(x,t)$ and $g(t)$

$$\begin{aligned} \partial_t u - \partial_{xx} u &= 0, \quad x < g(t) && (1.13)\\ \partial_t S_1 - \partial_{xx} S_1 &= -\lambda\partial_x u, \quad x < g(t) && (1.14)\\ \partial_t S_2 - \partial_{xx} S_2 &= 0, \quad x > g(t) && (1.15)\\ u &= e^{S_1}, \quad x = g(t) && (1.16)\\ g'(t) &= -e^{-S_1}\partial_x u, \quad x = g(t) && (1.17)\\ S_1 &= S_2, \quad x = g(t) && (1.18)\\ \partial_x S_1 - \partial_x S_2 &= \lambda u, \quad x = g(t) && (1.19)\\ u(x,0) &= u_0(x), && (1.20) \end{aligned}$$

and (1.9), (1.10), where $u_0(x) = \theta_0'(x)$.

https://doi.org/10.1017/S0017089503001538 Published online by Cambridge University Press

In the next section we prove the local classical existence and uniqueness of the solution to problem (1.9), (1.10) and (1.13)–(1.20). In Section 3 we prove the global classical existence of the solution when $\lambda = 0$. This is preliminary work for Section 4, in which we prove a global classical existence of the solution for sufficiently small λ. At the end of this paper we state a convergence result which holds as $\lambda \to 0$.

2. Classical solution locally in time. It is convenient to straighten the free boundary. Let

$$y = x - g(t), \quad t = t,$$

set

$$\begin{aligned} u(x,t) &= u(y+g(t),t) = v(y,t),\\ S_i(x,t) &= S_i(y+g(t),t) = w_i(y,t), \quad i=1,2 \end{aligned}$$

and then

$$\partial_x u = \partial_y v, \quad \partial_{xx} u = \partial_{yy} v, \quad \partial_t u = \partial_t v - g'(t)\partial_y v.$$

Therefore the problem (1.9), (1.10) and (1.13)–(1.20) becomes

$$\partial_t v - \partial_{yy} v - g'(t)\partial_y v = 0, \quad y<0 \tag{2.1}$$
$$\partial_t w_1 - \partial_{yy} w_1 - g'(t)\partial_y w_1 = -\lambda \partial_y v, \quad y<0 \tag{2.2}$$
$$\partial_t w_2 - \partial_{yy} w_2 - g'(t)\partial_y w_2 = 0, \quad y>0 \tag{2.3}$$
$$v = e^{w_1}, \quad y=0 \tag{2.4}$$
$$g'(t) = -e^{-w_1}\partial_y v, \quad y=0 \tag{2.5}$$
$$w_1 = w_2, \quad y=0 \tag{2.6}$$
$$\partial_y w_1 - \partial_y w_2 = \lambda v, \quad y=0 \tag{2.7}$$
$$v(y,0) = v_0(y), \tag{2.8}$$
$$w_1(y,0) = w_{1,0}(y), \quad w_2(y,0) = w_{2,0}(y) \tag{2.9}$$
$$g(0) = 0, \tag{2.10}$$

where $v_0(y) = u_0(x)$, $w_{i,0}(y) = S_{i,0}(x)$, $i = 1, 2$.

We assume, for $0 < \alpha < 1$,

$$v_0(y),\ w_{1,0}(y) \in C^{1+\alpha}(-\infty, 0], \tag{2.11}$$
$$w_{2,0}(y) \in C^{1+\alpha}[0, +\infty), \tag{2.12}$$

and the consistency condition

$$v_0(0) = e^{w_1(0)}, \quad w_{1,0}(0) = w_{2,0}(0), \tag{2.13}$$
$$w'_{1,0}(0) - w'_{2,0}(0) = \lambda v_0(0). \tag{2.14}$$

Define $D_{1,T} = (-\infty, 0) \times (0, T)$, $D_{2,T} = (0, +\infty) \times (0, T)$.

THEOREM 2.1. *Under the assumptions* (2.11)–(2.14)*, there is a* $T>0$*, such that the problem* (2.1)–(2.10) *has a unique solution* $(v, w_1, w_2, g) \in C^{1+\alpha,(1+\alpha)/2}(\overline{D}_{1,T}) \times C^{1+\alpha,(1+\alpha)/2}(\overline{D}_{1,T}) \times C^{1+\alpha,(1+\alpha)/2}(\overline{D}_{2,T}) \times C^{1+\alpha/2}[0, T]$*, moreover*

$$|v|_{C^{1+\alpha,(1+\alpha)/2}(\overline{D}_{1,T})} + |w_1|_{C^{1+\alpha,(1+\alpha)/2}(\overline{D}_{1,T})} + |w_2|_{C^{1+\alpha,(1+\alpha)/2}(\overline{D}_{2,T})} + |g|_{C^{1+\alpha/2}[0,T]} \leq C \tag{2.15}$$

where C, T *depend on* $|v_0|_{C^{1+\alpha}(-\infty,0]}$, $|w_{1,0}|_{C^{1+\alpha}(-\infty,0]}$ *and* $|w_{2,0}|_{C^{1+\alpha}[0,+\infty)}$*, but are independent of the lower bound of* $|\lambda|$.

Proof. Define

$$\mathcal{D}_1 = \{g(t) \in C^1[0, T];\ g(0)=0,\ g'(0) = -e^{-w_{1,0}(0)}v_0'(0)\},$$
$$\mathcal{D}_{1,M} = \{g(t) \in \mathcal{D}_1;\ |g'(t)|_{C[0,T]} \leq M\}$$

where $M = e^{-w_{1,0}(0)}|v_0'(0)| + 1$, and

$$\mathcal{D}_2 = \{v(y, t) \in C^{1,1/2}(\overline{D}_{1,T});\ v(y, 0) = v_0(y)\},$$
$$\mathcal{D}_{2,N} = \{v(y, t) \in \mathcal{D}_2;\ |v(y, t)|_{C^{1,1/2}(\overline{D}_{1,T})} \leq N\}$$

where $N = 2|v_0(y)|_{C^{1+\alpha}(-\infty,0]} + 1$ and T is determined later on. Set

$$\mathcal{D}_{M,N} = \mathcal{D}_{1,M} \times \mathcal{D}_{2,N},$$

then $\mathcal{D}_{M,N}$ is a closed convex set in $C^1[0, T] \times C^{1,1/2}(\overline{D}_{1,T})$.

For given $(g(t), v(y, t)) \in \mathcal{D}_{M,N}$, first we consider the diffraction problem (2.2), (2.3), (2.6), (2.7) and (2.9) for w_1, w_2. This problem has a unique solution $(w_1, w_2) \in C^{1+\alpha,(1+\alpha)/2}(\overline{D}_{1,T}) \times C^{1+\alpha,(1+\alpha)/2}(\overline{D}_{2,T})$ (see [**9**]), moreover

$$|w_1|_{C^{1+\alpha,(1+\alpha)/2}(\overline{D}_{1,T})} + |w_2|_{C^{1+\alpha,(1+\alpha)/2}(\overline{D}_{2,T})} \leq C_1 \tag{2.16}$$

where C_1 depends on M, N, $|w_{1,0}|_{C^{1+\alpha}(-\infty,0]}$ and $|w_{2,0}|_{C^{1+\alpha}[0,+\infty)}$.

Recalling (2.1), (2.4) and (2.8) we define $\overline{v}(y, t)$ as a solution of the problem

$$\partial_t\overline{v} - \partial_{yy}\overline{v} - g'(t)\partial_y\overline{v} = 0, \quad y<0 \tag{2.17}$$
$$\overline{v} = e^{w_1}, \quad y=0 \tag{2.18}$$
$$\overline{v}(y, 0) = v_0(y). \tag{2.19}$$

This problem also has a unique solution $\overline{v} \in C^{1+\alpha,(1+\alpha)/2}(\overline{D}_{1,T})$ (see [**10**]), and

$$|\overline{v}|_{C^{1+\alpha,(1+\alpha)/2}(\overline{D}_{1,T})} \leq C_2 \tag{2.20}$$

where C_2 depends on M, N, $|w_{1,0}|_{C^{1+\alpha}(-\infty,0]}$ and $|w_{2,0}|_{C^{1+\alpha}[0,+\infty)}$ by (2.16).

Finally we define a new free boundary $\overline{g}(t)$ by the conditions (2.5) and (2.10)

$$\overline{g}(t) = \int_0^t -\exp\{-w_{1,0}(0, \tau)\}\partial_y\overline{v}(0, \tau)\,d\tau,$$

therefore

$$\overline{g}'(t) = -\exp\{-w_{1,0}(0, t)\}\partial_y\overline{v}(0, t),$$

so $\overline{g}'(t) \in C^{\alpha/2}[0, T]$ and

$$|\overline{g}'(t)|_{C^{\alpha/2}[0,T]} \leq C_3, \tag{2.21}$$

where C_3 depends on M, N, $|w_{1,0}|_{C^{1+\alpha}(-\infty,0]}$ and $|w_{2,0}|_{C^{1+\alpha}[0,+\infty)}$ by (2.16) and (2.20).

Define a mapping $\mathcal{F} : \mathcal{D}_{M,N} \longrightarrow C^1[0, T] \times C^{1,1/2}(\overline{D}_{1,T})$ by

$$\mathcal{F}(g(t), v(y, t)) = (\overline{g}(t), \overline{v}(y, t)).$$

In the following we prove that $\mathcal{F}(\mathcal{D}_{M,N}) \subset \mathcal{D}_{M,N}$, in fact

$$\overline{g}(0) = 0, \quad \overline{g}'(0) = -e^{-w_{1,0}(0)} v_0'(0),$$

by the definition of $\overline{g}(t)$. Using (2.21) we arrive at

$$\begin{aligned} |\overline{g}'(t)|_{C[0,T]} &\leq |\overline{g}'(t) - \overline{g}'(0)|_{C[0,T]} + |\overline{g}'(0)| \\ &\leq T^{\alpha/2} |\overline{g}'(t) - \overline{g}'(0)|_{C^{\alpha/2}[0,T]} + e^{-w_{1,0}(0)} |v_0'(0)| \\ &\leq T^{\alpha/2} C_3 + e^{-w_{1,0}(0)} |v_0'(0)|, \quad \text{by (2.21)}. \end{aligned}$$

So, if we take $T \leq (\frac{1}{2C_3})^{2/\alpha}$, we have

$$|\overline{g}'(t)|_{C[0,T]} \leq e^{-w_{1,0}(0)} |v_0'(0)| + 1 = M,$$

which means that $\overline{g}'(t) \in \mathcal{D}_{1,M}$.

On the other hand, in a similar way, using interpolation inequalities and (2.20), for any $\sigma > 0$,

$$\begin{aligned} |\overline{v}(y, t)|_{C^{1,1/2}(\overline{D}_{1,T})} &\leq |\overline{v}(y, t) - \overline{v}(y, 0)|_{C^{1,1/2}(\overline{D}_{1,T})} + |\overline{v}(y, 0)|_{C^1(-\infty,0]} \\ &\leq \sigma |\overline{v}(y, t) - \overline{v}(y, 0)|_{C^{1+\alpha,(1+\alpha)/2}(\overline{D}_{1,T})} + C(\sigma) |\overline{v}(y, t) - \overline{v}(y, 0)|_{L^\infty(D_{1,T})} \\ &\quad + |v_0(y)|_{C^{1+\alpha}(-\infty,0]} \\ &\leq \sigma |\overline{v}(y, t)|_{C^{1+\alpha,(1+\alpha)/2}(\overline{D}_{1,T})} + C(\sigma) T^{(1+\alpha)/2} |\overline{v}(y, t)|_{C^{1+\alpha,(1+\alpha)/2}(\overline{D}_{1,T})} \\ &\quad + (\sigma + 1) |v_0(y)|_{C^{1+\alpha}(-\infty,0]} \\ &\leq \left[\sigma + C(\sigma) T^{(1+\alpha)/2}\right] C_2 + 2 |v_0(y)|_{C^{1+\alpha}(-\infty,0]} \quad \text{by (2.20)} \\ &= 2 |v_0(y)|_{C^{1+\alpha}(-\infty,0]} + 1 = N, \end{aligned}$$

if we let σ be sufficiently small, then let T be sufficiently small. Therefore $\overline{v}(y, t) \in \mathcal{D}_{2,N}$, so $\mathcal{F}$ maps $\mathcal{D}_{M,N}$ into itself. The proof for the continuity of $\mathcal{F}$ is standard and so we omit the details.

Since $\mathcal{F}(\mathcal{D}_{M,N})$ is precompact, as $\overline{g}(t) \in C^{1+\alpha/2}[0, T]$ and $\overline{v}(y, t) \in C^{1+\alpha,(1+\alpha)/2} \times (\overline{D}_{1,T})$ with the estimates (2.20) and (2.21), so from the Schauder fixed point theorem we know that there is some $(g(t), v(y, t)) \in \mathcal{D}_{M,N}$ such that $\mathcal{F}(g(t), v(y, t)) = (g(t), v(y, t))$. This means that $(v(y, t), w_1(y, t), w_2(y, t), g(t))$ is the solution of problem (2.1)–(2.10).

The estimate (2.15) is a consequence of the estimates (2.16), (2.20), (2.21) and interpolation inequalities.

Finally, we prove uniqueness. Suppose $(v(y, t), w_1(y, t), w_2(y, t), g(t))$ and $(\overline{v}(y, t), \overline{w}_1(y, t), \overline{w}_2(y, t), \overline{g}(t))$ are two solutions of the problem (2.1)–(2.10). Set

$$V = v - \overline{v}, \quad W_1 = w_1 - \overline{w}_1, \quad W_2 = w_2 - \overline{w}_2, \quad G = g - \overline{g}.$$

Notice that

$$\exp\{w_1\} - \exp\{\overline{w}_1\} = \int_0^1 \frac{d}{d\tau}\exp\{\tau w_1 + (1-\tau)\overline{w}_1\}\,d\tau$$
$$= \int_0^1 \exp\{\tau w_1 + (1-\tau)\overline{w}_1\}d\tau\ (w_1 - \overline{w}_1),$$

and so (V, W_1, W_2, G) satisfies

$$\partial_t V - \partial_{yy} V - g'(t)\partial_y V = \partial_y \overline{v} G', \quad y<0 \tag{2.22}$$
$$\partial_t W_1 - \partial_{yy} W_1 - g'(t)\partial_y W_1 = -\lambda\partial_y V + \partial_y \overline{w}_1 G', \quad y<0 \tag{2.23}$$
$$\partial_t W_2 - \partial_{yy} W_2 - g'(t)\partial_y W_2 = \partial_y \overline{w}_2 G', \quad y>0 \tag{2.24}$$
$$V = \int_0^1 \exp\{\tau w_1 + (1-\tau)\overline{w}_1\}d\tau\ W_1, \quad y=0 \tag{2.25}$$
$$G'(t) = -e^{-w_1}\partial_y V + \partial_y v\int_0^1 \exp\{-\tau w_1 - (1-\tau)\overline{w}_1\}d\tau\ W_1, \quad y=0 \tag{2.26}$$
$$W_1 = W_2, \quad y=0 \tag{2.27}$$
$$\partial_y W_1 - \partial_y W_2 = \lambda V, \quad y=0 \tag{2.28}$$
$$V(y,0)=0, \tag{2.29}$$
$$W_1(y,0)=0, \quad W_2(y,0)=0 \tag{2.30}$$
$$G(0)=0. \tag{2.31}$$

From (2.22), (2.25) and (2.29) we find that

$$|V|_{C^{1+\alpha,(1+\alpha)/2}(\overline{D}_{1,T})} \le C\big(|G'|_{C[0,T]} + |W_1|_{C^{1+\alpha,(1+\alpha)/2}(\overline{D}_{1,T})}\big). \tag{2.32}$$

(2.23), (2.24), (2.27), (2.28) and (2.30) imply that

$$|W_1|_{C^{1+\alpha,(1+\alpha)/2}(\overline{D}_{1,T})} + |W_2|_{C^{1+\alpha,(1+\alpha)/2}(\overline{D}_{2,T})} \le C\big(|V|_{C^{1,1/2}(\overline{D}_{1,T})} + |G'|_{C[0,T]}\big), \tag{2.33}$$

and from (2.26) it follows that

$$|G'|_{C[0,T]} \le C\big(|V|_{C^{1,1/2}(\overline{D}_{1,T})} + |W_1|_{C(\overline{D}_{1,T})}\big). \tag{2.34}$$

Substituting (2.34) into (2.33), we have

$$|W_1|_{C^{1+\alpha,(1+\alpha)/2}(\overline{D}_{1,T})} + |W_2|_{C^{1+\alpha,(1+\alpha)/2}(\overline{D}_{2,T})} \le C\big(|V|_{C^{1,1/2}(\overline{D}_{1,T})} + |W_1|_{C(\overline{D}_{1,T})}\big); \tag{2.35}$$

then substituting (2.34) and (2.35) into (2.32), we obtain

$$|V|_{C^{1+\alpha,(1+\alpha)/2}(\overline{D}_{1,T})} \le C\big(|V|_{C^{1,1/2}(\overline{D}_{1,T})} + |W_1|_{C(\overline{D}_{1,T})}\big). \tag{2.36}$$

Adding two equalities (2.35) and (2.36), using the interpolation inequality we get

$$|V|_{C^{1+\alpha,(1+\alpha)/2}(\overline{D}_{1,T})} + |W_1|_{C^{1+\alpha,(1+\alpha)/2}(\overline{D}_{1,T})} + |W_2|_{C^{1+\alpha,(1+\alpha)/2}(\overline{D}_{2,T})}$$
$$\le C\big(|V|_{C^{1,1/2}(\overline{D}_{1,T})} + |W_1|_{C(\overline{D}_{1,T})}\big)$$
$$\le CT\big(|V|_{C^{1+\alpha,(1+\alpha)/2}(\overline{D}_{1,T})} + |W_1|_{C^{1+\alpha,(1+\alpha)/2}(\overline{D}_{1,T})}\big),$$

and it follows that

$$V(y,t) = W_1(y,t) = W_2(y,t) = 0$$

if T is sufficiently small. Then $G(t) = 0$ is reduced by (2.26) and (2.31).

This completes the proof of Theorem 2.1.

REMARK. If we set consistency conditions

$$\begin{aligned} v_0''(0) + g'(0)v_0'(0) &= e^{w_{1,0}(0)}[w_{1,0}'' + g'(0)w_{1,0}'(0) - \lambda v_0'(0)], \\ w_{1,0}'' + g'(0)w_{1,0}' - \lambda v_0'(0) &= w_{2,0}'' + g'(0)w_{2,0}'(0), \end{aligned}$$

where $g'(0) = -e^{-w_{1,0}(0)}v_0'(0)$, then the solution $(v, w_1, w_2, g) \in C^{2+\alpha,1+\alpha/2}(\overline{D}_{1,T}) \times C^{2+\alpha,1+\alpha/2}(\overline{D}_{1,T}) \times C^{2+\alpha,1+\alpha/2}(\overline{D}_{2,T}) \times C^{1+(1+\alpha)/2}[0,T]$.

3. Global classical solution with $\lambda = 0$. If $\lambda = 0$, the problem (2.1)–(2.10) can be solved by defining a function $w(y,t)$

$$w(y,t) = \begin{cases} w_1(y,t), & \text{if } y \le 0, \\ w_2(y,t), & \text{if } y > 0. \end{cases}$$

Considering conditions (2.6) and (2.7), $(v(y,t), w(y,t), g(t))$ should be a solution of following system

$$\partial_t v - \partial_{yy} v - g'(t)\partial_y v = 0, \quad y < 0 \tag{3.1}$$

$$\partial_t w - \partial_{yy} w - g'(t)\partial_y w = 0, \quad x \in \mathbb{R}^1, \quad t > 0 \tag{3.2}$$

$$v = e^w, \quad y = 0 \tag{3.3}$$

$$g'(t) = -e^{-w}\partial_y v, \quad y = 0 \tag{3.4}$$

$$v(y,0) = v_0(y), \quad y < 0 \tag{3.5}$$

$$w(y,0) = w_0(y), \quad y \in \mathbb{R}^1, \tag{3.6}$$

$$g(0) = 0, \tag{3.7}$$

where $w_0(y) = w_{1,0}(y)$ if $y \le 0$ and $w_0(y) = w_{2,0}(y)$ if $y > 0$.

REMARK. If $w(y,t)$ is a constant, the problem for (v, g) is simply a one phase Stefan problem (see [**11**]–[**14**]).

Suppose that

$$w_0(y) \in C^{1+\alpha}(\mathbb{R}^1), \quad |w_0(y)|_{L^\infty(\mathbb{R}^1)} \le M_0, \quad w_0'(y) \le 0, \tag{3.8}$$

$$v_0(y) \in C^{1+\alpha}(-\infty, 0], \quad v_0(y) - e^{w_0(y)} \ge 0 \quad \text{for } y < 0, \tag{3.9}$$

$$v_0(0) = e^{w_0(0)}. \tag{3.10}$$

Global existence theorem depends on the following a priori estimate with respect to $\partial_y v(0,t)$.

LEMMA 3.1. *Under the assumptions of* (3.8)–(3.10)*, for any* $T > 0, g \in C^1[0,T]$ *and* $g'(t) \ge 0$. $(u, w) \in C^{1+\alpha,(1+\alpha)/2}(\overline{D}_{1,T}) \times C^{1+\alpha,(1+\alpha)/2}(\mathbb{R}^1 \times [0,T])$ *is the solution of* *the problem* (3.1)–(3.3)*,* (3.5) *and* (3.6)*, then*

$$-C_0 \le \partial_y v(0,t) \le 0. \tag{3.11}$$

where C_0 is a positive constant which only depends on $|v_0|_{C^1(-\infty,0]}$ and $|w_0|_{C^1(\mathbb{R}^1)}$, but is independent of T.

REMARK. The key point is that C_0 does not depend on $|g(t)|_{C^1[0,T]}$ as well.

Proof. First, by maximum principle,

$$|w(y,t)| \le \sup|w_0(y)| = M_0 \tag{3.12}$$
$$|v(y,t)| \le \sup|v_0(y)| + \sup\left|e^{w(y,t)}\right| \le \sup|v_0(y)| + e^{M_0}. \tag{3.13}$$

Also, if we differentiate the equation (3.2) with respect to y, use the condition (3.8) and maximum principle for $\partial_y w(y,t)$, then we obtain

$$\inf w_0'(y) \le \partial_y w(y,t) \le 0. \tag{3.14}$$

Letting

$$Z(y,t) = v(y,t) - e^{w(y,t)},$$

then, by (3.13),

$$|Z(y,t)| \le \sup|v(y,t)| + \sup\left|e^{w(y,t)}\right| \le \sup|v_0(y)| + 2e^{M_0}$$

and $Z(y,t)$ satisfies, by (3.1)–(3.3),

$$\partial_t Z - \partial_{yy} Z - g'(t)\partial_y Z = e^w(\partial_y w)^2, \quad y<0 \tag{3.15}$$
$$Z = 0, \quad y=0 \tag{3.16}$$
$$Z(y,0) = v_0(y) - e^{w_0(y)}. \tag{3.17}$$

From (3.15) we see that $Z(x,t)$ is a supersolution of the equation (3.1) and $Z(y,t)$ attains its minimum on the boundary $y=0$ by (3.9) and (3.16), so $\partial_y Z(0,t) < 0$. Considering

$$\partial_y Z(0,t) = \partial_y v(0,t) - e^{w(0,t)}\partial_y w(0,t)$$

and (3.14), we have

$$\partial_y v(0,t) \le 0.$$

In order to prove that $\partial_y v(0,t)$ has a lower bound which is independent of T, we construct a comparison function in the domain $Q = \{(y,t) \in D_{1,T};\ -1<y<0\}$,

$$K(y,t) = C\ \ln(1-y),$$

where $C>0$ is determined later. Since

$$\partial_t K = 0, \quad \partial_y K = C\frac{-1}{1-y}, \quad \partial_{yy} K = C\frac{-1}{(1-y)^2},$$

https://doi.org/10.1017/S0017089503001538 Published online by Cambridge University Press

so for $(y, t) \in Q$

$$\partial_t(K-Z) - \partial_{yy}(K-Z) - g'(t)\partial_y(K-Z) = C\frac{1}{(1-y)^2} + C\frac{g'(t)}{1-y} - e^w(\partial_y w)^2$$
$$\geq \frac{C}{(1-y)^2} - e^w(\partial_y w)^2 \quad (\text{by } g'(t) \geq 0)$$
$$\geq \frac{C}{4} - e^w(\partial_y w)^2 \geq 0,$$

if $C \geq 4 \ \sup \ e^w(\partial_y w)^2$.
Obviously $K - Z = 0$ on $y = 0$. Also notice that, if $-1 < y < 0$,

$$\partial_y K(y, 0) = \frac{-C}{1-y} \leq -\frac{C}{2},$$

so $K(y, 0) \geq Z(y, 0) = v_0(y) - e^{w_0(y)}$, if we take C is big enough such that

$$-\frac{C}{2} \leq \ \inf \ \left[v_0(y) - e^{w_0(y)}\right]'.$$

This means that $K(y, 0) - Z(y, 0) \geq 0$. On the other hand, on the boundary $y = -1$

$$K(y, t) - Z(y, t) = C \ln 2 - Z(-1, t) \geq 0$$

if we let $C \geq (\ln 2)^{-1} \ \sup \{Z(y, t)\} = (\ln 2)^{-1}(\sup |v_0(y)| + 2e^{M_0})$. These calculations imply that $K(y, t) - Z(y, t)$ attains its minimum on $y = 0$, therefore

$$\partial_y[K(y, t) - Z(y, t)] \leq 0 \quad \text{on} \ \ y = 0,$$

i.e.,

$$\partial_y Z(0, t) \geq \partial_y K(0, t) = -C,$$

where

$$C = \max\left\{2 \sup e^w(\partial_y w)^2, \ -2 \inf \ \left[v_0(y) - e^{w_0(y)}\right]', \ (\ln 2)^{-1}\left(\sup |v_0(y)| + 2e^{M_0}\right)\right\}$$

is independent of T. So,

$$\partial_y v(0, t) - e^{w(0,t)}\partial_y w(0, t) \geq -C.$$

Therefore

$$\partial_y v(0, t) \geq -C + e^{M_0} \inf w_0'(y) := -C_0$$

where C_0 is independent of T.

THEOREM 3.2. *Under the assumptions of* (3.8)–(3.10)*, for any* $T > 0$*, the problem* (3.1)–(3.7) *has a unique solution*

$$(v, w, g) \in C^{1+\alpha,(1+\alpha)/2}(\overline{D}_{1,T}) \times C^{1+\alpha,(1+\alpha)/2}(\mathbb{R}^1 \times [0, T]) \times C^{1+\alpha/2}[0, T].$$

Moreover

$$g'(t) \geq 0. \tag{3.18}$$
$$|g'(t)|_{C[0,T]} \leq \overline{C}, \tag{3.19}$$
$$|v|_{C^{1+\alpha,(1+\alpha)/2}(\overline{D}_{1,T})} \leq M_1, \tag{3.20}$$
$$|w|_{C^{1+\alpha,(1+\alpha)/2}(\mathbb{R}^1 \times [0,T])} \leq M_2. \tag{3.21}$$

where $\overline{C}$, M_1 *and* M_2 *are independent of* T.

https://doi.org/10.1017/S0017089503001538 Published online by Cambridge University Press

Proof. The uniqueness is included in Theorem 2.1. Once we have an a priori estimate (3.11), the global existence is easy to prove.

In fact for any $T > 0$, define a compact convex set in $C[0, T]$

$$\mathcal{D} = \left\{g(t) \in C^1[0, T];\ g(0) = 0,\ g'(0) = -e^{w_0(0)} v_0'(0),\ 0 \le g'(t) \le \overline{C}\right\}$$

where $\overline{C} = e^{M_0} C_0$ and C_0 is from the priori estimate (3.11).

For given $g(t) \in \mathcal{D}$, let $w \in C^{1+\alpha,(1+\alpha)/2}(\mathbb{R}^1 \times [0, T])$ be the unique solution of the Cauchy problem (3.2) and (3.6) with the estimates

$$|w|_{L^\infty(\mathbb{R}^1 \times [0,T])} \le \sup |w_0| = M_0, \tag{3.22}$$

$$\inf w_0'(y) \le \partial_y w \le 0 \tag{3.23}$$

by (3.8) and maximum principle, moreover

$$|w|_{C^{1+\alpha,(1+\alpha)/2}(\mathbb{R}^1 \times [0,T])} \le C|w_0|_{C^{1+\alpha}(\mathbb{R}^1)}. \tag{3.24}$$

where C depends on $\overline{C}$ and is independent of T since the maximum of $|w|$ is independent of T.

Then we define $v(y, t)$ is the unique solution of the problem (3.1), (3.3) and (3.5) with the estimate

$$|v|_{L^\infty(\mathbb{R}^1 \times [0,T])} \le \sup |v_0| + e^{M_0} \tag{3.25}$$

$$-C_0 \le \partial_y v(0, t) \le 0 \tag{3.26}$$

by maximum principle and Lemma 3.1, moreover

$$\begin{aligned} |v|_{C^{1+\alpha,(1+\alpha)/2}(\overline{D}_{1,T})} &\le C\big(|v_0|_{C^{1+\alpha}(-\infty,0]} + |w|_{C^{1+\alpha,(1+\alpha)/2}(\mathbb{R}^1 \times [0,T])}\big) \\ &\le C\big(|v_0|_{C^{1+\alpha}(-\infty,0]} + |w_0|_{C^{1+\alpha}(\mathbb{R}^1)}\big). \end{aligned} \tag{3.27}$$

where C depends on $\overline{C}$ and is independent of T because that the maximum of $|v|$ is also independent of T.

Now we define a new free boundary $\overline{g}(t)$ by

$$\overline{g}(t) = \int_0^t -\exp\{-w(0, \tau)\}\partial_y v(0, \tau)\, d\tau,$$

therefore

$$\overline{g}'(t) = -\exp\{-w(0, t)\}\partial_y v(0, t), \tag{3.28}$$

from (3.26) we have

$$0 \le \overline{g}'(t) \le e^{M_0} C_0 = \overline{C}.$$

We define a mapping $\mathcal{F} : \mathcal{D} \to \mathcal{D}$ by $\mathcal{F}(g) = \overline{g}$. The proof of continuity of $\mathcal{F}$ is standard, we omit the details. Now we use the Schauder fixed point theorem: (see [**15**])

Let $\mathcal{D}$ be a compact convex set in Banach space and let $\mathcal{F}$ be a continuous mapping of $\mathcal{D}$ into itself. Then $\mathcal{F}$ has a fixed point.

It means $(v(y, t), w(y, t), g(t))$ is the solution of problem (3.1)–(3.7).

The estimates (3.19)–(3.21) are the consequences of the estimates (3.24), (3.27) and (3.28).

4. Global classical solution. Using the result of Theorem 3.2 we can prove following result.

THEOREM 4.1. *Under the assumptions of Theorem 2.1 and Lemma 3.1, for any* $T>0$, *there exists a* $\lambda_0>0$, *such that if* $0<|\lambda|\le\lambda_0$, *the problem* (2.1)–(2.10) *has a unique solution* $(v^\lambda(y,t), w_1^\lambda(y,t), w_2^\lambda(y,t), g^\lambda(t)) \in C^{1+\gamma,(1+\gamma)/2}(\overline{D}_{1,T}) \times C^{1+\gamma,(1+\gamma)/2}(\overline{D}_{1,T}) \times C^{1+\gamma,(1+\gamma)/2}(\overline{D}_{2,T}) \times C^{1+\gamma/2}[0,T]$, *where* $0<\gamma<\alpha$, *with the estimate*

$$|v^\lambda|_{C^{1+\gamma,(1+\gamma)/2}(\overline{D}_{1,T})} + \left|w_1^\lambda\right|_{C^{1+\gamma,(1+\gamma)/2}(\overline{D}_{1,T})} + \left|w_2^\lambda\right|_{C^{1+\gamma,(1+\gamma)/2}(\overline{D}_{2,T})} + |g^\lambda|_{C^{1+\gamma/2}[0,T]} \le C \tag{4.1}$$

where C *depends on* $|v_0|_{C^{1+\alpha}(-\infty,0]}$, $|w_{1,0}|_{C^{1+\alpha}(-\infty,0]}$, $|w_{2,0}|_{C^{1+\alpha}[0,+\infty)}$ *and* T, *but is independent of the lower bound of* $|\lambda|$.

Proof. We observe that the length of the interval $[0,\sigma]$ for the existence of solution in Theorem 2.1 depends on $|v_0|_{C^{1+\alpha}(-\infty,0]} + |w_{1,0}|_{C^{1+\alpha}(-\infty,0]} + |w_{2,0}|_{C^{1+\alpha}[0,+\infty)}$. When we extend the solution to $t>\sigma$, $t=\sigma$ is the initial time, so we should control $|v(y,\sigma)|_{C^{1+\alpha}(-\infty,0]} + |w_1(y,\sigma)|_{C^{1+\alpha}(-\infty,0]} + |w_2(y,\sigma)|_{C^{1+\alpha}[0,+\infty)}$.

We denote the solution (v, w_1, w_2, g) of the problem (2.1)–(2.10) by $(v^\lambda, w_1^\lambda, w_2^\lambda, g^\lambda)$. From the uniform estimate (2.15) we have

$$|v^\lambda|_{C^{1+\alpha,(1+\alpha)/2}(\overline{D}_{1,\sigma})} + \left|w_1^\lambda\right|_{C^{1+\alpha,(1+\alpha)/2}(\overline{D}_{1,\sigma})} + \left|w_2^\lambda\right|_{C^{1+\alpha,(1+\alpha)/2}(\overline{D}_{2,\sigma})} + |g^\lambda|_{C^{1+\alpha/2}[0,\sigma,]} \le C. \tag{4.2}$$

It follows that, possibly taking subsequences,

$$g^\lambda(t) \longrightarrow g^*(t) \ \text{ in } C^{1+\beta/2}[0,\sigma], \quad \gamma<\beta<\alpha \tag{4.3}$$
$$v^\lambda(y,t) \longrightarrow v^*(y,t) \ \text{ in } C^{1+\beta,(1+\beta)/2}(\overline{D}_{1,\sigma}), \tag{4.4}$$
$$w_1^\lambda(y,t) \longrightarrow w_1^*(y,t) \ \text{ in } C^{1+\beta,(1+\beta)/2}(\overline{D}_{1,\sigma}), \tag{4.5}$$
$$w_2^\lambda(y,t) \longrightarrow w_2^*(y,t) \ \text{ in } C^{1+\beta,(1+\beta)/2}(\overline{D}_{2,\sigma}), \tag{4.6}$$

where $(v^*(y,t), w_1^*(y,t), w_2^*(y,t), g^*(t))$ is the unique solution of the problem (2.1)–(2.10) with $\lambda=0$, i.e., if we define

$$w^*(y,t) = \begin{cases} w_1^*(y,t), & \text{if } y\le 0,\\ w_2^*(y,t), & \text{if } y>0, \end{cases}$$

then $(v^*(y,t), w^*(y,t), g^*(t))$ is the unique solution of the problem (3.1)–(3.7), so from Theorem 3.2 we have

$$(g^*)'(t) \ge 0.$$
$$|(g^*)'(t)|_{C[0,T]} \le \overline{C},$$
$$|v^*|_{C^{1+\alpha,(1+\alpha)/2}(\overline{D}_{1,T})} \le M_1,$$
$$|w_1^*|_{C^{1+\alpha,(1+\alpha)/2}(\overline{D}_{1,T})} \le M_2,$$
$$|w_2^*|_{C^{1+\alpha,(1+\alpha)/2}(\overline{D}_{2,T})} \le M_2.$$

From (4.3)–(4.6) we obtain

$$\begin{aligned}(g^\lambda)'(\sigma) &\longrightarrow (g^*)'(\sigma)\\ v^\lambda(y,\sigma) &\longrightarrow v^*(y,\sigma) \ \text{ in } C^{1+\beta}(-\infty,0],\\ w_1^\lambda(y,\sigma) &\longrightarrow w_1^*(y,\sigma) \ \text{ in } C^{1+\beta}(-\infty,0],\\ w_2^\lambda(y,\sigma) &\longrightarrow w_2^*(y,\sigma) \ \text{ in } C^{1+\beta}[0,+\infty).\end{aligned}$$

So there is a $\lambda_1 > 0$ such that if $0 < |\lambda| \le \lambda_1$,

$$\begin{aligned}|(g^\lambda)'(\sigma)| &\le \overline{C} + 1,\\ |v^\lambda(y,\sigma)|_{C^{1+\beta}(-\infty,0]} &\le M_1 + 1,\\ \left|w_1^\lambda(y,\sigma)\right|_{C^{1+\beta}(-\infty,0]} &\le M_2 + 1,\\ \left|w_2^\lambda(y,\sigma)\right|_{C^{1+\beta}[0,+\infty)} &\le M_2 + 1.\end{aligned}$$

In this way if we let $v^\lambda(y,\sigma)$, $w_1^\lambda(y,\sigma)$ and $w_2^\lambda(y,\sigma)$ be the initial values, then we can extend the solution of the problem (2.1)–(2.10) to the time interval $[\sigma, 2\sigma]$. Especially we have, by Theorem 2.1,

$$|v^\lambda|_{C^{1+\beta,(1+\beta)/2}(\overline{D}_{1,\sigma,2\sigma})} + \left|w_1^\lambda\right|_{C^{1+\beta,(1+\beta)/2}(\overline{D}_{1,\sigma,2\sigma})} + \left|w_2^\lambda\right|_{C^{1+\beta,(1+\beta)/2}(\overline{D}_{2,\sigma,2\sigma})} + |g^\lambda|_{C^{1+\beta/2}[\sigma,2\sigma]} \le C, \tag{4.7}$$

where $D_{i,\sigma,2\sigma} = D_{i,2\sigma} \backslash D_{i,\sigma}$, $i = 1, 2$. C depends on M_1 and M_2.

Combining the estimates (4.2) and (4.7) we obtain the estimate (4.1) in the interval $[0, 2\sigma]$ in which γ ia replaced by β. After finite steps we arrive at the estimate (4.1) for any finite $T > 0$, but C depends on T as well.

We complete the proof of Theorem 4.1.

The following result is the consequences of the uniform estimate (4.1).

THEOREM 4.2. *Under the assumptions of Theorem 4.1, as $\lambda \to 0$, the solutions $(v^\lambda, w_1^\lambda, w_2^\lambda, g^\lambda)$ of the problem* (2.1)–(2.10) *converge, possibly taking subsequences, to (v^*, w_1^*, w_2^*, g^*) in $C^{1+\gamma,(1+\gamma)/2}(\overline{D}_{1,T}) \times C^{1+\gamma,(1+\gamma)/2}(\overline{D}_{1,T}) \times C^{1+\gamma,(1+\gamma)/2}(\overline{D}_{2,T}) \times C^{1+\gamma/2}[0,T]$, where $0 < \gamma < \alpha$, (v^*, w_1^*, w_2^*, g^*) is the solution of problem* (2.1)–(2.10) *with $\lambda = 0$.*

Conclusion. We established local existence and uniqueness of the solution of a free boundary problem for a parabolic system. We also proved the global existence of a solution if λ is sufficiently small. As for general λ, it is difficult to control the $C^{1+\alpha}$-norms of v, w_1 and w_2. We shall consider this problem in the future. Another problem which we shall consider is the solvability and convergence of the problem in the multidimensional case.

REFERENCES

1. B. J. Matkowsky and G. I. Sivashinsky, An asymptotic derivation of two models in ame theory associated with the constant density approximation, *SIAM J. Appl. Math* **37** (1979), 686–699.

2. C. M. Brauner and A. Lunardi, Instabilities in a two-dimensional combustion model with free boundary, *Arch. Rational Mech. Anal.* **154** (2000), 157–182.

3. J. D. Buckmaster and G. S. S. Ludford, *Theory of laminar flames* (Cambridge University Press, 1982).

4. P. Clavin, Dynamic behavior of premixed flame fronts in laminar and turbulent flows, *Prog. Energy Combust. Sci.* **11** (1985), 1–59.

5. A. Langlois and M. Marion, Asymptotic analysis of a parabolic system arising in combustion theory, Preprint.

6. C. M. Brauner, A. Lunardi and C. S. Laine, Une nouvelle formulation de modeles de fronts en problemes totalment non lineaires, *C. R. Acad. Sci. Paris Sér.* 1 *Math.* **311** (1990), 597–602.

7. H. Berestycki, L. A. Caffarelli and L. Nirenberg, Uniform estimates for regularization of free boundary problems, in *Analysis and Partial Differential Equations* (Marcel Dekker, New York, 1990), 567–619.

8. L. A. Caffarelli and J. L. Vazquez, A free boundary problem for the heat equation arising in flame propagation, *Trans. Amer. Math. Soc.* **347** (1995), 411–441.

9. O. A. Ladyzenskaja, V. Ja. Rivkind and N. N. Ural'ceva, On the classical solvability of diffraction problems for equations of elliptic and parabolic types, *Proc. Steklov Inst. Math.* **92** (1966), 132–166.

10. O. A. Ladyzenskaja, V. A. Solonnikov and N. N. Ural'ceva, *Linear and quasi-linear equations of parabolic type* (American Math. Soc., Providence, Rhode Island, 1968).

11. J. R. Cannon, *The one-dimensional heat equation* (Addison-Wesley Publishing Company, 1984).

12. A. Friedman, *Partial differential equations of parabolic type* (Prentice-Hall, Inc. Englewood Cliffs, N.J., 1964).

13. L. Jiang, Unknown boundary problem for parabolic equation, *Advances in Math*. **5** (1962), 208–220.

14. A. M. Meirmanov, *The Stefan problem* (Walter de Gruyter, 1992).

15. D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, Second Edition (Springer-Verlag, 1992).