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LOCAL DEFORMATIONS OF ISOLATED SINGULARITIES ASSOCIATED WITH NEGATIVE LINE BUNDLES OVER ABELIAN VARIETIES

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Introduction

Let V be an analytic space with an isolated singularity p. In [1] M. Kuranishi approached the problem of deformations of isolated singularities (c.f. [2] and [3]) as follows; Let M be a real hypersurface in the complex manifold $V - \{p\}$. Then one has the induced CR-structure ${}^{\circ}T''(M)$ on M by the inclusion map $i: M \to V - \{p\}$ (c.f. Def. 1.6). Then deformations of the isolated singularity (V, p) give rise to ones of the induced CR-structure ${}^{\circ}T''(M)$. He established in § 9 in [1] the universality theorem for deformations of the induced CR-structure ${}^{\circ}T''(M)$, when M is compact strongly pseudo-convex (Def. 1.5) of $\dim M \geqq 5$. Form this theorem we can know CR-structures on M which appear in deformations of ${}^{\circ}T''(M)$.

Here we assume that V is 1-convex in the sense of Andoreotti-Grauert such that $\dim_C V \geqq 3$ and that M is a compact real hypersurface in $V - \{p\}$ defined by strictly plurisubharmonic function ρ on V such that $\rho \geqq 0$, that is, $M = \{q \in V; \rho(q) = c\}$, here c is a constant. Then as $\text{Prof}_p V \geqq 2$, we find in terms of [2] that the infinitesimal deformation $H^1(V, \Theta)$ (c.f. [1]) of the isolated singularity (V, p) is regarded as a subspace of the infinitesimal deformation $H^1(M, {}^{\circ}T''(M))$ of ${}^{\circ}T''(M)$ (c.f. § 3). Therefore in order to solve the problem of local deformations of (V, p), it is enough to determine the infinitesimal deformations $H^1(M, {}^{\circ}T''(M))$ and complex structure on a neighborhood of M in $V - \{p\}$, which induce CR-structures on M appearing in deformations of ${}^{\circ}T''(M)$.

In this paper we shall prove, using the above Kuranishi's theory, the following.

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THEOREM 5.1. *Let T be an abelian variety of* $\dim T \geqq 2$ *and B a negative line bundle over T. Let (B, T) be the isolated singularity defined by the exceptional variety $O(T)$ in B, here $O(T)$ denotes the zero section of B. Then local deformations of (B, T) are also isolated singularities (B', T'), where T' and B' are abelian varieties and negative line bundles over T', respectively.*

Remark. (B, T) is 1-convex with $\dim_C B \geqq 3$.

The above theorem has been essentially proved by M. Schlessinger [2]. However we want to publish this paper as an example of applications of Kuranishi's theory to deformations of isolated singularities.

In § 1 we describe basic notions of CR-structures and in § 2 determine the induced CR-structure $^\circ T''(B_1)$ on the unit sphere bundle B_1 of the negative line bundle B over the abelian variety T using a normalized automorphic factor for B. In § 3 we calculate the infinitesimal deformation $H^1(B, {}^\circ T''(B))$.

In § 4 and § 5 we show that $H^1(B, {}^\circ T''(B))$ has basis which are integrable CR-structures and that these integrable CR-structures are induced from some negative line bundles B' over abelian varieties T'.

§ 1 Basic definitions

Let M be a real oriented smooth manifold of dimension $2n+1$, $n = 1, 2, \cdots$, and let $^\circ T''$ be a subbundle of the complexified tangent bundle CTM. Let E be a vector bundle over M. We denote by $\Gamma(E)$ the set of C^∞-sections of E.

DEFINITION 1.1. The subbundle $^\circ T''$ is called an almost CR-structure on M, when the following condition is satisfied;

$$(1.1) \qquad {}^\circ T'' \cap {}^\circ \overline{T}'' = 0 \,, \qquad \dim_C {}^\circ T'' = n \,.$$

Moreover an almost CR-structure $^\circ T''$ is a CR-structure, provided that

(1.2) $^\circ T''$ is integrable in the sense of Frobenius, i.e. if Z_1, Z_2 are sections of $^\circ T''$, then so is their Lie bracket $[Z_1, Z_2]$.

Now let $^\circ T''$ be a CR-structure on M. Since $^\circ T''$ is the complex vector subbundle of CTM of complex fiber dimension n and is invariant under complex conjugation, there is a real line bundle F of TM such that

$$CTM = {}^{\circ}T'' \oplus {}^{\circ}\bar{T}'' \oplus CF\,. \tag{1.3}$$

From now on we fix this decomposition of CTM. Put

$$T' = {}^{\circ}T'' \oplus CF\,, \tag{1.4}$$

and we denote by $\pi'(\pi)$ the projection of CTM onto $T'({}^{\circ}T'')$, respectively.

DEFINITION 1.2. An almost CR-structure E'' on M is said to be of finite distance to ${}^{\circ}T''$ if $\pi' \circ i_{E''}\colon E'' \to T'$ is an isomorphism, where $i_{E''}\colon E'' \to CTM$ is the inclusion map.

PROPOSITION 1.3 [1]. *Let E'' be an almost CR-structure on M of finite distance to ${}^{\circ}T''$. Then there is a unique element φ of $\Gamma(M, \mathrm{Hom}({}^{\circ}T'', T'))$ such that*

$$E'' = \{X - \varphi(X);\, X \in {}^{\circ}T''\}\,. \tag{1.5}$$

Conversely let $\varphi \in \Gamma(M, \mathrm{Hom}({}^{\circ}T'', T'))$ and we write

$$\varphi = \varphi_1 + \varphi_2$$

where $\varphi_1 \in \Gamma(M, \mathrm{Hom}({}^{\circ}T'', {}^{\circ}\bar{T}''))$ and $\varphi_2 \in \Gamma(M, \mathrm{Hom}({}^{\circ}T'', CF))$. For any φ satisfying $\varphi_1(\overline{\varphi_1(X)}) \neq \bar{X}$ or $\varphi_2(\overline{\varphi_1(X)}) \neq \overline{\varphi_2(X)}$ for all $X \in {}^{\circ}T''$, $(X \neq 0)$, the formula (1.5) *defines an almost CR-structure E'' of finite distance to ${}^{\circ}T''$.*

The almost CR-structure E'' defined by (1.5) in terms of φ will be denoted by ${}^{\varphi}T''$. We identify ${}^{\varphi}T''$ with φ, and φ is also called an (almost) CR-structure on M, when ${}^{\varphi}T''$ is an (almost) CR-structure.

Next we will examine when T'' is integrable, i.e., a CR-structure on M. For any $X \in \Gamma(CTM)$ we put

$$X_{T'} = \pi'(X) \quad \text{and} \quad X_{{}^{\circ}T''} = \pi(X)\,.$$

The following formulation for the integrable condition is due to T. Akahori [4].

PROPOSITION 1.4. *Let $\varphi \in \Gamma(\mathrm{Hom}({}^{\circ}T'', T'))$ be an almost CR-structure on M. Let $P(\varphi)$ be a map of $\Gamma(\wedge^2 {}^{\circ}T'')$ into $\Gamma(T')$ defined by*

$$P(\varphi)(X, Y) = [X - \varphi(X), Y - \varphi(Y)]_{T'} + \varphi([X - \varphi(X), Y - \varphi(Y)]_{{}^{\circ}T''}) \quad \textit{for } X, Y \in \Gamma({}^{\circ}T'')\,. \tag{1.6}$$

Then φ is integrable if and only if

$$P(\varphi) \equiv 0\,.$$

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In order to compute the infinitesimal deformation of $°T''$, we must define the operator $\bar{\partial}_b^{(p)}: \Gamma(T' \otimes \wedge^p (°T'')^*) \to \Gamma(T' \otimes \wedge^{p+1} (°T'')^*)$, $p = 0, 1, 2, \cdots$. At first for $p = 0$, $\bar{\partial}_b^{(0)}: \Gamma(T') \to \Gamma(T' \otimes °T'')$ is defined by

$$(1.7) \qquad (\bar{\partial}_b^{(0)} u)(X) = [X, u]_{T'}\,, \qquad \text{for } u \in \Gamma(T') \text{ and } X \in \Gamma(°T'')\,.$$

For convenience sake we set

$$Xu = [X, u]_{T'}\,.$$

For $p \geqq 1$,

$$(1.8) \qquad \begin{aligned} &(\bar{\partial}_b^{(p)} \psi)(X_1, \cdots, X_{p+1}) \\ &\quad = \sum_{i=1}^{p} (-1)^{i-1} X_i \psi(X_1, \cdots, \hat{X}_i, \cdots, X_{p+1}) \\ &\quad\quad + \sum_{i<j} (-1)^{j+k} \psi([X_i, X_j], X_1, \cdots, \hat{X}_i, \cdots, \hat{X}_j, \cdots, X_{p+1})\,, \end{aligned}$$

where $X_1, \cdots, X_{p+1} \in \Gamma(°T'')$, and $\psi \in \Gamma(T' \otimes \Lambda^p(°T'')^*)$. From definitions of $\bar{\partial}_b^{(p)}$ it is clear that $\bar{\partial}_b^{(p+1)} \cdot \bar{\partial}_b^{(p)} = 0$, i.e.,

$$0 \longrightarrow \Gamma(T') \xrightarrow{\bar{\partial}_b^{(0)}} \Gamma(T' \otimes (°T'')^*) \xrightarrow{\bar{\partial}_b^{(1)}} (T' \otimes \wedge^2(°T'')^*) \longrightarrow \cdots$$

is a complex.

We explain the convexity of the CR-structure $°T''$ on M. Let $X_1, \cdots, X_n$ be a frame of $\Gamma(°\bar{T}'' | U)$ for some open set U in M. Then $\bar{X}_1, \cdots, \bar{X}_n$ become the frame of $\Gamma(°T'' | U)$. Take a cross-section S of $\Gamma(F | U)$ such that $S(p) \neq 0$ for any $p \in U$. Hence we obtain smooth functions C_{jk} on U defined by

$$(1.9) \qquad \sqrt{-1}[X_j, \bar{X}_k] \equiv C_{jk} S \,(\text{mod } °T'' \oplus °\bar{T}'')\,, \qquad \text{for } 1 \leq j,\ k \leq n\,.$$

It is trivial that the functional matrix

$$\| C_{jk} \| \text{ is hermitian}\,.$$

DEFINITION 1.5. $°T''$ is strongly pseudo-convex, when for any $p \in M$, there is a cross-section S of $\Gamma(F | U)$ such that

$$\| C_{jk} \| > 0\,, \qquad \text{on some neighborhood } U \text{ of } P\,.$$

Finally let us consider a complex manifold V and an imbedding $i: M \to V$, where $\dim_C V = n + 1$ and $\dim_R M = 2n + 1$. Then a complex subbundle $°T''$ on M is defined as follows: For any point $p \in M$,

$$(1.10) \qquad {}^{\circ}T''_p = Ci_{p*}TM \cap T_{i(p)}V \,,$$

where $T_{i(p)}V$ denotes the holomorphic tangent space of V at $i(P)$. Since $i(M)$ is a real hypersurface of V, ${}^{\circ}T''(M)$ becomes the subbundle of CTM of complex fiber dimension n. Clearly ${}^{\circ}T''$ is integrable, i.e., this subbundle defines a CR-structure on M.

DEFINITION 1.6. Let $i: M \to V$ be an imbedding as above. Then the CR-structure ${}^{\circ}T''$ on M defined by (1.10) is called the induced CR-structure by i, or simply the induced CR-structure.

§ 2. *CR*-structures on negative line bundles over abelian varieties

2.1. Let T be an abelian variety with an $n \times 2n$-matrix $\omega = (\omega^i_\alpha)_{1 \le i \le n, 1 \le \alpha \le 2n}$ as a period matrix, that is, let C^n be the space of n complex variables $(z_1, \cdots, z_n)$ and let $Z^{2n} = \overbrace{Z \times \cdots \times Z}^{2n}$. The elements of C^n and Z^{2n} are written as column vectors of length n and $2n$, respectively. For any element $\boldsymbol{d} = {}^t(d^1, \cdots, d^{2n})$ of Z^{2n}, we put

$$\omega \cdot \boldsymbol{d} = {}^t\left(\sum_{\alpha=1}^{2n} \omega^1_\alpha d^\alpha, \cdots, \sum_{\alpha=1}^{2n} \omega^n_\alpha d^\alpha\right) .$$

if Λ denotes the lattice $\{\omega \cdot \boldsymbol{d}; \boldsymbol{d} \in Z^{2n}\}$ in C^n, then

$$T = C^n/\Lambda \,.$$

Now let B be a negative line bundle over T and let π be the projection of C^n onto T. Then the induced bundle $\pi^{-1}(B)$ of B under π is isomorphic to the trivial bundle $C^n \times C$. From this fact there exists a holomorphic map $f: C^{2n} \times Z^{2n} \to C - \{0\}$, called an "automorphic factor" of B, satisfying the following conditions (6);

(C.1) For $\boldsymbol{d}_1, \boldsymbol{d}_2$ in Z^{2n}, and $z \in C^n$,
$$f(z, \boldsymbol{d}_1 + \boldsymbol{d}_2) = f(z + \omega \boldsymbol{d}_1, \boldsymbol{d}_2) f(z, \boldsymbol{d}_2) \,.$$

(C.2) let $\sim$ be the equivalence relation in $C^n \times C$ defined by
$(z_1, \zeta_1) \sim (z_2, \zeta_2) \Longleftrightarrow$ there is $\boldsymbol{d} \in Z^{2n}$ such that
$(z_2, \zeta_2) = (z_1 + \omega \boldsymbol{d}, f(z_1, \boldsymbol{d})\zeta_1)$.

Then the line bundle B over T is isomorphic to $C^n \times C/\sim$. It is clear that automorphic factors of B depend on the choice of isomorphisms of $\pi^{-1}(B)$ onto $C^n \times C$. And we can take an automorphic factor f of B which has the following form (c.f. pp. 111, [5]);

For any $z \in C^n$ and $d \in Z^{2n}$,

$$f(z, d) = \exp\{2\pi\sqrt{-1}({}^t z Q\bar{\omega} d + \tfrac{1}{2} d A^\circ d + {}^t b \cdot d)\} \tag{2.1}$$

where

$$Q = \begin{bmatrix} Q_{11}, & \cdots, & Q_{1n} \\ \vdots & & \vdots \\ Q_{n1}, & \cdots, & Q_{nn} \end{bmatrix}$$

is an $(n \times n)$-matrix such that ${}^t\bar{Q} = -Q$, and A° is a $(2n \times 2n)$-matrix and b denotes a real column vector of length $2n$. We fix the above automorphic factor f with (2.1). Let $h: C^n: \to R$ be the smooth positive function defined by

$$h(z) = \exp(-2\pi\sqrt{-1}\,{}^t z Q \bar{z}) . \tag{2.2}$$

Then we have $h(z) = h(z + \omega d)\,|f(z, d)|^2$ for every $d \in Z^{2n}$ and $z \in C^n$, so that h induces the hermitian metric $\tilde{h}$ on the line bundle B over T. Hence the Chern class $c(B)$ of B equals to the de Rham cohomology class of

$$\left(\frac{1}{2\pi\sqrt{-1}}\bar{\partial} \log h = \sum_{i,j=1}^{n} Q_{i\bar{j}} dz^i \wedge d\bar{z}^j\right) .$$

However since B is negative, we have $\sqrt{-1}Q < 0$, that is, the hermitian matrix $\sqrt{-1}Q$ is negative definite.

2.2. From the negativity of B we know that if T is regarded as the zero-section of B, then there exists an analytic variety $\tilde{T}$ and a holomorphic map g of B onto $\tilde{T}$ such that for some point $\tilde{t}_0 \in \tilde{T}$.

g is a bi-holomorphic map of $B - T$ onto $\tilde{T} - \{\tilde{t}_0\}$, and $g(T) = \tilde{t}_0$. (c.f. [6])

Clearly $\tilde{T}$ has the isolated singularity point $\tilde{t}_0$, which is denoted by $(\tilde{T}, \tilde{t}_0)$. Let $\tilde{S}$ be a real hypersurface around $\tilde{t}_0$ in $\tilde{T} - \{\tilde{t}_0\}$. Then local deformations of isolated singularity $(\tilde{T}, \tilde{t}_0)$ induce ones of the induced CR-structure on $\tilde{S}$ by the inclusion $i_{\tilde{S}}: \tilde{S} \to \tilde{T} - \{\tilde{t}_0\}$. However in terms of the biholomorphic map $g: B - T \to \tilde{T} - \{\tilde{t}_0\}$ we shall consider local deformations of the induced CR-structure on a real hypersurface around T in $B - T$.

Now let B_1 be the unit circle bundle over T defined by the hermitian metric h on B, i.e.,

$$B_1 = \{e \in B; \tilde{h}(e) = 1\} .$$

PROPOSITION 2.1. *Let* $^\circ T''(B_1)$ *be the induced CR-structure on* B_1 *by the inclusion* $\iota_{B_1}\colon B_1 \to B$. *Then* $^\circ T''(B_1)$ *is strongly pseudo-convex, that is,* B_1 *is the real* $(2n+1)$*-dimensional compact strongly pseudo-convex manifold.*

This proposition is proved by the following two lemmas. At first let $\tilde{\psi}$ be the natural projection of $C^n \times C$ onto $B = C^n \times C/\sim$ as in § 1. Here put

$$V = \tilde{\psi}^{-1}(B_1)(\subset C^n \times C)\,.$$

Then it is trivial that

$$V = \{(z, \zeta) \in C^n \times C;\, h(z)\,|\zeta|^2 = 1\}\,.$$

Moreover let ψ be the diffeomorphism of $C^n \times S^1$ onto V defined by

$$\psi(z, \theta) = \left(z, \frac{e^{\sqrt{-1}\theta}}{\sqrt{h(z)}}\right), \qquad \text{for } (z, \theta) \in C^n \times S^1\,,$$

here θ is the angular coordinate of S^1.

LEMMA 2.2. *Let* $^\circ T''(V) = CTV \cap T^{0,1}(C^n \times C)$ *be the induced CR-structure on* V. *Let* $Z_{\bar 1}, \cdots,$ *and* $Z_{\bar n}$, *be vector fields on* $C^n \times S^1$ *defined by*

$$Z_{\bar j} = \frac{\partial}{\partial \bar z^j} - \frac{\sqrt{-1}}{2}\,\frac{\partial \log h}{\partial \bar z^j}\,\frac{\partial}{\partial \theta}\,, \qquad j = 1, \cdots, n\,. \tag{2.3}$$

Then $\{\psi_*(Z_{\bar j})\}_{j=1}^n$ *become the global basis of* $^\circ T''(V)$.

Proof. By direct calculations we obtain

$$\psi_{*(z,\theta)}\left(\frac{\partial}{\partial \bar z^j}\right) = \frac{\partial}{\partial \bar z^j} - \frac{1}{2}\,\frac{\partial \log h}{\partial \bar z^j}\,\frac{e^{\sqrt{-1}\theta}}{\sqrt{h}}\,\frac{\partial}{\partial \zeta} - \frac{1}{2}\,\frac{\partial \log h}{\partial \bar z^j}\,\frac{e^{-\sqrt{-1}\theta}}{\sqrt{h}}\,\frac{\partial}{\partial \bar\zeta}$$

and

$$\psi_{*(z,\theta)}\left(\frac{\partial}{\partial \theta}\right) = \sqrt{-1}\left(\frac{e^{\sqrt{-1}\theta}}{\sqrt{h}}\,\frac{\partial}{\partial \zeta} - \frac{e^{-\sqrt{-1}\theta}}{\sqrt{h}}\,\frac{\partial}{\partial \bar\zeta}\right).$$

Here we have

$$\psi_*(Z_{\bar j}) = \frac{\partial}{\partial \bar z^j} - \bar\zeta\frac{\partial \log h}{\partial \bar z^j}\,\frac{\partial}{\partial \bar\zeta}\,, \qquad j = 1, \cdots, n\,. \tag{2.4}$$

This means $\psi_{*(z,\theta)}(Z_{\bar j}) \in T^{0,1}_{\psi(z,\theta)}(C^n \times C)$, for any $(z, \theta) \in C^n \times S^1$, so that the $\psi_*(Z_j)$ are cross-sections of $^\circ T''(V)$. It is clear that $\{\psi^*(Z_{\bar j})\}_{j=1}^n$ is the global base of $^\circ T''(V)$. Q.E.D.

Next let $^{\circ}T''(C^n \times S^1)$ be the complex subspace of $CT(C^n \times S^1)$ generated by the basis $\{Z_{\bar{j}}\}_{j=1}^{n}$. Then from the above lemma it is trivial that $^{\circ}T''(C^n \times S^1) = {}^{\circ}T''(V)$, and that $^{\circ}T''(C^n \times S^1)$ is the CR-structure on $C^n \times S^1$. $CT(C^n \times S^1)$ has the following decomposition; let $F = \{R(\partial/\partial\theta)\}$ be the real line bundle over $C^n \times S^1$, spanned by $\partial/\partial\theta$. Then we get

$$CT(C^n \times S^1) = {}^{\circ}T''(C^n \times S^1) \oplus {}^{\circ}\bar{T}''(C^n \times S^1) \oplus CF\,.$$

LEMMA 2.3. *$^{\circ}T''(C^n \times S^1)$ is strongly convex.*

Proof. Put $Z_j = \bar{Z}_{\bar{j}}$ $(j = 1, \cdots, n)$. By (2.3) and (2.2) it follows that

$$[Z_{\bar{j}}, Z_i] = \sqrt{-1}\,\frac{\partial^2 \log h}{\partial \bar{z}^j \partial z^i}\frac{\partial}{\partial\theta} = 2\pi Q_{i\bar{j}}\frac{\partial}{\partial\theta} \qquad (1 \le i, j \le n) \tag{2.5}$$

On the other hand since Q is negative definite, $^{\circ}T''(C^n \times S^1)$ is strongly convex. Q.E.D.

Remark. The next formulas are trivial;

$$[Z_{\bar{j}}, Z_{\bar{i}}] = [Z_j, Z_i] = \left[Z_{\bar{j}}, \frac{\partial}{\partial\theta}\right] = \left[Z_j, \frac{\partial}{\partial\theta}\right] = 0 \tag{2.6}$$
$$\text{for } 1 \le i, j \le n\,.$$

We shall express the induced CR-structure $^{\circ}T''(B_1)$ by using $\{Z_{\bar{j}}\}_{j=1}^{n}$. Let $\tilde{\psi}$ be the canonical projection of $C^n \times C$ onto $B = C^n \times C/\sim$ and $\hat{\psi}$ the composite map of ψ and $\tilde{\psi}$;

$$\hat{\psi} = \tilde{\psi} \circ \psi : C^n \times S^1 \xrightarrow{\psi} C^n \times C \xrightarrow{\tilde{\psi}} B\,.$$

It follows from (C.2) in 2.1 and the definition of ψ that for $(z, \theta), (z', \theta') \in C^n \times S^1$, $\hat{\psi}(z, \theta) = \hat{\psi}(z', \theta')$ means that there is a $d \in Z^{2n}$ such that

$$z' = z + \omega d, \quad \text{and} \quad \theta' = \theta + \arg f(z, d)\,.$$

LEMMA 2.4. *For $(z, \theta) \in C^n \times S^1$, and $d \in Z^{2n}$*

$$\hat{\psi}_{*(z,\theta)}(Z_{\bar{j}}) = \hat{\psi}_{*(z+\omega d, \theta + \arg f(z,d))}(Z_{\bar{j}})\,, \qquad (j = 1, \cdots, n)\,,$$

and

$$\hat{\psi}_{*(z,\theta)}\left(\frac{\theta}{\partial\theta}\right) = \hat{\psi}_{*(z+\omega d, \theta + \arg f(z,d))}\left(\frac{\partial}{\partial\theta}\right).$$

Therefore $\{\hat{\psi}_*(Z_{\bar{j}})\}_{j=1,\dots,n}$ and $\hat{\psi}_*(\partial/\partial\theta)$ become vector fields on B_1.

Proof. It follows that

$$\hat{\psi}_{*(z+\omega d,\theta+\arg f(z,d))}(Z_j)$$
$$= \tilde{\psi}_{*\left(z+\omega d,\frac{e^{\sqrt{-1}(\theta+\arg f(z,d))}}{\sqrt{h(z+\omega d)}}\right)}\left(\frac{\partial \bar{z}^j}{\partial} - \bar{\zeta}\frac{\log h(z+\omega d)}{\partial \bar{z}^j}\frac{\partial}{\partial \bar{\zeta}}\right).$$

Here set

$$\zeta_0 = \frac{e^{\sqrt{-1}(\theta+\arg f(z,d))}}{\sqrt{h(z+\omega d)}}$$

and

$$A = \hat{\psi}_{*((z+\omega d,\theta+\arg f(z,d)))}(Z_{\bar{j}}).$$

Using $h(z) = h(z+\omega d)\,|f(z,d)|^2$ as in § 1, we get

$$A = \tilde{\psi}_{*(z+\omega d,\zeta_0)}\left(\frac{\partial}{\partial \bar{z}^j} + \frac{\partial \log f(z,d)}{\partial \bar{z}^j}\bar{\zeta}\frac{\partial}{\partial \bar{\zeta}} - \frac{\partial \log h(z)}{\partial \bar{z}^j}\bar{\zeta}\frac{\partial}{\partial \bar{\zeta}}\right). \tag{2.7}$$

On the other hand let g_d be the bi-holomorphic map of $C^n \times C$ onto $C^n \times C$ defined by

$$g_d(z,\zeta) = (z+\omega d, f(z,d)\zeta), \qquad \text{for any } d \in Z^{2n}.$$

Then we have

$$g_{d(z,f(z,d)^{-1}\zeta_0)}\left(\frac{\partial}{\partial \bar{z}^j}\right) = \left(\frac{\partial}{\partial \bar{z}^j} + \frac{\partial \log f(z,d)}{\partial \bar{z}^j}\zeta_0\left(\frac{\partial}{\partial \zeta}\right)\right)_{gd(z,f(z,d)^{-1}\zeta_0}\,,$$

and

$$g_{d(z,f(z,d)^{-1}\zeta_0)}\left(\bar{\zeta}\frac{\partial}{\partial \bar{\zeta}}\right) = \bar{\zeta}_0\left(\frac{\partial}{\partial \bar{\zeta}}\right)_{gd(z,f(z,d)^{-1}\zeta_0)}.$$

Hence from (2.7) it follows that

$$A = (\tilde{\psi}\cdot g_d)_{(z,f(z,d)^{-1}\zeta_0)}(\psi_{*(z,\theta)}Z_{\bar{j}}) \qquad \text{(c.f. (2.4))}$$

But as $\tilde{\psi} g_d = \tilde{\psi}$, we have finally

$$A = \hat{\psi}_{*(z,\theta)}Z_{\bar{j}}.$$

Similarly it is proved

$$\hat{\psi}_{*(z,\theta)}\left(\frac{\partial}{\partial \theta}\right) = \hat{\psi}_{*(z+\omega d,\theta+\arg f(z,d))}\left(\frac{\partial}{\partial \theta}\right). \qquad \text{Q.E.D.}$$

Let us return to the proof of Proposition 2.1. By virtue of Lemmas 2.2 and 2.4. the induced CR-structure ${}^{\circ}T''(B_1)$ on B_1 is spanned by $\hat{\psi}_*(Z_{\bar{l}})$,

$\cdots$, and $\hat{\psi}_*(Z_{\bar{n}})$, denoted by $\{\{\hat{\psi}_*(Z_{\bar{1}}), \cdots, \hat{\psi}_*(Z_{\bar{n}})\}\}$. Furthermore we have the next decomposition of CTB_1;

$$CTB_1 = {}^\circ T''(B_1) \oplus {}^\circ \bar{T}''(B_1) \oplus CF(B_1)\,, \tag{2.8}$$

where

$$\begin{cases} {}^\circ T''(B_1) = \{\{\hat{\psi}_*(Z_{\bar{1}}), \cdots, \hat{\psi}_*(Z_{\bar{n}}),\}\}\,, \\ {}^\circ \bar{T}''(B_1) = \{\{\hat{\psi}_*(Z_1), \cdots, \hat{\psi}_*(Z_n)\}\}\,, \\ CF(B_1) = \{\{\hat{\psi}_*(\partial/\partial\theta)\}\}. \end{cases} \tag{2.9}$$

Thus our proposition is completely proved.

§ 3. Infinitesimal deformations of ${}^\circ T''(B_1)$

Notations being as in § 2, let us first consider relations between almost CR-structures of finite distance to ${}^\circ T''(C^n \times S^1)$ on $C^n \times S^1$ (c.f. Lemma 2.3) and ones of finite distance to ${}^\circ T''(B_1)$ on B_1. For this purpose we set

$${}^\circ T'(C^n \times S^1) = {}^\circ \bar{T}''(C^n \times S^1) \oplus CF$$

and

$$T'(B_1) = {}^\circ \bar{T}''(B_1) \oplus CF(B_1)\,.$$

From Proposition 1.3 it is enough to consider the correspondence between $\Gamma(\mathrm{Hom}\,({}^\circ T''(C^n \times S^1),\ T'(C^n \times S^1)))$ and $\Gamma(\mathrm{Hom}\,({}^\circ T''(B_1), T'(B_1)))$. Here we put for simplicity

$${}^\circ T'' = {}^\circ T''(C^n \times S^1) \quad \text{and} \quad T' = T'(C^n \times S^1)\,.$$

Let $Z_{\bar{1}}, \cdots$, and $Z_{\bar{n}}$ be the basis of ${}^\circ T''$ defined by Lemma 2.2. Then we have the following

Proposition 3.1. *For any $\varphi \in \Gamma\,(\mathrm{Hom}\,({}^\circ T'', T'))$ we can write*

$$\varphi(Z_{\bar{j}}) = \sum_{k=1}^{n} \varphi_{\bar{j}}^k Z_k + \varphi_{\bar{j}} \frac{\partial}{\partial\theta} \qquad (j = 1, \cdots, n)\,, \tag{3.1}$$

where $\varphi_{\bar{j}}^k$ and $\varphi_{\bar{j}}$ are smooth functions on $C^n \times S^1$. Then φ induces an element of $\Gamma(\mathrm{Hom}\,({}^\circ T''(B_1), T'(B_1)))$ if and only if the following condition (C) *is satisfied;*

$$(\mathrm{C}) \begin{cases} \varphi_{\bar{j}}^k(z, \theta) = \varphi_{\bar{j}}^k(z + \omega \boldsymbol{d}, \theta + \arg f(z, \boldsymbol{d})) \\ \varphi_{\bar{j}}(z, \theta) = \varphi_{\bar{j}}(z + \omega \boldsymbol{d},\ + \arg f(z, \boldsymbol{d}))\,, \end{cases}$$

for each $d \in Z^{2n}$ *and* $(z, \theta) \in C^n \times S^1$.

Proof. Let τ be the linear map from $\Gamma(\mathrm{Hom}\,({}^\circ T''(B_1), T'(B_1)))$ to $\Gamma(\mathrm{Hom}\,({}^\circ T'', T'))$ defined as follows; let $\tilde{\varphi}$ be any element of $\Gamma(\mathrm{Hom}\,({}^\circ T''(B_1), T'(B_1)))$. Then we put, for any $(z, \theta) \in C^n \times S^1$,

$$[(\tau\tilde{\varphi})(Z_{\bar{j}})]_{(z,\theta)} = \hat{\psi}^{-1}_{*\hat{\psi}(z,\theta)}(\tilde{\varphi}(\hat{\psi}_{*(z,\theta)}(Z_{\bar{j}}))) .$$

If $\tilde{\varphi}$ is an element of $\Gamma(\mathrm{Hom}\,({}^\circ T''(B_1), T'(B_1)))$ with the expression

$$\tilde{\varphi}(\hat{\psi}_*(Z_{\bar{j}}) = \sum_{k=1}^{n} \tilde{\varphi}^k_{\bar{j}}\hat{\psi}_*(Z_k) + \tilde{\varphi}_{j}\hat{\psi}_*\left(\frac{\partial}{\partial\theta}\right) ,$$

then it follows that

$$(\tau\tilde{\varphi})(Z_{\bar{j}}) = \sum_{k=1}^{n} (\tilde{\varphi}^k_{\bar{j}} \circ \hat{\psi})Z_k + (\tilde{\varphi}_j \circ \hat{\psi})\frac{\partial}{\partial\theta} .$$

Thus $\tau\tilde{\varphi}$ satisfies the condition (C). Conversely an arbitrary element $\varphi \in \Gamma(\mathrm{Hom}\,({}^\circ T'', T))$ satisfying (C) induces an element $\tilde{\varphi}$ of $\Gamma(\mathrm{Hom}\,({}^\circ T''(B_1), T'(B^1_1)))$, and we have

$$\varphi = \tau\tilde{\varphi} . \qquad \text{Q.E.D.}$$

We denote by $\Gamma_{(C)}(\mathrm{Hom}\,({}^\circ T'', T'))$ the set of all smooth-section of $\mathrm{Hom}\,({}^\circ T'', T')$ satisfying the condition (C). The above Proposition 3.1 shows that

$$\tau : \Gamma(\mathrm{Hom}\,({}^\circ T''(B_1), T'(B_1))) \to \Gamma_{(C)}(\mathrm{Hom}\,({}^\circ T'', T'))$$

is isomorphic.

More generally let $\Gamma_{(C)}(\wedge^k ({}^\circ T'')^* \otimes T')$ be the set of all smooth-sections φ of $\wedge^k ({}^\circ T'')^* \otimes T'$, $(k = 0, 1, \cdots, n)$ such that, when φ is expressed as

$$(Z_{\bar{j}_1} \wedge \cdots \wedge Z_{\bar{j}_k}) = \sum_{\ell=1}^{n} \varphi^{\ell}_{\bar{j}_1,\cdots,\bar{j}_k} Z_l + \varphi_{j_1,\cdots,j_k}\frac{\partial}{\partial\theta} , \qquad (1 \leq j_1 < \cdots < j_k \leq n) .$$

Then all coefficients $\varphi_{\bar{j}_1,\cdots,\bar{j}_k}$ and $\varphi_{j_1,\cdots,j_k}$ satisfy the condition (C). Then τ induces the isomorphism of $\Gamma(\wedge^k ({}^\circ T''(B_1))^* \otimes T'(B_1))$ onto $\Gamma_{(C)}(\wedge^k ({}^\circ T'')^* \otimes T')$. Here we have the following commutative diagram;

$$\begin{array}{ccccc} 0 \longrightarrow & \Gamma(T'(B_1)) & \xrightarrow{\bar\partial_b^{(0)}} & \Gamma(\mathrm{Hom}\,({}^\circ T''B_1),\ T'(B_1)) & \\ & \downarrow \tau & & \downarrow \tau & \\ 0 \longrightarrow & \Gamma_{(C)}(T') & \xrightarrow{\bar\partial_b^{(0)}} & \Gamma_{(C)}(\mathrm{Hom}\,({}^\circ T'', T')) & \end{array}$$

$$\begin{array}{ccc} \xrightarrow{\bar\partial_b^{(1)}} & \Gamma(\wedge^2({}^\circ T''(B_1) \otimes T'(B)) & \longrightarrow \\ & \downarrow \tau & \\ \xrightarrow{\bar\partial_b^{(1)}} & \Gamma_{(C)}(\wedge^2({}^\circ T'')^* \otimes T') & \longrightarrow \end{array}$$

where the $\bar\partial_b^{(k)}$ denote the operators defined by (1.7) and (1.8).

Let $H^k({}^\circ T''(B_1))$ and $H^k_{(C)}({}^\circ T'')$ be the k-th cohomologies of complexes $\{\Gamma(\wedge^k({}^\circ T''(B_1))^* \otimes T'(B_1)), \bar\partial_b^{(k)}\}_{k=0}^n$ and $\{\Gamma_{(C)}(\wedge^k({}^\circ T'')^* \otimes T'), \bar\partial_b^{(k)}\}_{k=0}^n$, respectively. Then we know that

$$H^k({}^\circ T''(B_1)) \cong H^k_{(C)}({}^\circ T'')\,.$$

3.1. We shall determine explicitly a basis of the first cohomology $H^1_{(C)}({}^\circ T'')$, that is, the infinitesimal deformation of ${}^\circ T''(B_1)$. First of all let φ be an element of $\Gamma_{(C)}(\mathrm{Hom}\,({}^\circ T'', T')$ with

$$\varphi(Z_{\bar j}) = \sum_{k=1}^n \varphi_{\bar j}^k Z_k + \varphi_{\bar j}\frac{\partial}{\partial\theta} \qquad (j = 1, \cdots, n)\,. \tag{3.1}$$

Then we obtain the following

PROPOSITION 3.2. *It follows that* $\bar\partial_b^{(1)}\varphi = 0$ *if and only if for all* i, $j \in \{1, \cdots, n\}$,

$$Z_{\bar i}\varphi_{\bar j}^k - Z_{\bar j}\varphi_{\bar i}^k = 0\,, \qquad (k = 1, \cdots, n) \tag{3.2}$$

and

$$\sum_{k=1}^n (\varphi_{\bar j}^k \Phi_{k\bar i} - \varphi_{\bar i}^k \Phi_{k\bar j}) + Z_{\bar i}\varphi_{\bar j} - Z_{\bar j}\varphi_{\bar i} = 0\,, \tag{3.3}$$

where we put

$$\Phi_{i\bar j} = 2\pi Q_{i\bar j} \qquad (\text{c.f. } (2.1))\,. \tag{3.4}$$

Proof. By (1.8) and (2.6) we see that

$$\begin{aligned} \bar\partial_b^{(1)}\varphi(Z_{\bar i}, Z_{\bar j}) &= [Z_{\bar i}, \varphi(Z_{\bar j})]_{T'} - [Z_{\bar j}, \varphi(Z_{\bar i})]_{T'} \\ &= \sum_{k=1}^n (Z_{\bar i}\varphi_{\bar j}^k - Z_{\bar j}\varphi_{\bar i}^k)Z_k + \sum_{k=1}^n (\varphi_{\bar j}^k[Z_{\bar i}, Z_k]_{T'} - \varphi_{\bar i}^k[Z_{\bar j}, Z_k]_{T'}) \\ &\quad + (Z_{\bar i}\varphi_{\bar j} - Z_{\bar j}\varphi_{\bar i})\frac{\partial}{\partial\theta}\,. \end{aligned}$$

However as $[Z_{\bar{j}}, Z_i] = 2\pi Q_{i\bar{j}}(\partial/\partial\theta)$ (c.f. (2.5)), it is trivial that

$$\bar{\partial}_b^{(1)}\varphi(Z_{\bar{i}}, Z_{\bar{j}}) = \sum_{n=1}^{n}(Z_{\bar{i}}\varphi_{\bar{i}}^k - Z_{\bar{j}}\varphi_{\bar{i}}^k)Z_k$$
$$+ \left\{\sum_{k=1}^{n}(\varphi_{\bar{j}}^k 2\pi Q_{k\bar{j}} - \varphi_{\bar{i}}^k 2\pi Q_{k\bar{i}}) + Z_{\bar{i}}\varphi_j - Z_{\bar{j}}\varphi_i\right\}\frac{\partial}{\partial\theta}\,.$$

This fact proves Proposition 3.2. Q.E.D.

In order to study properties of the $\{\varphi_{\bar{j}}^k, \varphi_j\}$ $k, j = 1, \cdots, n$ on $C^n \times S^1$ satisfying the differential equations (3.2) and (3.3), we denote by $\mathfrak{S}(f)$ the Fourier expansion of any function f on $C^n \times S^1$ with respect to the angular parameter of S^1. Let us put

$$\mathfrak{S}(\varphi_{\bar{j}}^k)(z, \theta) = \sum_{m\in Z}\varphi_{\bar{j},m}^k(z)e^{\sqrt{-1}m\theta}\,. \tag{3.5}$$

Then from the uniqueness of Fourier expansions and the condition (C) it follows that

$$\varphi_{\bar{j},m}^k(z + \omega d) = \varphi_{\bar{j},m}^k(z)e^{-\sqrt{-1}m\arg f(z,d)} \tag{3.6}$$

for any $d \in Z^{2n}$ and $m = 0, \pm 1, \pm 2, \cdots$.

At first we consider the differential systems (3.2). By (2.3), (3.2) means that

$$\frac{\partial\varphi_{\bar{j}}^k}{\partial\bar{z}^i} - \frac{\partial\varphi_{\bar{i}}^k}{\partial\bar{z}^j} - \frac{\sqrt{-1}}{2}\left(\frac{\partial\log h(z)}{\partial\bar{z}^i}\frac{\partial\varphi_{\bar{j}}^k}{\partial\theta} - \frac{\partial\log h(z)}{\partial\bar{z}^j}\frac{\partial\varphi_{\bar{i}}^k}{\partial\theta}\right) = 0\,, \tag{3.7}$$
$$(1 \le i, j, k \le n)\,.$$

LEMMA 3.3. *For each $m \in Z$, the $\{\varphi_{\bar{j},m}^k\}$ $k, j = 1, \cdots, n$ satisfy the following equation;*

$$\frac{\partial\varphi_{\bar{j},m}^k}{\partial\bar{z}^i} - \frac{\partial\varphi_{\bar{i},m}^k}{\partial\bar{z}^j} + \frac{m}{2}\left(\frac{\partial\log h}{\partial\bar{z}^i}\varphi_{\bar{j},m}^k - \frac{\partial\log h}{\partial\bar{z}^j}\varphi_{\bar{i},m}^k\right) = 0\,. \tag{3.8}$$

Proof. This lemma is trivial from (3.7) and definitions of the $\varphi_{\bar{j},m}^k$. Q.E.D.

Moreover we have

LEMMA 3.4. *Let $\hat{\varphi}_m$ be the element of $\Gamma_{(C)}(\mathrm{Hom}\,({}^\circ T'', T'))$ defined by*

$$\hat{\varphi}_m(Z_{\bar{j}}) = \sum_{k=1}^{n}e^{\sqrt{-1}m\theta}\varphi_{\bar{j},m}^k(z)Z_k \qquad (m \in Z)\,.$$

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Then there exists an element ζ_m of $\Gamma_{(C)}(T')$ for every non-zero integer m such that

$$\zeta_m = \sum_{k=1}^{n} \zeta_m^k(z) Z_k ,$$

and

$$\bar{\partial}_b^{(0)} \zeta_m(Z_{\bar{j}}) = \hat{\varphi}_m(Z_{\bar{j}}) + \sum_{k=1}^{n} \zeta_m^k \Phi_{k\bar{j}} \frac{\partial}{\partial \theta} . \tag{3.9}$$

Proof. Put $\psi_{\bar{j},m}^k(z) = \varphi_{\bar{j},m}^k(z) h^{m/2}(z)$. Then we find from (2.2) and (3.6) that

$$\psi_{\bar{j},m}^k(z + \omega d) = \psi_{\bar{j},m}^k(z) e^{-\sqrt{-1} m \arg f(z,d)} j(z, d)^{-m} = \psi_{\bar{j},m}^k(z) f(z, d)^{-m} , \qquad (d \in Z^{2n}) .$$

Therefore if we set

$$\psi_m^k = \sum_{j=1}^{n} \psi_{\bar{j},m}^k d\bar{z}^j ,$$

ψ_m^k is regarded as the cross-section of the vector bundle $B^{-m} \otimes \wedge^{0,1}(T)$ over the abelian variety T, where $\wedge^{0,1}(T)$ represents the bundle consisting of (0, 1)-type differential forms on T. Let $\bar{\partial}$ be the usual exterior derivation of type (0, 1) on T. Then it is clear that

$$\bar{\partial} \psi_m^k = \sum_{i<j} \left(\frac{\partial \psi_{\bar{j},m}^k}{\partial \bar{z}^i} - \frac{\partial \psi_{\bar{i},m}^k}{\partial \bar{z}^j} \right) d\bar{z}^i \wedge \bar{z}^j .$$

But

$$\frac{\partial \psi_{\bar{j},m}^k}{\partial \bar{z}^i} = \left[\frac{\partial \varphi_{\bar{j},m}^k}{\partial \bar{z}^i} + \frac{m}{2} \varphi_{\bar{j},m}^k \frac{\partial \log h}{\partial \bar{z}^j} \right] h^{m/2} ,$$

so that using (3.8), we obtain, for any $m \in Z$,

$$\bar{\partial} \psi_m^k = 0 , \qquad (k = 1, \cdots, n) .$$

Now let $m \neq 0$. Then, since B is a negative line bundle and the holomorphic tangent bundle of T is analytically trivial ($\dim_C T \geqq 2$), there is an element $\tilde{\eta}_m^k$ of $\Gamma(B^{-m})$ such that, for any $m (\neq 0)$,

$$\bar{\partial} \tilde{\eta}_m^k = \psi_m^k , \qquad (k = 1, \cdots, n) .$$

If we write η_m^k the pull-back of $\tilde{\eta}_m^k$ by the projection $\hat{\psi}: C^n \to T$, η_m^k is the cross-section of the trivial bundle $C^n \times C$ over C^n and satisfies the following relations;

$$(3.10)\qquad \begin{cases} \eta_m^k(z+\omega \boldsymbol{d}) = \eta_m^k(z) f(z, \boldsymbol{d})^{-m} & (\boldsymbol{d} \in Z^{2n}), \\ \bar{\partial}\eta_m^k = \psi_m^k . \end{cases}$$

Furthermore let ζ_m^k be the smooth function on $C^n \times S^1$ defined by

$$\zeta_m^k(z, \theta) = e^{\sqrt{-1}m\theta}\eta_m^k(z) h^{-m/2}(z) .$$

We have then from (3.10)

$$\zeta_m^k(z+\omega \boldsymbol{d}, \theta + \arg f(z, \boldsymbol{d})) = \zeta_m^k(z, \theta) ,$$

that is, the vector field ζ_m defined by

$$\zeta_m = \sum_{k=1}^n \zeta_m^k Z_k$$

belongs to $\Gamma_{(C)}(T')$.

This ζ_m satisfies (3.9). Indeed it follows that

$$\begin{aligned} (\bar{\partial}_b \zeta_m)(Z_{\bar{j}}) &= \sum_{k=1}^n (Z_{\bar{j}}\zeta_m^k) Z_k + \sum_k \zeta_m^k [Z_{\bar{j}}, Z_k] \\ &= \sum_k e^{\sqrt{-1}m\theta} \frac{\partial \eta_m^k}{\partial \bar{z}^j} h^{-m/2} Z_k + \sum_k \zeta_m^k \Phi_{k\bar{j}} \frac{\partial}{\partial \theta} , \qquad (j = 1, \cdots, n) . \end{aligned}$$

Here using $\bar{\partial}\eta_m^k = \psi_m^k$ in (3.10), we have

$$\begin{aligned} (\bar{\partial}_b \zeta_m)(Z_{\bar{j}}) &= \sum_k e^{\sqrt{-1}m\theta} \psi_{\bar{j},m}^k h^{-m/2} Z_k + \sum_k \zeta_m^k \Phi_{k\bar{j}} \frac{\partial}{\partial \theta} \\ &= \sum_k e^{\sqrt{-1}m\theta} \varphi_{\bar{j},m}^k Z_k + \sum_k \zeta_m^k \Phi_{k\bar{j}} . \end{aligned}$$

Q.E.D.

Next for (3.3), we set

$$(\mathfrak{S}\varphi_j)(z, \theta) = \sum_{m \in Z} \varphi_{j,m}(z) e^{\sqrt{-1}m\theta} , \qquad (j = 1, \cdots, n) .$$

Let $\hat{\varphi}_m \in \Gamma_{(C)}(\mathrm{Hom}(^\circ T'', T'))$ be as in Lemma 3.4. If we define the element $\varphi_m \in \Gamma_{(C)}(\mathrm{Hom}(^\circ T'', T'))$ for each $m \in Z$, by

$$\varphi_m(Z_{\bar{j}}) = \hat{\varphi}_m(Z_{\bar{j}}) + \varphi_{j,m} e^{\sqrt{-1}m\theta} \frac{\partial}{\partial \theta} , \qquad (j = 1, \cdots, n) ,$$

then each φ_m is the $e^{\sqrt{-1}m\theta}$-component of the Fourier expansion $\mathfrak{S}(\varphi)$ of φ, that is,

$$\mathfrak{S}(\varphi) = \sum_{m \in Z} \varphi_m .$$

Here $\mathfrak{S}(\varphi)$ is defined by

$$\mathfrak{S}(\varphi)(Z_j) = \sum_k \mathfrak{S}(\varphi_j^k)Z_k + \mathfrak{S}(\varphi_j)\frac{\partial}{\partial\theta}, \qquad (j = 1, \cdots, n).$$

Since $\bar{\partial}_b^{(1)}\varphi = 0$, we find

$$\bar{\partial}_b^{(1)}\varphi_m = 0, \qquad \text{for every } m \in Z.$$

Moreover we can write, for any non-zero integer m, using ζ_m in Lemma 3.4,

$$(3.11) \qquad (\varphi_m - \bar{\partial}_b^{(0)}\zeta_m)(Z_j) = \tilde{\varphi}_{j,m}(z)e^{\sqrt{-1}m\theta}\frac{\partial}{\partial\theta}, \qquad j = 1, \cdots, n,$$

where the $\tilde{\varphi}_{j,m}(z)$ are smooth functions on C^n such that

$$\tilde{\varphi}_{j,m}(z + \omega d) = \tilde{\varphi}_{j,m}(z), \qquad \text{for all } (z, d) \in C^n \times Z^{2n}.$$

LEMMA 3.5. *Let m by any non-zero integer. Then $\varphi_m - \bar{\partial}_b^{(0)}\zeta_m$ in (3.11) is $\bar{\partial}_b^{(0)}$-boundary, i.e., there exists an element $\eta_m\frac{\partial}{\partial\theta}$ of $\Gamma_{(C)}(T')$ such that*

$$\bar{\partial}_b^{(0)}\left(\eta_m\frac{\partial}{\partial\theta}\right) = \varphi_m - \bar{\partial}_b^{(0)}\zeta_m.$$

Proof. As $\bar{\partial}_b^{(1)}(\varphi_m - \bar{\partial}_b^{(0)}\zeta_m) = 0$, the family of functions $\{\tilde{\varphi}_{j,m}\}_{j=1,\cdots,n}$ in the right hand side of (3.11) satisfies the following relations;

$$\frac{\partial\tilde{\varphi}_{j,m}}{\partial\bar{z}^i} - \frac{\partial\tilde{\varphi}_{i,m}}{\partial\bar{z}^j} + \frac{m}{2}\left(\frac{\partial\log h}{\partial\bar{z}^i}\tilde{\varphi}_{j,m} - \frac{\partial\log h}{\partial\bar{z}^j}\tilde{\varphi}_{i,m}\right) = 0,$$
$$1 \leq 1, j \leq n.$$

Therefore this lemma is proved in the same way as Lemma 3.4. Q.E.D.

We obtain from Lemma 3.5 the following

PROPOSITION 3.6. *Let φ be an arbitrary element of $\Gamma_{(C)}(\mathrm{Hom}\,(^\circ T'', T'))$ such that $\bar{\partial}_b^{(1)}\varphi = 0$. Moreover let $\mathfrak{S}(\varphi)$ be the Fourier expansion of φ with respect to the parameter θ of S^1;*

$$\mathfrak{S}(\varphi) = \sum_{m\in Z}\varphi_m$$

where the φ_m are elements of $\Gamma_{(C)}(\mathrm{Hom}\,(^\circ T'', T'))$ defined by

$$\varphi_m(Z_j) = e^{\sqrt{-1}m\theta}\left(\sum_k \varphi_{j,m}^k(z)Z_k + \varphi_{j,m}(z)\frac{\partial}{\partial\theta}\right).$$

Then for all $m\,(\neq 0) \in Z$, we have,

$$\varphi_m = \bar{\partial}_b^{(0)}\zeta_m\,, \qquad \textit{for some } \zeta_m \in \Gamma_{(C)}(T')\,.$$

Furthermore we can prove the following

PROPOSITION 3.7. *Let all notations be as in the above proposition. Let* $\varphi \in \Gamma_{(C)}(\mathrm{Hom}\,({}^\circ T'', T'))$, *with* $\bar{\partial}_b^{(1)}\varphi = 0$. *Then* φ *is* $\bar{\partial}_b$*-cohomologous to* φ_0, *where* $\mathfrak{S}(\varphi) = \varphi_0 - \sum_{Z-(0)\ni m}\varphi_m$.

Proof. In general let f be any C^∞-function on $C^n \times S^1$. Then $\mathfrak{S}(f)$ converges uniformly on every compact subset of $C^n \times S^1$. Now for any $\varphi \in \Gamma_{(C)}(\mathrm{Hom}\,({}^\circ T'', T'))$, we define the norm, denoted by $|\varphi|$, as follows; Let $\varphi(Z_{\bar{j}}) = \sum \varphi_{\bar{j}}^k Z_k - \varphi_j \dfrac{\partial}{\partial\theta}$. Then we have

$$|\varphi| = \max_{1\le j,k\le n}\{\sup|\varphi_{\bar{j}}^k|, \sup|\varphi_j|\}\,.$$

This is well-defined because of the condition (C). Here if we put $\mathfrak{S}_k(\varphi) = \sum_{|m|\le k}\varphi_m$, for any non-negative integer k, we find that for any $\varepsilon > 0$, there exists an integer $k(\varepsilon) \geqq 0$, such that

$$|\varphi - \mathfrak{S}_{k(\varepsilon)}(\varphi)| < \varepsilon\,.$$

Let τ be the isomorphism of the complex $\{\Gamma(\wedge^k({}^\circ T''(B_1))^* \otimes T'(B_1)), \bar{\partial}_b^k\}$ onto $\{\Gamma_{(C)}(\wedge^k({}^\circ T'')^* \otimes T'), \bar{\partial}_b(k)\}$ as in the proof of Proposition 3.1.

On the other hand we impose the hermitian innerproduct, $\langle\,,\rangle$ on $CT(B_1)$ such that, $\psi_*(Z_{\bar{1}}), \cdots, \psi_*(Z_{\bar{n}}), \psi_*(Z_1), \cdots, \psi^*(Z_n)$ and $\psi_*(\partial/\partial\theta)$ are orthonormal basis. For every $\tilde{\varphi} \in \Gamma(\wedge^k({}^\circ T(B_1)^* \otimes T'(B_1))$, we set

$$|||\tilde{\varphi}||| = \sum_{1\le j_1<\cdots j_k\le n}\langle\tilde{\varphi}(\psi_*(Z_{\bar{j}_1}), \cdots, \psi_*(Z_{j_k}), \tilde{\varphi}(\psi_*(Z_{\bar{j}_1}), \cdots, \psi_*(Z_{\bar{j}}{}^k))\rangle^{\frac{1}{2}}\,.$$

The L_2-norm, denoted by $\|\;\|_{B_1}$, on $\Gamma(\wedge^k({}^\circ T''(B_1))^* \otimes T'(B_1))$ is defined by

$$\|\tilde{\varphi}\|_{B_1} = \int_{B_1}|||\tilde{\varphi}|||dv$$

where dv denotes the volume element associated with the hermitian inner product $\langle\,,\rangle$ on $CT(B_1)$.

We can further form the formal adjoint

$$\bar{\partial}_2^{(k)*}\colon \Gamma(\wedge^k({}^\circ T''(B_1))^* \otimes T'(B_1)) \to \Gamma(\wedge^{k-1}({}^\circ T''(B_1))^* \otimes T'(B_1))$$

of $\bar{\partial}_b^{(k-1)}$ with respect to the above norm $\|\;\|_{B_1}$, $(k = 1, \cdots, n)$.

Now take an element φ of $\Gamma_{(C)}(({}^\circ T'')T')$ with $\bar{\partial}_\beta^{(1)}\varphi = 0$. For any $\varepsilon > 0$. there exirts an integer $k(\varepsilon)$ such that

(3.12) $$\|\tau^{-1}\varphi - \tau^{-1}(\mathfrak{S}_{k(\varepsilon)}(\varphi))\| < \varepsilon .$$

Moreover it is clear that $\bar{\partial}_b^{(1)}\tau^{-1}\varphi = 0$. Since ${}^\circ T''(B_1)$ is strongly pseudoconvex and $\dim_R B_1 \geqq 5$, we know from § 6 [1] that there exists $\tilde{\zeta} \in \Gamma(T'(B_1))$ and $\tilde{\eta} \in \Gamma({}^\circ T''(B_1)^* \otimes T'(B_1))$ such that

$$\tau^{-1}\varphi - \tau^{-1}\varphi_0 = \tilde{\eta} + \bar{\partial}_b^{(0)}\tilde{\zeta} ,$$

where $(\bar{\partial}_b^{(2)*}\cdot\bar{\partial}_b^{(1)} + \bar{\partial}_b^{(0)}\bar{\partial}_b^{(1)*})\tilde{\eta} = 0$.

We shall show $\tilde{\eta} = 0$. Indeed suppose $\tilde{\eta} \neq 0$. Then it follows that

$$\|\tau^{-1}\varphi - \tau^{-1}\varphi_0\|_{B_1} \geqq \|\tilde{\eta}\|_{B_1} > \varepsilon_1 , \qquad \text{for some } \varepsilon_1 > 0 .$$

Here for any $\varepsilon > 0$ with $\varepsilon < \varepsilon_1$, we choose an integer $k(\varepsilon)$ satisfying (3.12), and put

$$\mathfrak{S}'_{k(\varepsilon)}(\varphi) = \mathfrak{S}_{k(\varepsilon)}(\varphi) - \varphi_0 .$$

It follows from Proposition 3.6 that there is an element $\varepsilon \in \Gamma(T')$ such that

$$\mathfrak{S}'_{k(\varepsilon)}(\varphi) = \bar{\partial}_b^{(0)}\zeta ,$$

so that we have

$$\varepsilon > \|\tau^{-1}\varphi - \tau^{-1}\varphi_0 - \tau^{-1}\mathfrak{S}'_{k(\varepsilon)}(\varphi)\|_{B_1} = \|\tau^{-1}\varphi - \tau^{-1}\varphi_0\bar{\partial}_b^{(0)}\tau^{-1}\zeta\|_{B_1} \geqq \|\tilde{\eta}\|_{B_1} .$$

This is a contradiction and so our proposition is proved. Q.E.D.

By virtue of the above arguments, in order to determine the infinitesimal deformation $H^1_{(C)}({}^\circ T'')$ $(\cong H^1({}^\circ T''(B_1)))$, it is enough to consider the following subcomplex (3.13) of $\{\Gamma_{(C)}(\wedge^p ({}^\circ T'')^* \otimes T'),\ \bar{\partial}_b^{(p)}\}$; Let

$$\varphi \in \Gamma(\wedge^k ({}^\circ T'')^* \otimes T')$$

and set

$$\varphi(Z_{\bar{j}_1}, \cdots, Z_{\bar{j}_k}) = \sum_{i=1}^{n} \varphi^i_{\bar{j}_1,\dots,\bar{j}_k} Z_i + \varphi_{\bar{j}_1,\cdots\bar{j}_k}\frac{\partial}{\partial\theta} \qquad (1 \leq j_1, \cdots, j_k \leqq n) .$$

We denote by $\Gamma_T(\wedge^k ({}^\circ T'')^* \otimes T')$ the set of all $\varphi \in \Gamma(\wedge^k ({}^\circ T'')^* \otimes T')$ such that $\varphi^i_{\bar{j}_1,\dots,\bar{j}_k}$ and $\varphi_{\bar{j}_1,\dots,\bar{j}_k}$ are smooth functions on the abelian variety T. Then the complex

(3.13) $$0 \longrightarrow \Gamma_T(T') \xrightarrow{\bar{\partial}_b^{(0)}} \Gamma_T(({}^\circ T'')^* \otimes T') \xrightarrow{\bar{\partial}_b^{(1)}} \Gamma_T(\wedge^2 ({}^\circ T'')^* \otimes T') \xrightarrow{\bar{\partial}_b^{(2)}} \cdots ,$$

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is the required one.

THEOREM 3.8. *Let* $°T''$ *be the strongly pseudo-convex CR-structure on* $C^n \times S^1$ $(n \geqq 2)$ *induced from the negative line bundle* B *over the abelian variety* T $(\dim_C T = n)$, *as before. Let* $\varphi \in \Gamma_{(C)}((°T'')^* \otimes T')$ *with*

$$\varphi(Z_{\bar{j}}) = \sum_{k=1}^{n} C_{\bar{j}}^{k} Z_k + a_j \frac{\partial}{\partial \theta}, \qquad (j = 1, \cdots, n), \tag{3.14}$$

where the $C_{\bar{j}}^k$ *and the* a_j *are arbitrary constants such that*

$$\sum_{k=1}^{n} C_{\bar{j}}^{k} Q_{k\bar{i}} = \sum_{k=1}^{n} C_{\bar{i}}^{k} Q_{k\bar{j}}, \qquad (1 \leq i, j \leq n). \tag{3.15}$$

Then any element of $H^1_{(C)}(°T'')$ *can be represented by some element* ρ *as above.*

Proof. Take $\varphi \in \Gamma_T((°T'')^* \otimes T')$ such that $\bar{\partial}_b^{(1)}\varphi = 0$ and

$$\varphi(Z_{\bar{j}}) = \sum_{k=1}^{n} \varphi_{\bar{j}}^{k} Z_k + \varphi_j \frac{\partial}{\partial \theta}, \qquad (j = 1, \cdots, n). \tag{3.16}$$

Recall that $\varphi_{\bar{j}}^k$ and φ_j are C^∞-functions on T. By Proposition 3.2, $\bar{\partial}_0^{(1)}\varphi = 0$ means that φ satisfies equations (3.2) and (3.3). Here using $\varphi_{\bar{j}}^k$ and φ_j in (3.16), we put

$$\psi = \sum_{j,k=1}^{n} \varphi_{\bar{j}}^{k} \frac{\partial}{\partial z^k} \otimes d\bar{z}^j, \qquad \omega = \sum_{j=1}^{n} \varphi_j d\bar{z}^j,$$

and

$$\Phi = \sum_{i,j=1}^{n} \Phi_{i\bar{j}} dz^j \wedge d\bar{z}^j \,(= \sqrt{-1}\partial\bar{\partial}\log h), \qquad (\text{c.f. } (1.2)),$$

where $\Phi_{i\bar{j}} = 2\pi Q_{i\bar{j}}$.

Let $\bigwedge^{(p,q)}(T)$ be the set of all differential forms of type (p, q) on T. Moreover let $\barwedge$ be the generalized interior product, that is, for any $\tilde{\psi} \in \bigwedge^{(p,q)}(T) \otimes \Gamma(T(T))$ with

$$\tilde{\psi} = \sum_{j_1,\cdots,j_q} \sum_{i_1,\cdots,i_p} \sum_{k} A^k_{i_1,\cdots,i_p,\bar{j}_1,\cdots,\bar{j}_q} dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q} \otimes \frac{\partial}{\partial z^k}$$

the linear map $\tilde{\psi}\barwedge : \bigwedge^{(\ell,m)}(T) \to \bigwedge^{(\ell+p-1,m+q)}(T)$ is defined by

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$$\tilde{\psi} \barwedge \Big(\sum_{\substack{\alpha_1, \cdots, \alpha_\ell \\ \beta_1, \cdots, \beta_m}} B_{\alpha_1, \cdots, \alpha_\ell, \beta_1, \cdots, \beta_m} dz^{\alpha_1} \wedge \cdots \wedge dz^{\alpha_\ell} \wedge d\bar{z}^{\beta_1} \wedge \cdots d\bar{z}^{\beta_m} \Big)$$
$$= \sum_{i_1, \cdots, i_p} \sum_{j_1, \cdots, j_q} \sum_{\alpha_1, \cdots, \alpha_\ell} \sum_{\beta_1, \cdots, \beta_m} \sum_k \Big\{ \sum_{\ell'=1}^{\ell} (-1)^{\ell'+1} A^k_{i_1, \cdots, i_p, \bar{j}_q} dz^{i_1} \wedge$$
$$\cdots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q} \wedge B_{\alpha_1, \cdots, \alpha_\ell \beta_1, \cdots, \beta_m} dz^{\alpha_1} \wedge$$
$$\cdots \wedge dz^{\alpha_{\ell'-1}} \wedge \Big\langle \frac{\partial}{\partial z^k} dz^{\alpha_{\ell'}} \Big\rangle \wedge \cdots \wedge dz^{\alpha_\ell} \wedge d\bar{z}^{\beta_1} \wedge \cdots \wedge d\bar{z}^{\beta_m} \Big\} .$$

Then (3.2) and (3.3) are rewritten as follows;

(3.17) $$\bar{\partial}\psi = 0 ,$$

and

(3.18) $$\psi \barwedge \Phi - \bar{\partial}\omega = 0 .$$

Next we examine conditions for $\bar{\partial}_b$-boundary in the complex (3.13). For this aim let

$$\psi = \sum_{j=1}^{n} \psi^j Z_j + \eta \frac{\partial}{\partial \theta}$$

be an element of $\Gamma_T(T')$ such that $\varphi = \bar{\partial}_b^{(0)}\psi$. Then we obtain

(3.19) $$\begin{cases} \varphi^k_{\bar{j}} = \dfrac{\partial \psi^k}{\partial \bar{z}^j} & (1 \le j, k \le n) , \\ \varphi_{\bar{j}} = \displaystyle\sum_{k=1}^{n} \psi^k \Phi_{k\bar{j}} + \frac{\partial \eta}{\partial \bar{z}^j} , & (j = 1, \cdots, n) . \end{cases}$$

Set here

$$\xi = \sum_{k=1}^{n} \psi^k \frac{\partial}{\partial z^k} .$$

Then (3.19) means that

(3.20) $$\psi = \bar{\partial}\xi \quad \text{and} \quad \omega = \xi \barwedge \Phi + \bar{\partial}\eta .$$

Therefore in terms of (3.17), (3.18) and (3.20) the complex (3.13) reduces to the following one;

$$0 \longrightarrow \Gamma(T(T)) \oplus C^\infty(T) \xrightarrow{\bar{\partial}_\Phi^{(0)}} (\wedge^{(0,1)}(T) \otimes \Gamma(T(T))) \oplus \wedge^{(0,1)}(T)$$
$$\xrightarrow{\bar{\partial}_\Phi^{(1)}} (\wedge^{(0,2)}(T) \otimes \Gamma(T(T))) \oplus \wedge^{(0,2)}(T) ,$$

where

$$\bar\partial_\Phi^{(0)}(\xi, \eta) = (\bar\partial\xi, \xi \barwedge \Phi + \bar\partial\eta)$$

and

$$\bar\partial_\Phi^{(1)}(\psi', \omega') = (\bar\partial\psi', \psi' \barwedge \Phi - \bar\partial\omega') ,$$

for $(\xi, \eta) \in \Gamma(T(T)) \oplus C^\infty(T)$ and $(\psi', \omega') \in (\wedge^{(0,1)}(T) \otimes T(T)) \oplus \wedge^{(0,1)}(T)$.

Now let $(\psi', \omega') \in \operatorname{Ker} \bar\partial_\Phi^{(1)}$. As $\bar\partial\psi' = 0$, it follows that there exist constants $\{C_j^k\}$ $1 \leq j, k \leq n$ and $\xi \in \Gamma(T(T))$ such that

$$\psi' = \sum_{j,k=1}^n C_j^k d\bar z^j \otimes \frac{\partial}{\partial z^k} + \bar\partial\xi .$$

Moreover we have

$$\psi' \barwedge \Phi - \bar\partial\omega' = \sum_{i,j,k} C_j^k \Phi_{k\bar i} d\bar z^j \wedge d\bar z^i + \bar\partial(\xi \barwedge \Phi - \omega') = 0 \tag{3.21}$$

Therefore the $\bar\partial$-cohomology class of $\sum_{i,j,k} C_j^k \Phi_{k\bar i} d\bar z^j \wedge d\bar z^i$ is zero. However since C_j^k and $\Phi_{k\bar j}$ are constants, we get

$$\sum_{i,j,k} C_j^k \Phi_{k\bar i} d\bar z^j \wedge d\bar z^i = 0 \tag{3.22}$$

This fact shows (3.15). Moreover it follows from (3.21) and (3.22) that $\bar\partial(\xi \barwedge \Phi - \omega') = 0$, so that we are able to write

$$\xi \barwedge \Phi - \omega' = \sum_{j=1}^n a_j d\bar z^j + \bar\partial\eta_0 , \tag{3.23}$$

where the a_j are constants and η_0 is a smooth function on T. Clearly

$$\left(\sum_{j,k} C_j^k \frac{\partial}{\partial z^k} \otimes d\bar z^j, \sum_j a_j d\bar z^j\right)$$

belongs to $\operatorname{Ker} \bar\partial_\Phi^{(1)}$. It follows that (ψ', ω') and

$$\left(\sum_{j,k} C_j^k \frac{\partial}{\partial z^k} \otimes d\bar z^j, \sum_j a_j d\bar z^j\right)$$

are $\bar\partial_\Phi^{(0)}$-cohomologous. Indeed we have

$$(\psi', \omega') - \left(\sum_{j,k} C_j^k \frac{\partial}{\partial z^k} \otimes d\bar z^j, \sum_j a_j d\bar z^j\right)$$
$$= (\bar\partial\xi, \xi \barwedge \Phi - \bar\partial\eta_0) = \bar\partial_\Phi^{(0)}(\xi, -\eta_0) .$$

Thus our theorem is completely proved. Q.E.D.

§ 4. Integrable CR-structures on $C^n \times S^1$.

Let all notations be as before. Recall that ${}^\circ T''$ is the strongly pseudoconvex CR-structure on $C^n \times S^1$ induced by $\psi: C^n \times S^1 \to C^n \times C$ (see the bottom of Proposition 2.1). Let now $\varphi \in \Gamma_{(C)}(({}^\circ T'')^* \otimes T')$ be at finite distance from ${}^\circ T''$. Then ${}^\varphi T'' = \{X - \varphi(X); X \in {}^\circ T''\}$ becomes an almost CR-structure on $C^n \times S^1$. It follows from Proposition 1.4 that φ is integrable if and only if $P(\varphi) = 0$, where the linear map $P: \Gamma(({}^\circ T'')^* \otimes T') \to \Gamma(\wedge^2 ({}^\circ T'')^* \otimes T')$ is defined by (1.6).

PROPOSITION 4.1. *Let $\varphi \in \Gamma_{(C)}(({}^\circ T'')^* \otimes T')$ be as in Theorem 3.8. Then φ is integrable.*

Proof. This is trivial from the definition of P and (3.14). Q.E.D.

For convenience sake we write $\mathscr{H}^1$ the set of all $\varphi \in \Gamma_{(C)}(({}^\circ T'')^* \otimes T')$ satisfying (3.14) and (3.15) in Theorem 3.8, so that $\mathscr{H}^1$ is isomorphic to $H^1_{(C)}({}^\circ T'')$. Let φ be any element of $\mathscr{H}^1$ with

$$\varphi(Z_{\bar{j}}) = \sum_{k=1}^{n} C_{\bar{j}}^{k} Z_k - \frac{\sqrt{-1}}{2} a_j \frac{\partial}{\partial \theta}, \qquad (j = 1, \cdots, n), \tag{4.1}$$

where the norm $|\varphi|$ of φ is sufficiently small. Then φ is at finite distance to ${}^\circ T''$ and we get

$$\begin{aligned} Z_{\bar{j}} - \varphi(Z_{\bar{j}}) = &\left(\frac{\partial}{\partial \bar{z}^j} - \sum_{k=1}^{k} C_{\bar{j}}^{k} \frac{\partial}{\partial z^k}\right) \\ &- \frac{\sqrt{-1}}{2}\left(\frac{\partial \log h}{\partial \bar{z}^j} + \sum_{k=1}^{n} C_{\bar{j}}^{k} \frac{\partial \log h}{\partial z^k} - a_j\right)\frac{\partial}{\partial \theta}, \\ & j = 1, \cdots, n. \end{aligned} \tag{4.2}$$

And the CR-structure $\varphi T''$ is generated by $\{Z_{\bar{j}} - \varphi(Z_{\bar{j}})\}$ $j = 1, \cdots, n$.

We shall next determine complex structures on $C^n \times C$ which induce CR-structures $\varphi \in \mathscr{H}^1$ on $C^n \times S^1$ by the map $\psi: C^n \times S^1 \to C^n \times C$. Remember that ψ is defined by

$$\psi(z, \theta) = \left(z, \frac{e^{\sqrt{-1}\theta}}{\sqrt{h(z)}}\right) \qquad \text{for } (z, \theta) \in C^n \times S^1 .$$

PROPOSITION 4.2. *Let $\varphi \in \mathscr{H}^1$ with* (4.1). *Let $z^1, \cdots, z^n$ and ζ be the canonical coordinates of $C^n \times C$, and put*

$$(4.3)\qquad \begin{cases} \check{Z}_{\bar{j}} = \left(\dfrac{\partial}{\partial \bar{z}^j} - \sum_{k=1}^{n} C_{\bar{j}}^{\bar{k}} \dfrac{\partial}{\partial \bar{z}^k}\right) + \left(\sum_{k=1}^{n} C_{\bar{j}}^{\bar{k}} \dfrac{\partial \log h}{\partial z^k} - a_j\right)\zeta \dfrac{\partial}{\partial \zeta}\,, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad (j = 1, \cdots, n)\,. \\ \dfrac{\partial}{\partial \bar{\zeta}} \end{cases}$$

Then the CR-structure ${}^{\varphi}T''$ on $C^n \times S^1$ is the one which is induced from the complex structure $\{\{\check{Z}^{\varphi}_{\bar{1}}, \cdots, \check{Z}^{\varphi}_{\bar{n}}, \partial/\partial\bar{\zeta}\}\}$ on $C^n \times C$ by ψ, that is,

$$(4.4)\qquad \psi_*(\varphi T'') = \psi_*(CT(C^n \times S^1)) \cap \left\{\left\{\check{Z}^{\varphi}_{\bar{1}}, \cdots, \check{Z}^{\varphi}_{\bar{n}}, \frac{\partial}{\partial \bar{\zeta}}\right\}\right\}.$$

Proof. First of all we see the system (4.3) is integrable. In fact it follows that

$$\begin{aligned}[\check{Z}^{\varphi}_{\bar{i}}, \check{Z}^{\varphi}_{\bar{j}}] &= -\sum_{k=1}^{n} \left(C_{\bar{i}}^{\bar{k}} \frac{\partial^2 \log h}{\partial \bar{z}^j \partial z^k} - C_{\bar{j}}^{\bar{k}} \frac{\partial^2 \log h}{\partial \bar{z}^i \partial z^k}\right)\zeta \frac{\partial}{\partial \zeta} \\ &= -\sqrt{-1} \sum_{k=1}^{n} (C_{\bar{i}}^{\bar{k}} Q_{k\bar{j}} - C_{\bar{j}}^{\bar{k}} Q_{k\bar{i}})\zeta \frac{\partial}{\partial \zeta} = 0\,,\end{aligned}$$

and

$$\left[\check{Z}^{\varphi}_{\bar{j}}, \frac{\partial}{\partial \bar{\zeta}}\right] = 0\,, \qquad (1 \geq i, j \leq n)\,.$$

Thus (4.3) is integrable. We next show the relation (4.4). For this aim it is enough to prove that

$$\psi_*({}^{\varphi}T'') \subset \left\{\left\{\check{Z}^{\varphi}_{\bar{1}}, \cdots, \check{Z}^{\varphi}_{\bar{n}}, \frac{\partial}{\partial \bar{\zeta}}\right\}\right\}.$$

However it follows that for all $(z, \theta) \in C^n \times S^1$,

$$\begin{aligned}&\psi_{*(z,\theta)}(Z_{\bar{j}} - \varphi(Z_{\bar{j}})) \\ &\quad = \left(\frac{\partial}{\partial \bar{z}^j} - \sum_k C_{\bar{j}}^{\bar{k}} \frac{\partial}{\partial \bar{z}^k}\right)_{\psi(z,\theta)} + \left\{\left(\sum_k C_{\bar{j}}^{\bar{k}} \frac{\partial \log h}{\partial z^k} - a_j\right)\zeta \frac{\partial}{\partial \zeta}\right\}_{\psi(z,\theta)} \\ &\qquad - \left\{\left(\frac{1}{2}\, \frac{\partial \log h}{\partial \bar{z}^j} - \sum_k C_{\bar{j}}^{\bar{k}} \frac{\partial \log h}{\partial z^k} - a_j\right)\bar{\zeta} \frac{\partial}{\partial \bar{\zeta}}\right\}_{\psi(z,\theta)}, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad (j = 1, \cdots, n)\,.\end{aligned}$$

Q.E.D.

Now for any $\varphi \in \mathscr{H}^1$ with (4.1), we denote by $T''_{\varphi}(C^n \times C)$ the subbundle of $CT(C^n \times C)$ generated by the system (4.3) in Proposition 4.2, or the complex structure on $C^n \times C$ defined by (4.3). Moreover let T^{φ} be the diffeomorphism of $C^n \times C$ onto $C^n \times C$ defined by

$$
(4.5)\quad \begin{cases} z^j \circ T^\varphi(w, t) = w^j - \sum_{k=1}^n C_k^j \bar{w}^k , \qquad (j = 1, \cdots, n) , \\ \zeta \circ T^\varphi(w, t) = t \exp \Big\{ 2\pi\sqrt{-1} \sum_{i,j,k,\ell=1}^n Q_{k\bar{j}} C_i^k C_{\bar{\ell}}^i w^\ell \bar{w}^j \\ \qquad\qquad - \sqrt{-1}\pi \sum_{k,i,j=1}^n Q_{k\bar{j}} C_{\bar{i}}^k \bar{w}^i \bar{w}^j + \sum_{j=1}^n a_j \bar{z}^j \end{cases}
$$

where (w, t) and (z, ζ) are usual coordinates on $C^n \times C$.

Then we have the following

PROPOSITION 4.3. *Let φ be an element of $\mathscr{H}^1$ with* (4.1). *Then the complex structure $T''_\varphi(C^n \times C)$ is induced from the standard complex structure on $C^n \times C$ by the diffeomorphism T^φ: $C^n \times C \to C^n \times C$ defined by* (4.5), *that is,*

$$
T''\varphi(C^n \times C) = \left\{ \left\{ T_*^\varphi\left(\frac{\partial}{\partial \bar{w}^1}\right), \cdots, T_*^\varphi\left(\frac{\partial}{\partial \bar{w}^n}\right), T_*^\varphi\left(\frac{\partial}{\partial \bar{t}}\right) \right\} \right\} .
$$

Proof. By direct calculations we have

$$
\begin{aligned} T_*^\varphi\left(\frac{\partial}{\partial \bar{w}^j}\right) = \left(\frac{\partial}{\partial \bar{z}^j} - \sum_{k=1}^n C_{\bar{j}}^k \frac{\partial}{\partial z^k}\right) \\ - \Big\{ -2\pi\sqrt{-1} \sum_{i,k=1}^n Q_{k\bar{j}} C_\ell^k (\bar{z}^i T\varphi) - a_j \Big\} \zeta \circ T^\varphi \frac{\partial}{\partial \zeta} \\ + \frac{\partial \zeta \circ T^\varphi}{\partial \bar{w}^j} \frac{\partial}{\partial \bar{\zeta}} , \qquad (j = 1, \cdots, n) . \end{aligned}
$$

However from $h(z) = \exp (2\pi\sqrt{-1} \sum_{j,1=1}^n Q_{i\bar{j}} z^i \bar{z}^j)$, it follows that

$$
\sum_{k=1}^n C_{\bar{j}}^k \frac{\partial \log h}{\partial z^k} = -2\pi\sqrt{-1} \sum_{i,k=1}^n Q_{k\bar{j}} C_{\bar{i}}^k \bar{z}^i ,
$$

so that we find, for every $(w, t) \in C \times C$,

$$
\begin{aligned} T_{*(u,t)}^\varphi\left(\frac{\partial}{\partial \bar{w}^j}\right) = \left(\frac{\partial}{\partial \bar{z}^j} - \sum_{k=1}^n C_{\bar{j}}^k \frac{\partial}{\partial z^k}\right)_{T^\varphi(w,t)} \\ + \left\{ \left(\sum_{k=1}^n C_{\bar{j}}^k \frac{\partial \log h}{\partial z^k} + a_j \right) \zeta \frac{\partial}{\partial \zeta} \right\}_{T^\varphi(w,t)} + \left(\frac{\partial \bar{\zeta}}{\partial \bar{w}^j} \frac{\partial}{\partial \bar{\zeta}} \right)_{T^\varphi(w,t)} , \\ (j = 1, \cdots, n) \end{aligned}
$$

On the other hand it is clear that

$$
T_*^\varphi\left(\frac{\partial}{\partial \bar{t}}\right) = \frac{\partial \bar{\zeta} \circ T^\varphi}{\partial \bar{t}} \frac{\partial}{\partial \bar{\zeta}} .
$$

Then the above equations show

$$T''_\varphi(C^n \times C) = \left\{\left\{T^\varphi_*\left(\frac{\partial}{\partial \overline{w}^1}\right), \cdots, T^\varphi_*\left(\frac{\partial}{\partial \overline{w}^n}\right), T^\varphi_*\left(\frac{\partial}{\partial \overline{t}}\right)\right\}\right\}. \qquad \text{Q.E.D.}$$

§ 5. The main theorem

We shall complete in this section the proof of the following.

THEOREM 5.1. *Let T be an abelian variety of complex dimension n ($n \geqq 2$), and let B be a negative line bundle over T. We denote by (B, T) the isolated singularity defined from B and T, as stated at 2.2 in § 2. Then any local deformation of (B, T) is also (B', T'), where T' is an abelian variety of $\dim_C T' = n$, and B' represents a negative line bundle over T'.*

First of all recall that the infinitesimal deformation $H^1({}^\circ T''(B_1))$ of (B, T) is isomorphic to $\mathscr{H}^1$ as in the previous section, and that any element of $\mathscr{H}^1$ is integrable (c.f. Proposition 4.1.). Therefore we shall at first prove that small CR-structures in $\mathscr{H}^1$ are induced ones from negative line bundles over abelian varieties.

Now let us take any element φ of $\mathscr{H}^1$ with $\varphi(Z_{\bar{j}}) = \sum_{k=1}^n C^k_{\bar{j}} Z_k - \sqrt{-1} a_j(\partial/\partial\theta)$, $(j = 1, \cdots, n)$, whose norm $|\varphi|$ is sufficiently small, and fix φ. Then the CR-structure ${}^\varphi T''(C^n \times S^1)$ on $C^n \times S^1$ defined by (4.2) is induced from the complex structure $T''_\varphi(C^n \times C)$ on $C^n \times C$ determined in terms of (4.3) in Proposition 4.2. Moreover from Proposition 4.3, $T''_\varphi(C^n \times C)$ arises out of the standard complex structure on $C^n \times C$ by the map $T^\varphi: C^n \times C \to C^n \times C$ which is defined by (4.5). We regard this map T^φ as the bundle map between two trivial bundles $C^n \times C$ over C^n. Thus in order to prove the above statement it is enough to show that there exist an abelian variety T' and a negative line bundle B' over T' such that the bundle map $T^\varphi: C^n \times C \to C^n \times C$ induces canonically a bundle map $\tilde{T}^\varphi: B \times B$.

From now on we shall show the above statements. For this aim let us put

$$C = \begin{pmatrix} C^1_{\bar{1}}, & \cdots, & C^1_{\bar{n}} \\ & \cdots\cdots & \\ C^n_{\bar{1}}, & \cdots, & C^n_{\bar{n}} \end{pmatrix}, \qquad a = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

and

$$Q = \begin{pmatrix} Q_{1\bar{1}}, & \cdots, & Q_{1\bar{n}} \\ & \cdots\cdots\cdots & \\ & \cdots\cdots\cdots & \\ Q_{n\bar{1}}, & \cdots, & Q_{n\bar{n}} \end{pmatrix}$$

Then the condition (3.15) becomes

$$(3.15)' \qquad {}^tCQ = {}^tQC$$

where t denotes the transpose of matrices. Furthermore the map T^φ is also represented as follows;

$$(4.5)' \qquad T^\varphi(w, t) = (w - C - \overline{w}, t\cdot \exp\{2\pi\sqrt{-1}{}^t\overline{w}{}^tQC\overline{C}w - \sqrt{-1}\pi{}^t\overline{w}{}^tQC\overline{w} + {}^ta\overline{w}\}) ,$$

where

$$w = \begin{pmatrix} w^1 \\ \vdots \\ w^n \end{pmatrix} \in C^n .$$

Here we set

$$(5.1) \qquad g(w) = \exp\{2\pi\sqrt{-1}{}^t\overline{w}{}^tQC\overline{C}w - \sqrt{-1}\pi{}^t\overline{w}{}^tQC\overline{w} + {}^ta\overline{w}\} .$$

so that it follows that

$$(5.2) \qquad T^\varphi(w, t) = (w - C\overline{w}, g(w)t) .$$

Now we write ω' and $f'(w, \boldsymbol{d})$ a periodic matrix and an automorphic factor respectively, corresponding to an abelian variety T' and a negative line bundle B' over T' to be required for the given CR-structure $\varphi \in \mathscr{H}^1$. Then since T^φ in (5.2) induces a bundle map of B' onto B, ω' and f' must satisfy the following conditions;

$$(5.3) \qquad \omega' - c\overline{\omega}' = \omega$$

and

$$(5.4) \qquad g(w + \omega'\boldsymbol{d})f'(w, \boldsymbol{d}) = g(w)f(w - C\overline{w}, \boldsymbol{d}) , \quad \text{for all } (w, \boldsymbol{d}) \in C^n \times Z^{2n} ,$$

where ω is the periodic matrix of T and f denotes the automorphic factor of B in § 2. Since $|\varphi|$ is sufficiently small, there exists a unique ω' satisfying (5.3), and we fix ω'. Furthermore we obtain the following.

PROPOSITION 5.2. *Let f' be the map of $C^n \times Z^{2n}$ into C defined by* (5.4). *Then f' becomes an automorphic factor for the periodic matrix, ω', that is,*

(5.5) *f' is the holomorphic map of $C^n \times Z^{2n}$ into $C - \{0\}$,* *and*

$$f'(w, d_1 + d_2) = f'(w + \omega' d_1, d_2) f'(w, d_1)\,, \quad \text{for } w \in C^n \text{ and } d_i \in Z^{2n}\ (i = 1, 2)\,. \tag{5.6}$$

The proof of this proposition is due to the following two lemmas.

LEMMA 5.3. *For an arbitrary* $(w, d) \in C^n \times Z^{2n}$, *we have*

$$f(w, d) = f(w, d) g(\omega' d)^{-1} \exp\,(-2\pi\sqrt{-1}\,{}^t(\bar{\omega}' d)\,{}^t Q C \bar{C} w\}\,. \tag{5.7}$$

Proof. At the beginning we get, using (5.1), (5.3) and ${}^tQC = {}^tCQ$,

$$g(w + \omega' d) = g(w) g(\omega' d) \exp\,\{2\pi\sqrt{-1}\,{}^t\bar{w}\,{}^tQC(\bar{C}\omega' - \bar{\omega}')d + 2\pi\sqrt{-1}\,{}^t(\bar{\omega}' d)\,{}^tQC\bar{C}w\}\,. \tag{5.8}$$

Therefore it follows from (5.4) that

$$f'(w, d) = f(w - C\bar{w}, d) g(\omega' d)^{-1} \exp\,\{-2\pi\sqrt{-1}\,{}^t\bar{w}\,{}^tQC(\bar{C}\omega' - \bar{\omega}')d - 2\pi\sqrt{-1}\,{}^t(\bar{\omega}' d)\,{}^tQC\bar{C}s\}\,.$$

On the other hand we find from (2.1)

$$f(w - C\bar{w}, d) = \exp\,\{2\pi\sqrt{-1}\,{}^t(w - C\bar{w})Q\bar{\omega} d + A(d)\}\,,$$

where we put $A(d) = \frac{1}{2}{}^t d A^\circ d + {}^t b d$, so that we obtain

$$\begin{aligned} f'(w, d) &= g(\omega' d)^{-1} \exp\,\{-2\pi\sqrt{-1}\,{}^t(\bar{\omega}' d)\,{}^tQC\bar{C}w\} \\ &\quad \times \exp\,\{-2\pi\sqrt{-1}\,{}^t\bar{w}\,{}^tQC(\bar{C}\omega' - \bar{\omega}')d\} \\ &\quad \times \exp\,\{2\pi\sqrt{-1}\,{}^t(w - C\bar{w})Q(\bar{\omega}' - \bar{C}\omega')d + A(d)\} \\ &= g(\omega' d) \exp\,\{-2\pi\sqrt{-1}\,{}^t(\bar{\omega}' d)\,{}^tQC\bar{C}w\} f(w, d)\,. \end{aligned}$$

Q.E.D.

Moreover we have

LEMMA 5.4. f' *satisfies* (5.6).

Proof. From (5.7) it is clear that

$$\begin{aligned} f'(w, d_1 + d_2) &= f(w, d_1 + d_2) g(\omega'(d_1 + d_2))^{-1} \\ &\quad \times \exp\,\{-2\pi\sqrt{-1}\,{}^t(\bar{\omega}'(d_1 + d_2))\,{}^tQC\bar{C}w\}\,. \end{aligned}$$

Using the fact that f is the automorphic factor, we get from (5.8)

$$\begin{aligned} g(\omega' d_1 + \omega' d_2) &= g(\omega' d_1) g(\omega' d_2) \exp\,\{2\pi\sqrt{-1}\,{}^t(\bar{\omega}' d_1)\,{}^tQC(\bar{C}\omega' - \bar{\omega}')d_2\} \\ &\quad + 2\pi\sqrt{-1}\,{}^t(\bar{\omega}' d_2)\,{}^tQC\bar{C}\omega' d_1\}\,. \end{aligned}$$

Hence it follows that

$$
\begin{aligned}
f'(w, \boldsymbol{d}_1 + \boldsymbol{d}_2) &= f(w + \omega \boldsymbol{d}_1, \boldsymbol{d}_2) f(w, \boldsymbol{d}_1) g(\omega' \boldsymbol{d}_1)^{-1} g(\omega' \boldsymbol{d}_2)^{-1} \\
&\quad \times \exp(-2\pi\sqrt{-1}\,{}^t(\bar{\omega}' \boldsymbol{d}_1)\,{}^t Q C(\bar{C}\omega' - \bar{\omega}')\boldsymbol{d}_2 - 2\pi\sqrt{-1}\,{}^t(\bar{\omega}\boldsymbol{d}_1)\,{}^t QC\bar{C}\omega\} \\
&\quad \times \exp\{-2\pi\sqrt{-1}\,{}^t(\bar{\omega}'\boldsymbol{d}_2)\,{}^t QC\bar{C}(w + \omega'\boldsymbol{d}_1)\} .
\end{aligned} \tag{5.9}
$$

But we get

$$
f(w + \omega' \boldsymbol{d}_1, \boldsymbol{d}_2) = f(w + \omega' \boldsymbol{d}_1, \boldsymbol{d}_2) \exp\{2\pi\sqrt{-1}\,{}^t(-C\bar{\omega}'\boldsymbol{d}_1) Q(\bar{\omega}' - \bar{C}\omega')\boldsymbol{d}_2\} .
$$

Finally noting ${}^t(C\bar{\omega}'\boldsymbol{d}_1)Q = {}^t(\bar{\omega}'\boldsymbol{d}_1)\,{}^tQC$, we find in terms of (5.9) and the above relation,

$$
\begin{aligned}
f'(w, \boldsymbol{d}_1 + \boldsymbol{d}_2) = f(w + \omega' \boldsymbol{d}_1, \boldsymbol{d}_2) g(\omega' \boldsymbol{d}_2)^{-1} \\
\times \exp\{-2\pi\sqrt{-1}\,{}^t(\bar{\omega}'\boldsymbol{d}_2)\,{}^tQC\bar{C}(w + \omega'\boldsymbol{d}_1)\} \\
\times f(w, \boldsymbol{d}_1) g(\omega'\boldsymbol{d}_1)^{-1} \exp\{-2\pi\sqrt{-1}\,{}^t(\bar{\omega}'\boldsymbol{d}_1)\,{}^tQC\bar{C}w\} .
\end{aligned}
$$

This equation shows (5.6). Q.E.D.

From the above lemmas, Proposition 5.2 is proved. Further let T' and B' the abelian variety and the line bundle over T', respectively, defined from ω' and f' in Proposition 5.2. Then we have that following.

PROPOSITION 5.5. *Let all notations be as above. If C is sufficiently small, then B' is negative. Moreover the map $T^\varphi: \boldsymbol{C}^n \times \boldsymbol{C} \to \boldsymbol{C}^n \times \boldsymbol{C}$ defined by* (4.5) (*or* (4.5)$'$) *induces canonically the bundle map $\tilde{T}^\varphi: B' \to B$.*

Proof. This is trivial from constructions of T^φ and B'. Q.E.D.

Thus all small CR-structures in $\mathscr{H}^1$ are induced ones from negative line bundles over abelian varieties.

Now from the universality theorem in § 9 [1], we know the following facts; Let $\iota_{B_1}: B_1 \to B$ be the inclusion map as before and let N be any neighborhood of B_1 in B. When N_ω is a deformation of the complex structure on N which is the open submanifold of B, where $\omega \in \Gamma(N, T''(N)^* \otimes T'(N))$ and an embedding $i: B_1 \to N$ is given, we denote by $\omega \circ i$ the induced CR-structure on V by $i: B_1 \to N$. Finally let $\mathscr{H}^1_K$ be the harmonic space in $\Gamma({}^\circ T''(B_1)^* \otimes T'(B_1))$ with respect to the operator $\bar{\partial}_b^{(2)*}\bar{\partial}_b^{(1)} + \bar{\partial}_b^{(0)}\bar{\partial}_b^{(1)*}$ ($\dim_R \mathscr{H}^1_K < \infty$). Then there exists a differential map

$\psi_K: \mathscr{H}^1_K \to \Gamma({}^\circ T''(B_1)^* \otimes T'(B_1))$ satisfying the following conditions;

(α) $\psi'_K(0)t = t$ for any $t \in \mathscr{H}^1_K$.

(β) If N_ω is a deformation of N such that n_1-Sobolev-norm $\|\omega\|_{n_1}$ of ω is sufficiently small (n_1 is a sufficiently large integer), then there are a point $t_\omega \in \mathscr{H}^1_K$ and an embedding $i_\omega: B_1 \to N_\omega$ such that

$$\omega \circ i_\omega = \psi_K(i_\omega)\ .$$

Moreover i_ω and t_ω are infinitely differentiable on ω, and when ω is zero, it follows that $t_\omega = 0$ and $\omega \circ {}^t\omega = \iota_V$.

Here we can prove the following

PROPOSITION 5.6. *For any sufficiently small point $t \in \mathscr{H}^1_K$, $\psi_K(t)$ is an induced CR-structure on V form a negative line bundle over an abelian variety.*

Proof. At first, for any small $\varphi \in \mathscr{H}^1$ with the expression (4.1) we have shown that the complex structure T''_φ on B defined by (4.3) becomes one of a negative line bundle B' over an abelian variety T' (c.f. Proposition 5.5) and that the CR-structure is induced from B'. Here let $\hat{T}$ be the smooth map of $\mathscr{H}^1$ into $\Gamma(T''(B)^* \otimes T'(B))$ defined by

$$\hat{T}(\varphi)\Big(\frac{\partial}{\partial \bar{z}^j}\Big) = \sum_{k=1}^n C^k_j \frac{\partial}{\partial z^k} + \Big(\sum_{k=1}^n \frac{\partial \log h}{\partial z^k} - a_j\Big)\zeta \frac{\zeta}{\partial \zeta}\,, \qquad (j = 1, \cdots, n)$$
$$\hat{T}(\varphi)\Big(\frac{\partial}{\partial \bar{\zeta}}\Big) = 0$$

for any $\varphi \in \mathscr{H}^1_\varepsilon$ with (4.1), (see (4.3)). Moreover let $\mathscr{H}^1$ be a sufficiently small neighborhood of 0 in $\mathscr{H}^1$ such that Proposition 5.5 and the above statement (β) hold good. Then we have from (β) a smooth map

$$\tau: \mathscr{H}^1_\varepsilon \longrightarrow \mathscr{H}^1_K \qquad \text{such that } \hat{T}(\varphi) \circ i_{T(\varphi)} = \psi_K(\tau(\varphi)) \text{ and } \tau(0) = 0\ .$$

In order to prove this proposition it is enough to show that the derivation $\tau^1_{(0)}$ of τ at 0 is injective, because of $\dim \mathscr{H}^1 = \dim \mathscr{H}^1_K$. Here let s be a small real number. Then it follows from Theorem 7.1 and (5.16) in [1] that

$$\lim_{s\to 0} \frac{\hat{T}(s\varphi) \circ i_{T(s\varphi)} - \hat{T}(s\varphi) \circ \iota B_1}{s} = \bar{\partial}^{(0)}_b \xi \tag{5.10}$$

where ξ is an element of $\Gamma(T'(B_1))$.

On the other hand we get from the definition of $\hat{T}$

$$\lim_{s\to 0} \frac{\hat{T}(s\varphi) \circ \iota B_1 - \hat{T}(0) \circ \iota B_1}{s} = \varphi\ . \tag{5.11}$$

Therefore we have from (5.10) and (5.11)

$$\frac{d\hat{T}(s\varphi) \circ i_{T(s\varphi)}}{ds}\bigg|_{s=0} = \varphi + \bar{\partial}^{(0)}_b \xi\ .$$

Furthermore it follows, using (β), that

$$\left.\frac{d\psi_K(\tau(s\varphi))}{ds}\right|_{s=0} = \tau'(0)\cdot\varphi\,,$$

so that

$$\tau'(0)\cdot\varphi = \bar{\partial}_b^{(0)}\xi + \varphi\,. \tag{5.12}$$

Now suppose that $\tau'(0)\cdot\varphi = 0$. Then it is clear from (5.12) that the $\bar{\partial}_b^{(1)}$-cohomology class of φ is zero, but as φ belongs to $\mathscr{H}^1$ we have $\varphi = 0$. Thus $\tau'(0)$ is injective. Q.E.D.

Our main Theorem 5.1 is obtained from the above Proposition 5.6.

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