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A CONSTRUCTIVE PROOF OF BRAUER'S THEOREM ON INDUCED CHARACTERS IN THE GROUP RING $\mathcal{R}[G]$

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Dedicated to Professor V. Shokurov on his sixtieth birthday

Abstract We provide an alternative constructive proof of the classical Brauer theorem for finite groups based on the well-known description of the complex irreducible representations of the symmetric groups S_n. The theorem is first proved for S_n and then for general G by embedding in S_n and applying the Mackey subgroup theorem.

Keywords: Brauer theorem; symmetric groups; long cycles; induced characters; Young diagrams

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1. Introduction

The group ring of a finite group is the set of integer sums of irreducible trace characters $\mathcal{R}[G] = \mathbb{Z}\text{-span}\{\chi_\pi : \pi \in G^\wedge\}$, which becomes a ring under the usual operations $\chi_\pi + \chi_\mu = \chi_{\pi\oplus\mu}$ and $\chi_\pi \cdot \chi_\mu = \chi_{\pi\otimes\mu}$; the identity element is the trivial character $\mathbf{1}_G$. The irreducible trace characters $\{\chi_\pi : \pi \in G^\wedge\}$ are, by definition, a $\mathbb{Z}$-basis for the representation ring.

Though the theory of representations of finite groups is a well-developed subject, the depth of our understanding of representations depends dramatically on the class of groups being considered. In this respect, the best-understood class is the family of permutation groups S_n; the theory for other classes of quasi-simple groups lags far behind.

The classical Brauer theorem [**2**] states that all elements in $\mathcal{R}[G]$, and, in particular, all irreducible characters χ_π, are integer linear combinations of trace characters induced from one-dimensional representations on *elementary subgroups*. These are direct products $E = A \times B$, where A is cyclic and B is a p-group for some prime such that $|B| = p^r$ is relatively prime to the order $|A|$. We write $\mathcal{E}(G)$ for the set of all elementary subgroups in G.

Theorem 1.1 (Brauer). *If G is a finite group, every element f of the group ring $\mathcal{R}[G]$ is a $\mathbb{Z}$-linear combination of induced characters $I_E^G(\phi)(g) = \mathrm{Tr}(\mathrm{Ind}_E^G(\phi)_g)$,*

$$f = \sum_i m_i I_{E_i}^G(\phi_i),$$

where $E_i \in \mathcal{E}(G)$ and ϕ_i is a one-dimensional representation on E_i.

We say that a particular group G is of *Brauer type* or *type B* if this result holds for G.

Many proofs of this result have been given (see [**1**, **3**]), perhaps the shortest being that of [**9**]. In this note we observe that the theory for symmetric groups S_n is so strong that we can give a constructive and straightforward proof of Brauer's theorem for such groups, and since every G is a subgroup of some S_n an application of the Mackey subgroup theorem, which describes the irreducible decomposition of restrictions of an induced representation, allows us to conclude that all finite groups are of Brauer type.

We approach Brauer's theorem for S_n by mostly constructive methods, involving induction from well-understood classes of groups (much simpler than the arbitrary p-groups appearing in the class $\mathcal{E}$ of elementary groups). We show first that S_n has a weaker Brauer-type property, property (B*), in which the class $\mathcal{E}(S_n)$ is replaced with the larger class $\mathcal{N}(S_n)$ of nilpotent subgroups, and induction is from irreducible rather than one-dimensional representations. This helps because the combinatorial properties of nilpotent groups are much better than those of the class $\mathcal{E}$; for instance, $\mathcal{E}$ is not closed under direct products. Next, we give a self-contained proof that all nilpotent groups are of Brauer type, from which it follows immediately by induction in stages that symmetric groups are actually of (strong) Brauer type.

Ultimately, the nilpotent case reduces to proving that particular small abelian groups of the form $\mathbb{Z}_{pq}^2 = \mathbb{Z}_p^2 \times \mathbb{Z}_q^2$ are of Brauer type, which we do using finite Fourier transforms.

An interesting aspect of the proof is its use of certain properties of the Sylow subgroups in S_n, when n is a prime power p^m, and their relation to 'long cycles' such as $\sigma_0 = (1, 2, \dots, n)$. Specifically, we show the following.

Theorem 1.2. *When $n = p^m$ for some prime, every n-cycle in S_n lies in a unique p-Sylow subgroup (although several n-cycles can lie in the same Sylow subgroup). For any Sylow subgroup Syl_p in S_n and any n-cycle $\sigma \in \mathrm{Syl}_p$, we have that*

$$x\sigma x^{-1} \in \mathrm{Syl}_p \implies x \in N_{S_n}(\mathrm{Syl}_p) \quad \textit{for all } x \in S_n,$$

and the intersection with Syl_p of the orbit under conjugation $C_n = S_n \cdot \sigma$ is the orbit of σ under the normalizer $N_{S_n}(\mathrm{Syl}_p)$.

This relation between 'long cycles' and Sylow subgroups in symmetric groups seems not to have been noted previously, and may prove useful in other investigations. When n is not a prime power, long cycles reappear in a different way. Given a relatively prime factorization $n = n_1 n_2$, a long cycle σ is contained in a unique subgroup H that is a copy of $S_{n_1} \times S_{n_2}$ embedded in S_n in a non-standard way via the Chinese remainder theorem. The representation $\mathrm{Ind}_H^{S_n}(\mathbf{1})$ induced from the trivial representation on H will play a crucial role in our analysis.

Our initial efforts in this paper are focused on proving that symmetric groups have a *weak Brauer property* (B*). First, we present some background regarding representations of S_n and their well-known characterization in terms of induced representations. For arbitrary G there exists a general formula for the trace character induced from a finite-dimensional representation ρ on a subgroup $H \subseteq G$. If C_g is a G-conjugacy class and

$\chi_\rho(h) = \mathrm{Tr}(\rho_h)$, the trace character of the induced representation $\mathrm{Ind}_H^G(\rho)$ is

$$I_H^G(\rho)(C_g) = \begin{cases} \left|\dfrac{G}{H}\right| \cdot \dfrac{1}{|C_g|} \displaystyle\sum_{h \in C_g \cap H} \chi_\rho(h) & \text{if } C_g \text{ meets } H, \\ 0 & \text{if } C_g \text{ is disjoint from } H. \end{cases} \tag{1.1}$$

When $G = S_n$ we write Λ_n for the set of partitions of an integer $n \geqslant 2$:

$$\lambda = (\lambda_1, \dots, \lambda_n), \quad \text{with } \lambda_1 \geqslant \cdots \geqslant \lambda_n \geqslant 0 \text{ and } \sum_i \lambda_i = n.$$

With each partition we associate a subgroup $S_\lambda = S_{\lambda_1} \times \cdots \times S_{\lambda_n}$ in S_n (with the convention that $S_\lambda = S_{\lambda_1} \times \cdots \times S_{\lambda_r}$ if $\lambda_r > 0$ and $\lambda_{r+1} = 0$). The representation $U_\lambda = \mathrm{Ind}_{S_\lambda}^{S_n}(\mathbf{1}_{S_\lambda})$ induced from the trivial character $\mathbf{1}_{S_\lambda}$ on S_λ has a trace character whose values are

$$\psi^{(\lambda)}(C_g) = I_{S_\lambda}^{S_n}(\mathbf{1}_{S_\lambda})(C_g) = \begin{cases} \left|\dfrac{S_n}{S_\lambda}\right| \cdot \left|\dfrac{C_g \cap S_\lambda}{C_g}\right| & \text{if } C_g \text{ meets } S_\lambda, \\ 0 & \text{if } C_g \text{ is disjoint from } S_\lambda. \end{cases} \tag{1.2}$$

Obviously, $U_{(n,0,\dots,0)} = \mathbf{1}_{S_n}$ (the trivial representation on S_n) and $U_{(1,\dots,1)}$ is the left regular representation $L = \mathrm{Ind}_E^{S_n}(\mathbf{1}_E)$, where $E = \{e\}$. The index $\lambda_* = (n, 0, \dots, 0)$ is exceptional in that all other S_λ are proper subgroups, while $S_{\lambda^*} = S_n$.

We impose a lexicographic order on partitions of n, letting

$$\lambda < \mu \text{ if } \lambda_i < \mu_i \text{ for the first index } i = 1, 2, \dots \text{ such that } \lambda_i \neq \mu_i,$$

so $\lambda_* = (n, 0, \dots, 0) < (n-1, 1, 0, \dots, 0) < \cdots < (1, 1, \dots, 1)$. The following well-known result (see [**4**, pp. 52–57]) regarding irreducible representations of S_n is the basis of our discussion of these groups.

Theorem 1.3 (Young's rule). *There exists a bijective correspondence between partitions Λ of n and irreducible representations $\pi_\lambda \in S_n^\wedge$ such that*

$$U_\lambda \cong \pi_\lambda \oplus \left(\bigoplus_{\mu < \lambda} m_\mu \pi_\mu \right) \quad (\textit{with } m_\mu \in \mathbb{Z}_+), \tag{1.3}$$

thereby associating each λ with a unique irreducible representation π_λ.

The result we actually need in our discussion of symmetric groups S_n follows immediately by a simple recursive argument.

Corollary 1.4. *The trace characters $\psi^{(\lambda)} = I_{S_\lambda}^{S_n}(\mathbf{1}_{S_\lambda})$ of the induced representations $\{U_\lambda \colon \lambda \in \Lambda\}$ form a $\mathbb{Z}$-basis for the representation ring $\mathcal{R}[S_n]$.*

We also note that (irreducible) trace characters on S_n can only have integer values. Obviously, they lie in $\mathbb{Q}$, by (2) and Corollary 1.4, but, as is well known, the values of

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trace characters on any finite group are algebraic integers; hence, they lie in $\mathbb{Z}$ when $G = S_n$.

All U_λ are induced from proper subgroups except for $U_{\lambda_*} = \mathbf{1}_{S_n}$. This poses a problem if we wish to reduce the study of the Brauer property for S_n to certain proper subgroups. We circumvent this by constructing a new trace character $\psi^{(0)}$ such that

$$\{\psi^{(0)}\} \cup \{\psi^{(\lambda)} \colon \lambda \in \Lambda,\ \lambda \neq \lambda_*\} \text{ is a } \mathbb{Z}\text{-basis for } \mathcal{R}[S_n]. \tag{1.4}$$

The $\psi^{(0)}$ we construct can take two forms, depending on whether or not n is a prime power. When $n = p^m$ our new character will be a sum of induced characters $I_{N_i}^{S_n}(\pi_i)$, where

- N_i is a p-Sylow in S_{p^m} and $\pi_i = \mathbf{1}$, or
- N_i is the cyclic subgroup generated by a long cycle σ in S_{p^m} and π_i is the canonical unitary character $\chi(\sigma^k) = \mathrm{e}^{2\pi\mathrm{i}k/p}$.

When n is not a prime power, we can write $n = n_1 n_2$ with relatively prime factors. In this case, we start with a long cycle σ and construct a trace character

$$\psi^{(0)} = I_H^{S_n}(\mathbf{1}_S),$$

where $H \subseteq S_n$ is a subgroup that contains σ and is *isomorphic* to a direct product of symmetric groups $S_{n_1} \times S_{n_2}$. However, the embedded subgroup is not a product of the subgroups $S_{A_i} \subseteq S_n$ acting on disjoint subsets $A_i \subseteq [1, n]$ of cardinality $|A_i| = n_i$, unlike the subgroups $S_\lambda = S_{\lambda_1} \times \cdots \times S_{\lambda_n}$ corresponding to partitions of $[1, n]$.

Finally, we recall that all finite p-groups are nilpotent; hence, elementary groups are nilpotent. Furthermore, any finite nilpotent group N is a direct product of its Sylow subgroups, which are unique (see [**5**, pp. 154–156]).

> *If a nilpotent group has order $n = \prod_{i=1}^{r} p_i^{m_i}$, each of its Sylow subgroups S_{p_i} is normal and N is their direct product $N = \prod_{i=1}^{r} S_{p_i}$.*

We need the following general facts regarding tensor and Kronecker products of representations.

(1) $\mathrm{Tr}(\mu \otimes \nu)_g = \mathrm{Tr}(\mu_g) \cdot \mathrm{Tr}(\nu_g)$ for representations of a group G.

(2) If μ is a representation of a group G and π a representation of a subgroup H, then $\mu \otimes \mathrm{Ind}_H^G(\pi) \cong \mathrm{Ind}_H^G((\mu \mid H) \otimes \pi)$ (see [**8**, § 4.3]).

(3) Kronecker products of representations on a direct product $G_1 \times G_2$ have the following properties:

   (i) $\mathrm{Tr}(\mu \times \nu)_{(a,b)} = \mathrm{Tr}\,\mu_a \cdot \mathrm{Tr}\,\nu_b$ for $(a, b) \in G_1 \times G_2$,

   (ii) $I_{H_1 \times H_2}^{G_1 \times G_2}(\mu \times \nu)_{(a,b)} = I_{H_1}^{G_1}(\mu)_a \cdot I_{H_2}^{G_2}(\nu)_b$ as functions on $G_1 \times G_2$,

   (iii) every irreducible finite-dimensional complex representation $\rho \in (A \times B)^\wedge$ is $\cong \mu \times \nu$ for some $\mu \in A^\wedge$, $\nu \in B^\wedge$.

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At the start of our discussion we consider the class $\mathcal{N}(G)$ of nilpotent subgroups in a finite group G, in place of the elementary subgroups $\mathcal{E}(G)$ that figure in Brauer's theorem. The class $\mathcal{N}(G)$ includes all abelian and elementary subgroups, and any $H \in \mathcal{N}(G)$ is the direct product of its p-Sylow subgroups. More importantly for our purposes, the class $\mathcal{N}$ is closed under formation of direct products and subgroups, while the class $\mathcal{E}$ is not.

We write $I_H^G(\rho)$ for the trace character of an induced representation $\mathrm{Ind}_H^G(\rho)$. For any class $\mathcal{C}(G)$ of subgroups we define the following subsets of the representation ring $\mathcal{R}[G]$:

$$I_*^G(\mathcal{C}(G)) = \mathbb{Z}\text{-span}\{I_H^G(\pi)\colon H \in \mathcal{C}(G),\ \pi \in H^\wedge\},$$
$$J_*^G(\mathcal{C}(G)) = \mathbb{Z}\text{-span}\{I_H^G(\phi)\colon H \in \mathcal{C}(G),\ \dim\phi = 1\}.$$

The following observation is fundamental to our discussion.

Lemma 1.5. *For any class of subgroups $\mathcal{C}(G)$ the functions $I_*^G(\mathcal{C}(G))$ form an ideal in the group ring $\mathcal{R}[G]$.*

Proof. As noted above, if $H \in \mathcal{C}(G)$, $\pi \in H^\wedge$, and μ is any finite-dimensional representation of G, then $(\mu \mid H) \otimes \pi$ decomposes into the irreducibles $\bigoplus_i \pi_i$ and

$$\mu \otimes \mathrm{Ind}_H^G(\pi) \cong \bigoplus_i \mathrm{Ind}_H^G(\pi_i).$$

Taking trace characters, we get $\chi_\mu \cdot I_*^G(\mathcal{C}(G)) \subseteq I_*^G(\mathcal{C}(G))$. $\square$

Since $I_*^G(\mathcal{C}(G))$ is an ideal, it equals $\mathcal{R}[G]$ if and only if it contains the trivial representation $\mathbf{1}_G$. The set of functions $J_*^G(\mathcal{C}(G))$ need not be an ideal, but, by the Mackey subgroup theorem [**6**], it is a subring if the class $\mathcal{C}(G)$ is closed under intersections and invariant under conjugation by elements of G.

2. Symmetric groups $\mathbf{S_n}$

Brauer's theorem asserts that

$$\mathcal{R}[G] = J_*^G(\mathcal{E}(G)).$$

Our first step towards a proof is to show that the symmetric groups have the weaker property (B*):

$$\text{for any permutation group } S_n \text{ we have that } \mathcal{R}[G] = I_*^G(\mathcal{N}(S_n)). \qquad (\mathrm{B}^*)$$

Thus, we have the following.

Theorem 2.1. *Every symmetric group S_n has property (B*): $\mathcal{R}[S_n] = I_*^{S_n}(\mathcal{N}(S_n))$, where $\mathcal{N}$ is the class of nilpotent subgroups.*

In § 3 we show that the nilpotent groups N have the strong Brauer property $\mathcal{R}[N] = J_*^N(\mathcal{E}(N))$, and then, by induction in stages,

$$\mathrm{Ind}_H^{S_n}(\phi) \cong \mathrm{Ind}_N^{S_n}(\mathrm{Ind}_H^N(\phi)) \quad (N \in \mathcal{N}(S_n),\ H \subseteq N),$$

we conclude that all symmetric groups are of strong Brauer type.

The following lemma underlies our discussion of the symmetric groups.

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Lemma 2.2. *Let C_n be the conjugacy class in S_n consisting of all n-cycles. For $f \in \mathcal{R}[S_n]$ define the unital ring homomorphism $P(f) = f(C_n)$. The kernel of P is precisely $\mathcal{M} = \mathbb{Z}\text{-span}\{\psi^{(\lambda)} \colon \lambda \neq \lambda_*$ in $\Lambda\}$.*

Proof. Obviously, $P(\mathcal{R}) = \mathbb{Z}$. Furthermore,

$$\ker P \supseteq \mathcal{M} = \mathbb{Z}\text{-span}\{\psi^{(\lambda)} \colon \lambda \neq \lambda_*\}.$$

In fact, by (1.2), $\psi^{(\lambda)}(C_n) = 0$, since $C_n \cap S_\lambda$ is empty for every $\lambda \neq \lambda_*$. In the reverse direction, if

$$f = \sum_{\pi \in S_n^\wedge} m_\pi \chi_\pi \quad (m_\pi \in \mathbb{Z}),$$

each χ_π is an integer combination of the $\psi^{(\lambda)}$, $\lambda \in \Lambda$, and so is f. Thus, if $P(f) = 0$, the coefficient of $\psi^{(\lambda_*)}$ must be 0 and $f \in \mathcal{M}$. □

We produce a new $\mathbb{Z}$-basis for $\mathcal{R}[S_n]$, as in (1.4), by adjoining one extra trace character to $\{\psi^{(\lambda)} \colon \lambda \neq \lambda_*\}$, which is already a $\mathbb{Z}$-basis for $\mathcal{M}$. By Lemma 2.2 we get a $\mathbb{Z}$-basis for $\mathcal{R}[S_n]$ if $\psi^{(0)} = \pm 1$ on C_n. We construct a vector $\psi^{(0)}$ having the following particular form:

$\psi^{(0)}$ is a sum $\sum_i m_i I_{H_i}^{S_n}(\pi_i)$ of trace characters induced from irreducible representations $\pi_i \in H_i^\wedge$, and $\psi^{(0)} = \pm 1$ on the maximal class C_n. (2.1)

We accomplish this using only pairs (H_i, π_i) of the following types.

- For general n we use $I_H^{S_n}(\mathbf{1})$, where H is a copy of $S_{n_1} \times S_{n_2}$ ($n = n_1 n_2$ relatively prime) constructed from a long cycle in S_n.
- For $n = p^m$ we use $I_H^{S_n}(\mathbf{1})$, where H is a p-Sylow subgroup of S_{p^m}, a class of p-groups whose structure is well understood.
- For $n = p^m$ we use $I_H^{S_n}(\chi)$, where $H = \langle \sigma \rangle$ is the cyclic group generated by a long cycle and χ is its canonical character $\chi(\sigma^k) = \mathrm{e}^{2\pi \mathrm{i} k/p}$.

For general n, the remaining characters $\psi^{(\lambda)}$ needed to generate $\mathcal{R}[S_n]$ are induced from the trivial character $\mathbf{1}$ on the products $S_\lambda = S_{\lambda_1} \times \cdots \times S_{\lambda_n}$ for $\lambda \in \Lambda^*$.

Constructing the extra character. In what follows we regard S_n as permutations of $X = \{1, 2, \ldots, n\}$ on which we impose the standard cyclic order; with this in mind it is convenient to identify X with $\mathbb{Z}_n = \{[1], [2], \ldots, [n]\}$. Let σ_0 be the particular 'long cycle' $(1, 2, \ldots, n)$ in C_n and let $H_0 = \langle \sigma_0 \rangle$ be the cyclic subgroup it generates in S_n.

Definition 2.3. A permutation $\tau \in S_n$ is a *cyclic k-shift* if $\tau(s) \equiv s + k \pmod n$ for all $s \in X$. Obviously, σ_0^k is the unique k-shift on X, and the various shifts comprise the cyclic group $H_0 \cong (\mathbb{Z}_n, +)$.

Not all powers σ_0^k are n-cycles; in fact, it is not hard to see that

$$\sigma_0^k \in C_n \iff o(\sigma_0^k) = n \iff k \in U_n = (\text{multiplicative units in } \mathbb{Z}_n).$$

Before proving the claims in (1.4) we establish a few facts we will need later.

Lemma 2.4. *If σ is an n-cycle in S_n and $H = \langle \sigma \rangle$, this subgroup is its own centralizer: $Z_{S_n}(H) = H$.*

Proof. We may assume that $\sigma = \sigma_0 = (1, 2, \ldots, n)$. Then $\tau \in S_n$ centralizes $H \iff \tau\sigma_0\tau^{-1} = (\tau(1), \tau(2), \ldots, \tau(n))$ is equal to σ_0, which means that the cyclically ordered list $(\tau(1), \ldots, \tau(n))$ is just $(1, 2, \ldots, n)$ subjected to a cyclic k-shift, which means that $\tau(s) \equiv s + k \pmod n$ for all $s \in [1, n]$. Thus, $\tau = \sigma_0^k$ and $Z_{S_n}(H) = H$. $\square$

One can also identify the normalizer of H_0 as an explicit subgroup of S_n, showing that it is the natural semi-direct product of $(\mathbb{Z}_n, +)$ acted upon by the multiplicative group of units $(U_n, \cdot)$, but we do not need this in the present work.

The second fact we need has already been posted as Theorem 1.2, which we now prove.

Proof of Theorem 1.2. When $n = p^m$, we prove the uniqueness of the p-Sylow containing a particular n-cycle by induction on the exponent m in p^m, the result being trivial if $m = 1$. Assuming that it holds for exponents $\leqslant m - 1$, we consider $n = p^m$; we may restrict our attention to the particular long cycle $\sigma_0 = (1, 2, \ldots, n)$. For brevity we write $r = p^{m-1}$ and $n = p^m$ below.

The cardinalities of Sylow subgroups in S_n are well known (see [**5**, pp. 81–83]). When $n = p^m$ we get that

$$|\mathrm{Syl}_p| = p^{(1+p+\cdots+p^{m-1})}$$

for any p-Sylow in S_{p^m}. This is related to the size of Sylow subgroups in $S_{p^{m-1}}$ by the identities

$$|\mathrm{Syl}_p(S_{p^m})| = |\mathrm{Syl}_p(S_{p^{m-1}})|^p \cdot p = p^r \cdot |\mathrm{Syl}_p(S_{p^{m-1}})|. \tag{2.2}$$

The first identity is immediate from the wreath product construction described in [**5**, pp. 81–83], where $\mathrm{Syl}_p(S_{p^m})$ is shown to be a semi-direct product

$$\mathrm{Syl}_p(S_{p^m}) \cong N \rtimes \mathbb{Z}_p, \quad \text{where } N \cong \mathrm{Syl}_p(S_{p^{m-1}}) \times \cdots \times \mathrm{Syl}_p(S_{p^{m-1}}).$$

The second identity follows because

$$p^{(1+p+\cdots+p^{m-1})} = p^r \cdot p^{(1+p+\cdots+p^{m-2})}.$$

Now consider the power σ_0^r ($r = p^{m-1}$), an element of order p in $H_0 = \langle \sigma_0 \rangle$. It decomposes into $r = p^{m-1}$ disjoint p-cycles

$$\sigma_0^r = \tau_1 \cdot \cdots \cdot \tau_r, \quad \text{where } \tau_k = (k, k + r, k + 2r, \ldots, k + (p - 1)r).$$

Orbits in $[1, p^m]$ under the action of $A = \langle \sigma_0^r \rangle$ are the supports

$$I_k = \mathrm{supp}(\tau_k) = \{k, k + r, \ldots, k + (p - 1)r\}, \quad 1 \leqslant k \leqslant r = p^{m-1},$$

of these cycles.

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Lemma 2.5. *If* Syl_p *is any* p*-Sylow subgroup in* S_{p^m} *that contains the long cycle* $\sigma_0 = (1, 2, \dots, p^m)$*, then* σ_0^r $(r = p^{m-1})$ *is in the centre* $Z = Z(\mathrm{Syl}_p)$ *and* $Z \subseteq H_0 = \langle \sigma_0 \rangle$*.*

Proof. Syl_p is nilpotent and has the non-trivial centre Z; this must be $\subseteq H_0 = \langle \sigma_0 \rangle$. In fact, if $\sigma_0 \in \mathrm{Syl}_p$, elements of Z commute with σ_0, but, by Lemma 2.4, the centralizer of σ_0 in S_{p^m} is H_0.

For each divisor d of $|H_0| = p^m$ there exists a unique subgroup such that $|H_d| = d$. Since σ_0^r has minimal order equal to p and $Z = H_d$ for some $d = p^k$ we get that $Z \supseteq \langle \sigma_0^r \rangle$, so σ_0^r is central in Syl_p. $\square$

It follows from Lemma 2.5 that Syl_p permutes the A-orbits in $[1, p^m]$, which are just the supports $I_k = \mathrm{supp}(\tau_k)$ of the cycles in σ_0^r. If X is the space of orbits and $\mathrm{Per}(X)$ the full group of permutations, we get the sequence of homomorphisms

$$e \to M_p \to \mathrm{Syl}_p \xrightarrow{\pi} \mathrm{Per}(X) \cong S_{p^{m-1}},$$

where M_p is the 'action kernel' $M_p = \{x \in \mathrm{Syl}_p \colon x(I_k) = I_k \text{ for all } k\}$. We identify $X \approx [1, r] = [1, p^{m-1}]$ via $I_k \to k$. The action of Syl_p is transitive: we know the cycles τ_k explicitly, from which it is clear that

$$\sigma_0 \tau_k \sigma_0^{-1} = \tau_{k+1} \quad (\text{reckoning subscripts mod } p^{m-1}),$$

but then $\sigma_0(I_k) = I_{k+1}$ and, since $\sigma_0 \in \mathrm{Syl}_p$, transitivity follows.

The action kernel M_p is the same subgroup of S_{p^m} for all p-Sylow subgroups containing σ_0. To see this, consider the kernel M_p for a particular p-Sylow and the subgroup

$$M_0 = \{x \in S_{p^m} \colon x \tau_k x^{-1} = \tau_k,\ \forall k \in [1, p]\}.$$

An element $x \in M_0$ can only act as a k-shift on the cyclic-ordered entries of τ_k (perhaps with a different shift on each cycle), and any element in S_{p^m} with this property is in M_0. Hence,

$$|M_0| = p^r \quad (r = p^{m-1}).$$

For a fixed p-Sylow let M_p be its action kernel. We first show that $M_p \subseteq M_0$. If $y \in M_p$, we have that $y(I_k) = I_k$, for all k, but σ_0^r is central in Syl_p, so

$$\prod_k \tau_k = \sigma_0^r = y \sigma_0^r y^{-1} = \prod_k y \tau_k y^{-1}.$$

Each conjugate is a p-cycle with the same support as τ_k, so by uniqueness of disjoint cycle decompositions we get that $y \tau_k y^{-1} = \tau_k$ for all k. Thus, $M_p \subseteq M_0$.

The equality $M_p = M_0$ follows if they have the same cardinality. Since $\pi(\mathrm{Syl}_p)$ is a p-group in $S_{p^{m-1}}$, we have that

$$|\pi(\mathrm{Syl}_p)| \leqslant |\mathrm{Syl}_p(S_{p^{m-1}})| = \frac{|\mathrm{Syl}_p|}{p^r} \quad (\text{by (2.2)}).$$

By transitivity and the fact that $|M_p| \leqslant |M_0|$, we also have that

$$|\pi(\mathrm{Syl}_p)| = \frac{|\mathrm{Syl}_p|}{|M_p|} \geqslant \frac{|\mathrm{Syl}_p|}{p^r};$$

hence, all these items are equal and, in particular, $|M_p| = |M_0| = p^r$. Thus, $M_p = M_0$ for *every* p-Sylow in S_{p^m} that contains σ_0.

The preceding calculation also shows that $|\pi(\mathrm{Syl}_p)| = |\mathrm{Syl}_p(S_{p^{m-1}})|$, so the image group $\tilde{S} = \pi(\mathrm{Syl}_p)$ is a p-Sylow in $S_{p^{m-1}}$. We obtain the exact sequence

$$e \to M_0 \to \mathrm{Syl}_p \xrightarrow{\pi} \tilde{S} \cong \mathrm{Syl}_p(S_{p^{m-1}}) \to e.$$

The image $\bar{\sigma}_0 = \pi(\sigma_0)$ is easily seen to be the long cycle $(1, 2, \dots, p^{m-1})$ under our identification $X \approx [1, p^{m-1}]$. By the induction hypothesis, $\tilde{S}$ is the unique p-Sylow containing $\bar{\sigma}_0$, regardless of which p-Sylow containing σ_0 we started with. It follows that a unique $\mathrm{Syl}_p \subseteq S_{p^m}$ is determined, and the first part of Theorem 1.2 is proved.

The second part is now easy. Let $\sigma \in C_n$ be any n-cycle and let Syl_p be the unique p-Sylow containing it. For $x \in S_n$,

$$x\sigma x^{-1} \in \mathrm{Syl}_p \implies x \text{ is in the normalizer } N_{S_n}(\mathrm{Syl}_p),$$

because $x\sigma x^{-1}$ lies in two Sylow subgroups Syl_p and $\mathrm{Syl}'_p = x\mathrm{Syl}_p x^{-1}$; hence, $\mathrm{Syl}'_p = \mathrm{Syl}_p$ and x normalizes Syl_p. It follows immediately that

$$C_n \cap \mathrm{Syl}_p = S_n \cdot \sigma \cap \mathrm{Syl}_p = N_{S_n}(\mathrm{Syl}_p) \cdot \sigma,$$

and that completes the proof. □

We now address the claims made in (1.4) and (2.1).

Proposition 2.6. *For $n \geqslant 3$, let σ_0 be the n-cycle $(1, 2, \dots, n)$ in the maximal conjugacy class C_n, and assume that $n = p^m$. If $S = \mathrm{Syl}_p(S_n)$ is the Sylow p-subgroup containing σ_0, and $\mathbf{1}$ is the trivial character on it, the value at C_n of the induced trace character $I^{S_n}_{\mathrm{Syl}_p(S_n)}(\mathbf{1})$ is a non-zero integer relatively prime to p.*

If H_0 is the cyclic group $\langle\sigma_0\rangle \cong (\mathbb{Z}_n, +)$ and χ is the canonical character

$$\chi(\sigma_0^j) = \mathrm{e}^{2\pi \mathrm{i} j/p},$$

the value of $I^{S_n}_{H_0}(\chi)$ at C_n is a power of p. Therefore, some $\mathbb{Z}$-linear combination $\psi^{(0)} \in \mathcal{R}[S_n]$ of $I^{S_n}_{\mathrm{Syl}_p}(\mathbf{1})$ and $I^{S_n}_{H_0}(\chi)$ has $\psi^{(0)}(C_n) = 1$.

Note. We can actually compute the exact value $I^{S_n}_{H_0}(\chi) = (-1)p^{m-1}$, but we will not need it here.

Proof. First consider the representation induced from the trivial representation $\mathbf{1}$ on the unique p-Sylow subgroup $\mathrm{Syl}_p(S_n)$ that contains the long cycle σ_0. By the usual induced character formula,

$$I^{S_n}_{\mathrm{Syl}_p(S_n)}(\mathbf{1})(C_n) = \left|\frac{S_n}{\mathrm{Syl}_p(S_n)}\right| \cdot \frac{1}{|C_n|} \sum_{C_n \cap \mathrm{Syl}_p(S_n)} 1 = n \cdot \frac{|C_n \cap \mathrm{Syl}_p(S_n)|}{|\mathrm{Syl}_p(S_n)|}$$

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since $|C_n| = (n-1)!$. It follows from Theorem 1.2 that

$$C_n \cap \mathrm{Syl}_p(S_n) = S_n \cdot \sigma_0 \cap \mathrm{Syl}_p(S_n) \text{ is the conjugation orbit } N_{S_n}(\mathrm{Syl}_p(S_n)) \cdot \sigma_0$$

and, hence, that $|\mathrm{Syl}_p|$ divides $|N_{S_n}(\mathrm{Syl}_p(S_n))|$, which divides $|S_n| = n!$. The stabilizer of σ_0 in S_n is H_0, and $H_0 \subseteq \mathrm{Syl}_p$ since $\sigma_0 \in \mathrm{Syl}_p$, so the stabilizer of σ_0 in $N = N_{S_n}(\mathrm{Syl}_p)$ is also H_0. Therefore, the orbit $\mathcal{O} = N \cdot \sigma_0$ has

$$|\mathcal{O}| = |N|/|H_0| = |N|/p^m,$$

and since $C_n \cap \mathrm{Syl}_p = N \cdot \sigma_0 = \mathcal{O}$ we get that

$$\frac{n}{|\mathrm{Syl}_p|} \cdot |C_n \cap \mathrm{Syl}_p| = \frac{p^m}{|\mathrm{Syl}_p|} \cdot \frac{|N|}{p^m} = \frac{|N|}{|\mathrm{Syl}_p|}.$$

This quotient is obviously relatively prime to p.

In the other case, we consider representations induced from $H_0 = \langle \sigma_0 \rangle \cong (\mathbb{Z}_n, +)$. The canonical surjective homomorphism of rings

$$\phi_p \colon [x]_n = x + n\mathbb{Z} \to [x]_p = x + p\mathbb{Z} \in \mathbb{Z}_p \cong \mathbb{Z}_{p^m}/p\mathbb{Z}_{p^m} \quad (x \in \mathbb{Z},\ n = p^m)$$

yields a natural unitary character on $(\mathbb{Z}_n, +)$ having values in the group of pth roots of unity $\Omega_p \subseteq \mathbb{C}$ if we take

$$\chi_p([x]_n) = \mathrm{e}^{2\pi \mathrm{i} x/p} \quad (x \in \mathbb{Z},\ n = p^m).$$

The value of the induced character on any class $C_g \subseteq S_n$ is

$$I_{H_0}^{S_n}(\chi_p)(C_g) = \begin{cases} \left|\dfrac{S_n}{H_0}\right| \cdot \dfrac{1}{|C_g|} \displaystyle\sum_{h \in H_0 \cap C_g} \chi_p(h) & \text{if } H_0 \cap C_g \text{ is non-trivial,} \\ 0 & \text{if } H_0 \cap C_g = \emptyset. \end{cases}$$

When $C_g = C_n$ the multiplier in front of the sum is equal to 1, since $|C_n| = |S_n/H_0| = (n-1)!$. The fact that the remaining sum is a power of p follows from a general observation about p-groups.

Lemma 2.7. *Let $|G| = p^m$. If $H = \langle \sigma \rangle$ for some element σ, and $G \cdot \sigma$ is its conjugacy class, then $|G \cdot \sigma \cap H|$ is a power of p.*

Proof. For $x \in G$, we have that

$$x\sigma x^{-1} \in H \quad \iff \quad x \text{ is in the normalizer } N_G(\sigma),$$

so $G \cdot \sigma \cap H$ is the orbit $N_G(\sigma) \cdot \sigma$ in G. Hence,

$$|G \cdot \sigma \cap H| = |N_G(H)|/|Z_G(H)| \text{ is a power of } p.$$

$\square$

This proves Proposition 2.6. $\square$

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We now take up construction of $\psi^{(0)}$ in the case where n is a product $n = MN$ with relatively prime factors $M, N < n$.

Proposition 2.8. *If $n = MN$ with relatively prime factors, there exists a subgroup $S \subseteq S_n$ algebraically isomorphic to $S_M \times S_N$ such that S contains an n-cycle σ_0 and the induced character $I_S^{S_n}(\mathbf{1}_S)$ is equal to 1 on the maximal class C_n.*

Proof. Obviously, S will contain $H_0 = \langle \sigma_0 \rangle$. We construct S so it has the property

$$x\sigma_0 x^{-1} \in S \implies x \in S \quad \text{for all } x \in S_n,$$

from which we get that $I_S^{S_n}(\mathbf{1}_S)(C_n) = 1$ by the following lemma.

Lemma 2.9. *Let $H_0 = \langle \sigma_0 \rangle$ be the subgroup generated by an n-cycle σ_0 in C_n, and suppose that $M \subseteq S_n$ is a subgroup containing H_0 that has the property*

$$x\sigma_0 x^{-1} \in M \implies x \in M \quad \text{for all } x \in S_n. \tag{2.3}$$

The trace character $\psi^{(0)} = I_M^{S_n}(\mathbf{1}_M)$ induced from the trivial character on M is then equal to 1 on C_n.

Proof. Conjugation by a suitable $y \in S_n$ yields $y\sigma_0 y^{-1} = \sigma_0' = (1, 2, \ldots, n)$. Since (2.3) holds for $M' = yMy^{-1}$ and σ_0', and since $\mathrm{Ind}_{M'}^{S_n}(\mathbf{1}_{M'}) \cong \mathrm{Ind}_M^{S_n}(\mathbf{1}_M)$, we may hereafter assume that $\sigma_0 = (1, 2, \ldots, n)$.

Next, consider an arbitrary subgroup of $H \subseteq S_n$ containing σ_0. The value of $I_H^{S_n}(\mathbf{1}_H)$ at C_n is given by the standard formula (1.2):

$$I_H^{S_n}(\mathbf{1}_H)(C_n) = \left|\frac{S_n}{H}\right| \cdot \frac{1}{|C_n|} \sum_{x \in H \cap C_n} 1 = \left|\frac{S_n}{H}\right| \cdot \frac{|H \cap C_n|}{(n-1)!} = n \cdot \frac{|H \cap C_n|}{|H|}.$$

When S_n acts on itself by conjugations, we have shown that $\mathrm{Stab}_{S_n}(\sigma_0) = Z_{S_n}(\sigma_0)$ is equal to H_0 (see Lemma 2.4), so $|\mathrm{Stab}_{S_n}(\sigma_0)| = n$. Since $H \supseteq H_0$, we get that $\mathrm{Stab}_H(\sigma_0) = H \cap H_0 = H_0$. Hence,

$$|\mathrm{Stab}_H(\sigma_0)| = |\mathrm{Stab}_{S_n}(\sigma_0)| = n \tag{2.4}$$

for any H containing σ_0.

Now assume that $H = M$, a subgroup having the property (2.3). In this case, we have $M \cap (S_n \cdot \sigma_0) = M \cap (M \cdot \sigma_0) = M \cdot \sigma_0$, so $M \cap C_n = M \cdot \sigma_0$ and

$$|M \cap C_n| = |M \cdot \sigma_0| = \frac{|M|}{|\mathrm{Stab}_M(\sigma_0)|} = \frac{|M|}{|\mathrm{Stab}_{S_n}(\sigma_0)|} = \frac{|M|}{n}. \tag{2.5}$$

Therefore,

$$I_M^{S_n}(\mathbf{1})(C_n) = n \cdot \frac{|M \cap C_n|}{|M|} = \frac{n}{|M|} \cdot \frac{|M|}{n} = 1,$$

as claimed. $\square$

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To construct S we encode $[1,n]$ as the Cartesian product space $I_M \times I_N = [1,M] \times [1,N]$ via any bijection $\phi\colon [1,n] \to I_M \times I_N$. This bijection of underlying spaces induces an isomorphism $\phi^*(\sigma) = \phi\sigma\phi^{-1}$ from S_n to $S_{I_M \times I_N}$ that sends k-cycles to k-cycles.

We may transfer the problem in our proposition over to $S_{I_M \times I_N}$, where we seek

(i) a subgroup $S \cong S_M \times S_N = S_{I_M} \times S_{I_N}$ in $S_{I_M \times I_N}$,

(ii) an n-cycle $\sigma \in S$ such that

$$x\sigma x^{-1} \in S \implies x \in S \quad (\forall x \in S_{I_M \times I_N}).$$

For $S \cong S_M \times S_N$ we take the subgroup

$$S = \{\tau \times \mu\colon \tau \in S_{I_M},\ \mu \in S_{I_N}\} \quad \text{in } S_{I_M \times I_N}, \tag{2.6}$$

where $\tau \times \mu(i,j) = (\tau(i), \mu(j))$. This subgroup includes the long cycle

$$\tilde{\sigma}_0 = ((11), (22), \dots, (nn)) \quad \text{in } S_{I_M \times I_N}.$$

Notation. In what follows, the intervals $[1,M]$ and $[1,N]$ should be regarded as cyclic-ordered lists, whose entries are reckoned $\bmod M$ and $\bmod N$, respectively. Thus $(i,j) = ([i]_M, [j]_N)$ for $i, j \in \mathbb{Z}$, where $[i]_M, [j]_N$ are congruence classes in $\mathbb{Z}_M, \mathbb{Z}_N$, respectively. By the Chinese remainder theorem, the pairs $(11), (22), \dots, (nn)$ run through all of $I_M \times I_N$ before the first repeat.

Our discussion employs the particular encoding map

$$\phi(k) = (kk), \quad \text{i.e. } \phi([k]_n) = ([k]_M, [k]_N) \text{ for } k \in \mathbb{Z}. \tag{2.7}$$

This is an n-cycle by the Chinese remainder theorem, and has the great advantage that ϕ^* maps the 'standard' n-cycle $\sigma_0 = (1, 2, \dots, n)$ in S_n to $\tilde{\sigma}_0 \in S$ in $S_{I_M \times I_N}$.

We now observe the following.

Lemma 2.10. *Every n-cycle $\sigma \in S_M \times S_N$ is conjugate within $S_M \times S_N$ to the standard n-cycle $\tilde{\sigma}_0$.*

Proof. Let $\sigma = \tau \times \mu$. If τ is not an M-cycle, there exists a proper τ-invariant subset A in I_M; then, $A \times I_N$ is a proper subset invariant under $\tau \times \mu$, which is impossible for an n-cycle. Thus, τ, μ are long cycles in S_M, S_N, respectively.

There exist $x \in S_M$, $y \in S_N$ such that $x\tau x^{-1} = (1, 2, \dots, M)$ and $y\mu y^{-1} = (1, 2, \dots, N)$, so if $z = (x, y)$ in $S_M \times S_N = S_{I_M} \times S_{I_N}$, we get that

$$z\sigma z^{-1} = (1, 2, \dots, M) \times (1, 2, \dots, N) = \tilde{\sigma}_0.$$

□

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As for (ii), if σ is any n-cycle in $S = S_M \times S_N$ and $x \in S_n$ is an element such that $x\sigma x^{-1} \in S$, we must show that $x \in S$. By the preceding lemma there exist $a, b \in S$ such that

$$a\tilde{\sigma}_0 a^{-1} = \sigma \quad \text{and} \quad (bxa)\tilde{\sigma}_0(bxa)^{-1} = \tilde{\sigma}_0,$$

so bxa is in the centralizer $Z_{S_{I_M \times I_N}}(\tilde{\sigma}_0)$. The particular isomorphism $\phi^*\colon S_n \to S_{I_M \times I_N}$ we have chosen sends σ_0 to $\tilde{\sigma}_0$, so the centralizer $Z_{S_n}(\sigma_0)$ maps to the centralizer of $\tilde{\sigma}_0$ in $S_{I_M \times I_N}$. By Lemma 2.4 we have that $Z_{S_n}(\sigma_0) = H_0 = \langle \sigma_0 \rangle$, so the corresponding centralizer of $\tilde{\sigma}_0$ is $\tilde{H}_0 = \langle \tilde{\sigma}_0 \rangle$. Therefore, bxa is a power $\tilde{\sigma}_0^r$, which lies in $\tilde{H}_0 \subseteq S_M \times S_N$. Then,

$$x = b^{-1}\tilde{\sigma}_0^r a^{-1} = b^{-1}a^{-1} \cdot a\tilde{\sigma}_0^r a^{-1}.$$

The first factor is in $S_M \times S_N$, and so is a; thus, the conjugate $y\tilde{\sigma}_0^r y^{-1}$ is in $S_M \times S_N$, and so is x. This completes the proof of Proposition 2.8. $\square$

We have now verified the claim of (1.4) when the extra character $\psi^{(0)}$ is defined as in Proposition 2.8 and Lemma 2.9.

Corollary 2.11. *If we define the extra character $\psi^{(0)}$ as in Proposition 2.8 and Lemma 2.9, the characters*

$$\{\psi^{(0)}\} \cup \{\psi^{(\lambda)}\colon \lambda \neq \lambda_* \text{ in } \Lambda_n\}$$

are a $\mathbb{Z}$-basis for the representation ring $\mathcal{R}[S_n]$.

3. S_n has the weak Brauer property (B*)

Here, we show that all symmetric groups have the weak Brauer property (B*): $\mathcal{R}[S_n] = I_*^{S_n}(\mathcal{N}(S_n))$.

Proposition 3.1. *For any n, we have that $\mathcal{R}[S_n] = I_*^{S_n}(\mathcal{N}(S_n))$, where $\mathcal{N}(S_n)$ is the class of all nilpotent subgroups, proper or not.*

Proof. We argue by induction on n, the result being trivial for $n = 1, 2$. By Corollary 2.11, if $\pi \in S_n^\wedge$, its trace character can be written as

$$\begin{aligned}\chi_\pi &= m_0\psi^{(0)} + \sum_{\lambda \neq \lambda_*} m_\lambda I_{S_\lambda}^{S_n}(\mathbf{1}_{S_\lambda}) \\ &= m_0\psi^{(0)} + \sum_{\lambda \neq \lambda_*} m_\lambda \psi^{(\lambda)} \quad (m_0, m_\lambda \in \mathbb{Z}),\end{aligned}$$

where, as in (1.4), the extra character is one of the following.

(i) When $n = p^m$, $\psi^{(0)}$ is a $\mathbb{Z}$-linear combination of characters $I_N^{S_n}(\pi)$ induced from irreducible representations of nilpotent subgroups.

(ii) When n is not a prime power the extra character is equal to $I_S^{S_n}(\mathbf{1}_S)$, where $S \subseteq S_n$ is a subgroup algebraically isomorphic to $S_M \times S_N$, with $n = MN$ and M, N relatively prime.

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Our task is to show that the induced characters $\psi^{(\lambda)}$ fall within $I_*^{S_n}(\mathcal{N}(S_n))$. The same sort of argument will apply to $\psi^{(0)}$ in the second case above, while in the first case we already have $\psi^{(0)} \in I_*^{S_n}(\mathcal{N}(S_n))$.

We observe that $\mathbf{1}_{S_\lambda}$ is a Kronecker product $\mathbf{1}_{S_{\lambda_1}} \times \cdots \times \mathbf{1}_{S_{\lambda_n}}$ on the subgroup $S_\lambda = S_{\lambda_1} \times \cdots \times S_{\lambda_n}$ in S_n. By the induction hypothesis (since $\lambda_k < n$ for all k if $\lambda \neq \lambda_*$), we have that

$$\mathbf{1}_{S_{\lambda_k}} \in \mathbb{Z}\text{-span}\{I_N^{S_{\lambda_k}}(\pi) \colon N \in \mathcal{N}(S_{\lambda_k}),\ \pi \in N^\wedge\}.$$

Thus, $\mathbf{1}_{S_\lambda}$ is a $\mathbb{Z}$-linear combination of functions on $S_{\lambda_1} \times \cdots \times S_{\lambda_n}$ having the following form. For $\sigma_k \in S_{\lambda_k}$,

$$\begin{aligned} f(\sigma_1, \dots, \sigma_n) &= I_{N_1}^{S_{\lambda_1}}(\pi_1)(\sigma_1) \cdot \cdots \cdot I_{N_n}^{S_{\lambda_n}}(\pi_n)(\sigma_n) \\ &= (I_{N_1 \times \cdots \times N_n}^{S_{\lambda_1} \times \cdots \times S_{\lambda_n}}(\pi_1 \times \cdots \times \pi_n))_{(\sigma_1, \dots, \sigma_n)} \\ &= I_{N_1 \times \cdots \times N_n}^{S_\lambda}(\pi_1 \times \cdots \times \pi_n)_\sigma, \end{aligned}$$

where $\sigma = \sigma_1 \cdot \cdots \cdot \sigma_n$ in S_λ. Each $N_k \subseteq S_{\lambda_k}$ is nilpotent and $\pi_k \in N_k^\wedge$, so $\pi = \pi_1 \times \cdots \times \pi_n$ is an irreducible representation of the nilpotent direct product $N = N_1 \times \cdots \times N_n$. (Note that this is where you would get in trouble working with characters induced from *elementary* subgroups; the class $\mathcal{E}$ is not closed under direct products, but class $\mathcal{N}$ is.)

We have shown that, for every $\lambda \neq \lambda_*$,

$$\mathbf{1}_{S_\lambda} \in \mathbb{Z}\text{-span}\{I_N^{S_\lambda}(\pi) \colon N \in \mathcal{N}(S_\lambda) \subseteq \mathcal{N}(S_n),\ \pi \in N^\wedge\}.$$

By induction in stages, $\psi^{(\lambda)} = I_{S_\lambda}^{S_n}(\mathbf{1}_{S_\lambda})$ is a $\mathbb{Z}$-linear combination of terms

$$I_{S_\lambda}^{S_n}(I_N^{S_\lambda}(\pi)) = I_N^{S_n}(\pi), \quad \text{where } N \in \mathcal{N}(S_n) \text{ and } \pi \in N^\wedge,$$

proving the proposition. □

We now observe that symmetric groups are of (strong) Brauer type if we can prove that all finite nilpotent groups are of Brauer type.

Lemma 3.2. *If all finite nilpotent groups are of Brauer type, so are all symmetric groups S_n.*

Proof. In Proposition 3.1 we showed that every irreducible character χ_π is a sum of induced characters $I_N^{S_n}(\mu)$, where $N \in \mathcal{N}(S_n)$ and $\mu \in N^\wedge$. If nilpotent groups are type B, each $\chi_\mu (\mu \in N^\wedge)$ is a sum of characters $I_H^N(\phi)$ with $H \in \mathcal{E}(N) \subseteq \mathcal{E}(S_n)$ and ϕ one dimensional. By induction in stages, S_n is of Brauer type. □

4. Nilpotent groups are Brauer type

In this section we show that the Brauer property holds for nilpotent groups if it can be established for abelian groups of the form $\mathbb{Z}_{pq}^2 \cong \mathbb{Z}_p^2 \times \mathbb{Z}_q^2$.

Proposition 4.1. *If the Brauer property holds for abelian groups of the form $\mathbb{Z}_{pq}^2$ (with p, q distinct primes), then all finite nilpotent groups are of Brauer type; hence, so are all symmetric groups S_n.*

We start with a lemma.

Lemma 4.2. *Let $\zeta\colon G \to \bar{G}$ be a surjective homomorphism between finite groups, let $\bar{\mathcal{P}}$ be the proper subgroups in $\bar{G}$, and let $\zeta^{-1}(\bar{\mathcal{P}})$ be their pullbacks to G. Then,*

$$I_*^G(\zeta^{-1}(\bar{\mathcal{P}})) = \mathbb{Z}\text{-span}\{I_H^G(\pi)\colon H \in \zeta^{-1}(\bar{\mathcal{P}}),\ \pi \in H^\wedge\}$$

is an ideal in $\mathcal{R}[G]$. If $\mathbf{1}_{\bar{G}} \in I_*^{\bar{G}}(\bar{\mathcal{P}})$, then

$$\mathbf{1}_G = \mathbf{1}_{\bar{G}} \circ \zeta \in I_*^G(\zeta^{-1}(\bar{\mathcal{P}}))$$

and $I_*^G(\zeta^{-1}(\bar{\mathcal{P}})) = \mathcal{R}[G]$.

Proof. If H is the pullback of a proper subgroup $\bar{H}$ in $\bar{G}$, and if $\pi \in H^\wedge$, then for any representation μ of G we have that [**8**]

$$\mu \otimes \operatorname{Ind}_H^G(\pi) \cong \operatorname{Ind}_H^G((\mu \mid H) \otimes \pi).$$

Hence, the trace character of this representation is

$$\sum_\beta I_H^G(\theta_\beta) \in I_*^G(\mathcal{P})$$

when we decompose $(\mu \mid H) \otimes \pi$ as a direct sum of irreducible representations $\theta_\beta \in H^\wedge$.

Pullbacks $\pi = \bar{\pi} \circ \zeta$ of irreducibles on a proper subgroup $\bar{H} \subseteq \bar{G}$ are irreducible on $H = \zeta^{-1}(H)$. Furthermore, we have that $I_H^G(\bar{\pi} \circ \zeta) = I_{\bar{H}}^{\bar{G}}(\bar{\pi}) \circ \zeta$. Thus,

$$I_*^G(\zeta^{-1}(\bar{\mathcal{P}})) \supseteq I_*^{\bar{G}}(\bar{\mathcal{P}}) \circ \zeta.$$

If $\mathbf{1}_{\bar{G}} \in I_*^{\bar{G}}(\bar{\mathcal{P}})$, then $\mathbf{1}_G = \mathbf{1}_{\bar{G}} \circ \zeta$ lies in $I_*^G(\zeta^{-1}(\bar{\mathcal{P}}))$, and this ideal is equal to $\mathcal{R}[G]$. $\square$

Corollary 4.3. *If every irreducible character $\bar{\rho}$ on $\bar{G}$ is a $\mathbb{Z}$-linear combination of characters $I_{\bar{H}}^{\bar{G}}(\bar{\pi})$ induced from irreducible characters on proper subgroups $\bar{H}$, then*

$$\mathcal{R}[G] = \mathbb{Z}\text{-span}\{I_H^G(\theta)\colon H = \zeta^{-1}(\bar{H}),\ \bar{H} \text{ a proper subgroup and } \theta \in H^\wedge\}.$$

Clearly, $\mathbb{Z}_{pq}^2 \cong \mathbb{Z}_{pq} \times \mathbb{Z}_{pq}$ is not elementary, but its proper subgroups can only have cardinalities p^2q, pq, pq^2, p, q, 1 and are all elementary. Thus, the Brauer property for $G = \mathbb{Z}_{pq}^2$ is equivalent to the statement

$$\mathcal{R}[\mathbb{Z}_{pq}^2] = \mathbb{Z}\text{-span}\{I_{H_i}^G(\phi_i)\colon H_i \text{ a proper subgroup of } G,\ \phi_i \in H_i^\wedge\}. \tag{4.1}$$

Assuming that $\mathbb{Z}_{pq}^2$ has the Brauer property, we prove Proposition 4.1 for a nilpotent group N by induction on $n = |N|$.

If N has $\mathbb{Z}_{pq}^2$ as a homomorphic image, then by Corollary 4.3 we have that $\mathcal{R}[N] = I_*^N(\zeta^{-1}(\bar{\mathcal{P}}))$, where $\bar{\mathcal{P}}$ are the proper subgroups in $\mathbb{Z}_{pq}^2$ and $\zeta\colon N \to \mathbb{Z}_{pq}^2$ is the quotient map. Proper subgroups in N are nilpotent, and of Brauer type by the induction hypothesis, so by induction in stages all elements in $\mathcal{R}[N]$ are sums of characters induced from one-dimensional characters on subgroups in $\mathcal{E}(N)$.

The only remaining possibility is that N has no surjective homomorphism onto $\mathbb{Z}_{pq}^2$.

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Lemma 4.4. *A finite nilpotent group N has either a surjective homomorphism to $\mathbb{Z}_{pq}^2$, or else N is already an elementary group.*

Proof. N is a direct product $N = \prod_p N_p$ of its Sylow subgroups (all normal) with $[N, N] = \prod_p [N_p, N_p]$. Then, $\prod_p N_p/[N_p, N_p] = \prod_p \bar{N}_p$ is the Sylow decomposition of the abelian group $\bar{N} = N/[N, N]$. If subgroups $\bar{N}_p$ and $\bar{N}_q$ both require at least two generators, for distinct primes $p \neq q$, we will show that $\bar{N}_p \times \bar{N}_q$ has $\mathbb{Z}_{pq}^2$ as a quotient, yielding a natural surjective homomorphism $N \to N_p \times N_q \to \bar{N}_p \times \bar{N}_q \to \mathbb{Z}_{pq}^2$.

In the remaining case, there exists at most one prime p_0 such that $\bar{N}_p$ fails to be cyclic. By the following lemma, N_p is itself cyclic for all $p \neq p_0$ (so $\prod_{p \neq p_0} N_p$ is cyclic), while N_{p_0} is a p-group. Thus, N is itself an elementary group in this case.

Lemma 4.5. *If N is a finite nilpotent group and H a subgroup such that $H[N, N] = N$, then $H = N$.*

Proof. Let $Z_0 = (e) \subset Z_1 = Z(N) \subset \cdots \subset Z_r = N$ be the ascending central series, and inductively define $H_0 = H$ and $H_{i+1} = Z_{i+1} H_i$. Then, H_i is normal in H_{i+1} and, if $H \neq N$, there exists a first index such that $H_i \neq N$ but $H_{i+1} = N$. Then, N/H_i is non-trivial abelian ($\cong$ a subgroup of Z_{i+1}/Z_i) and, therefore, $H_i \supseteq [N, N]$. But then $H[N, N] \subseteq H_i[N, N] = H_i \neq N$, contrary to the hypothesis. □

Corollary 4.6. *For any finite nilpotent group, the minimal number of generators is the same for both N and $\bar{N} = N/[N, N]$. In particular, if $\bar{N}$ is cyclic, N is also cyclic (and, in particular, abelian).*

Proof. If $\{\bar{x}_1, \dots, \bar{x}_r\}$ is a minimal set of generators for $\bar{N}$, let x_i be any preimage of $\bar{x}_i$ and let $H = \langle x_1, \dots, x_r \rangle$. Then, $H[N, N] = G$ because, if $y \in N$, we have that

$$\bar{y} = \phi(y) = \bar{u}_1 \cdots \bar{u}_s, \quad \text{where } \bar{u}_k \in \{\bar{x}_1^{\pm 1}, \dots, \bar{x}_r^{\pm 1}\}.$$

There then exists a $\gamma \in [N, N]$ such that $y = u_1 \cdots u_s \cdot \gamma \in H[N, N]$ and $H = G$ by Lemma 4.5. □

The last step in proving Lemma 4.4 is to exhibit a surjective homomorphism $\phi \colon \bar{N}_p \times \bar{N}_q \to \mathbb{Z}_{pq}^2$ when $\bar{N}_p$, $\bar{N}_q$ are not cyclic. By the fundamental structure theorem for abelian groups, if $|\bar{N}_p| = p^n$, we have that

$$\bar{N}_p = \bigoplus_{j=1}^{n} H_j, \quad \text{where } H_j = (\mathbb{Z}_{p^j} \oplus \cdots \oplus \mathbb{Z}_{p^j}) \ (n_j \text{ factors; } n_j \geqslant 0), \tag{4.2}$$

with $\sum_{j=1}^{n} j n_j = n$. Since $\bar{N}_p$ requires at least two generators, there exist at least two distinct factors in this direct sum, so there exists a subgroup of the form $\mathbb{Z}_{p^i} \times \mathbb{Z}_{p^j}$ ($i = j$ allowed if $n_i > 1$). In each of these factors there exists a subgroup C_i, C_j of index p, so if we kill all other direct summands and factor C_i, C_j out of $\mathbb{Z}_{p^i}, \mathbb{Z}_{p^j}$, we get $\mathbb{Z}_p \times \mathbb{Z}_p$ as a quotient of $\bar{N}_p$. Likewise for $\bar{N}_q$, yielding a surjective homomorphism from $\bar{N}_p \times \bar{N}_q$ to $\mathbb{Z}_p^2 \times \mathbb{Z}_q^2 \cong \mathbb{Z}_{pq}^2$. □

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5. Abelian groups $\mathbb{Z}_{pq}^2$ are of Brauer type

We prove this using a finite Fourier transform. Together with the preceding results, this establishes our fundamental result: that all S_n are of Brauer type. From this it follows easily (see § 6) that all finite groups are of Brauer type, by regarding them as subgroups of S_n and applying the Mackey subgroup theorem (see [**6**], [**7**, pp. 138–142] and [**8**]).

Theorem 5.1. *If p and q are distinct primes, the abelian group $G = \mathbb{Z}_{pq}^2$ is of Brauer type. It follows that all nilpotent groups and all symmetric groups S_n are of Brauer type.*

Proof. Proper subgroups of G can only have orders 1, p, q, pq, p^2q, pq^2; all are elementary. We parametrize elements of G as $(a, b, c, d) \in \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_q$.

The dual $G^\wedge$ for any finite abelian group consists of the multiplicative characters on G, and $\mathcal{R}[G]$ is just the ring of integer coefficient trigonometric polynomial functions $f = \sum_{\chi\in G^\wedge} a_\chi \chi$ ($a_\chi \in \mathbb{Z}$) with the usual pointwise $(+)$ and $(\cdot)$ operations. When $G = \mathbb{Z}_{pq}^2$, we show that the identity element $\mathbf{1}_G \in \mathcal{R}[G]$ is in the ideal

$$I_*^G[\mathcal{P}] = \mathbb{Z}\text{-span}\{I_H^G(\chi)\colon H \text{ a proper subgroup of } G,\ \chi \in H^\wedge\},$$

where $\mathcal{P}$ is the family of all proper subgroups; as previously noted, proper subgroups of G are elementary, so G is of Brauer type.

The Fourier transform on a finite abelian group is

$$\hat{f}(\chi) = \frac{1}{|G|} \sum_{g\in G} f(g)\overline{\chi(g)} \quad \text{for all } \chi \in G^\wedge.$$

By Fourier inversion,

$$f(g) = \sum_{\chi\in G^\wedge} \hat{f}(\chi)\chi(g) \quad \text{for all } g \in G \text{ and all } f,$$

so the Fourier transform $\hat{f}(\chi)$ provides the integer weights needed to write f as a sum of characters $\chi \in G^\wedge$. For instance, the identity element $\mathbf{1}_G$ is $\sum_\chi a_\chi \chi$, with $a_\chi = 1$ for $\chi = \mathbf{1}_G$, and $a_\chi = 0$ otherwise, so $(\mathbf{1}_G)^\wedge$ is the Dirac delta δ_{χ_0} at the trivial character $\chi_0 = \mathbf{1}_G \in G^\wedge$. The annihilator of a subgroup $H \subseteq G$ is $H^0 = \{\chi \in G^\wedge \colon \chi \mid H = 1\}$ in $G^\wedge$, which has the property

$$(\mathbf{1}_H)^\wedge = |H|/|G| \cdot \mathbf{1}_{H^0} \quad \text{(Poisson summation formula).}$$

In fact, if $f = \mathbf{1}_H$ and $\chi \in G^\wedge$, we have that

$$\begin{aligned} \hat{f}(\chi) &= \frac{1}{|G|} \sum_{g\in G} \mathbf{1}_H(g)\overline{\chi(g)} \\ &= \left|\frac{H}{G}\right| \cdot \frac{1}{|H|} \sum_{h\in H} \overline{\chi(h)} \\ &= \left|\frac{H}{G}\right| \cdot \begin{cases} 1 & \text{if } \chi \in H^0, \\ 0 & \text{if } \chi \notin H^0, \end{cases} \end{aligned}$$

by the orthogonality relations on H.

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The induced character corresponding to a multiplicative character χ_ρ on a subgroup H is

$$I_H^G(\chi_\rho)(g) = \begin{cases} \left|\dfrac{G}{H}\right| \cdot \chi_\rho(g) & \text{if } g \in H, \\ 0 & \text{if } g \notin H. \end{cases}$$

For the trivial representation on H, the class function $f = I_H^G(\mathbf{1}_H) = |G|/|H| \cdot \mathbf{1}_H$ is identified as an element of $\mathcal{R}[G]$ by taking the Fourier transform:

$$\hat{f} = \left|\frac{H}{G}\right| \cdot \left|\frac{G}{H}\right| \cdot \mathbf{1}_{H^0} = \mathbf{1}_{H^0} \quad \text{on } G^\wedge, \tag{5.1}$$

so $f = I_H^G(\mathbf{1}_H) = \sum_{\chi \in H^0} \chi$.

When $G = \mathbb{Z}_p^m$, its multiplicative characters are conveniently labelled by 'dual vectors' $\dot{a} \in \dot{\mathbb{Z}}_p^n$, m-tuples such that

$$\chi_{\dot{a}}(x) = \mathrm{e}^{2\pi\mathrm{i}(\dot{a}_1 x_1 + \cdots + \dot{a}_m x_m)/p}, \quad \dot{a} \in \dot{\mathbb{Z}}_p^m,\ x \in \mathbb{Z}_p^m,$$

so we identify $G^\wedge = (\dot{\mathbb{Z}}_p^n, +)$. The same sort of labelling is also convenient for the products $G = A \times B$, with $A = \mathbb{Z}_p^2$ and $B = \mathbb{Z}_q^2$. Non-trivial elements $x \in A$ generate cyclic groups H_x of order p, which are essentially disjoint: $H_x \cap H_y = (e)$ if $H_x \neq H_y$. There are $p + 1 = (p^2 - 1)/(p - 1)$ such subgroups. If $E = (e)$ is the trivial subgroup and we sum over representatives for the distinct H_x, the function

$$f = \left(\sum_x I_{H_x}^A(\mathbf{1}_{H_x})\right) - I_E^A(\mathbf{1}_E)$$

has Fourier transform

$$\hat{f} = \left(\sum_x \mathbf{1}_{H_x^0}\right) - \mathbf{1}_{\dot{A}} \quad \text{on } A^\wedge \cong \dot{A} = \dot{\mathbb{Z}}_p^2.$$

The annihilators H_x^0 are precisely the distinct proper cyclic subgroups in $\dot{\mathbb{Z}}_p^2$, and their union picks up each element in $\dot{\mathbb{Z}}_p^2$ once, except for the trivial character $(\dot{0}, \dot{0})$, which occurs $p + 1$ times. Thus,

$$\hat{f}(\dot{a}) = \begin{cases} 0 & \text{if } \dot{a} \neq (\dot{0}, \dot{0}), \\ (p+1) - 1 = p & \text{at } \dot{a} = (\dot{0}, \dot{0}). \end{cases}$$

Likewise for B upon replacing p by q.

In $G = A \times B = \mathbb{Z}_p^2 \times \mathbb{Z}_q^2$ we have subgroups and annihilators as follows (letting E be the trivial subgroup in A and $\dot{E}$ the trivial subgroup in $\dot{B} = B^\wedge$):

$$\begin{aligned} M_x = H_x \times B, &\qquad M_x^0 = H_x^0 \times B^0 = H_x^0 \times (\dot{0}, \dot{0}), \\ E \times B, &\qquad (E \times B)^0 = \dot{A} \times (\dot{0}, \dot{0}). \end{aligned}$$

Now consider

$$f = \left(\sum_x I^G_{H_x\times B}(\mathbf{1}_{H_x\times B})\right) - I^G_{E\times B}(\mathbf{1}_{E\times B}).$$

Writing $\chi_0 = (\dot{0},\dot{0},\dot{0},\dot{0})$ for the trivial character in $G^\wedge = \dot{\mathbb{Z}}_p \times \cdots \times \dot{\mathbb{Z}}_q$, the Fourier transform of f is

$$\begin{aligned}\hat{f}(\dot{a},\dot{b}) &= \left(\sum_x \mathbf{1}_{(H_x\times B)^0}(\dot{a},\dot{b})\right) - \mathbf{1}_{\dot{A}\times(\dot{0},\dot{0})}(\dot{a},\dot{b}) \\ &= \left[\left(\sum_x \mathbf{1}_{H_x^0}(\dot{a})\right) - \mathbf{1}_{\dot{A}}(\dot{a})\right]\cdot \mathbf{1}_{\dot{E}}(\dot{b}) \\ &= \begin{cases} 0 & \text{if } (\dot{a},\dot{b}) \neq (\dot{0},\dot{0},\dot{0},\dot{0}) \\ p & \text{if } (\dot{a},\dot{b}) = (\dot{0},\dot{0},\dot{0},\dot{0}) \end{cases} \\ &= p\delta_{\chi_0}.\end{aligned}$$

Reversing the roles of A and B and labelling the cyclic subgroups in B as H_y, we get a $\mathbb{Z}$-linear combination of indicator functions on annihilator subgroups in $G^\wedge$ such that

$$\begin{aligned}\hat{h}(\dot{a},\dot{b}) &= \left(\sum_y \mathbf{1}_{(A\times H_y)^0}(\dot{a},\dot{b})\right) - \mathbf{1}_{(\dot{0},\dot{0})\times\dot{B}}(\dot{a},\dot{b}) \\ &= \begin{cases} 0 & \text{if } (\dot{a},\dot{b}) \neq (\dot{0},\dot{0},\dot{0},\dot{0}), \\ q & \text{if } (\dot{a},\dot{b}) = (\dot{0},\dot{0},\dot{0},\dot{0}) \end{cases} = q\delta_{\chi_0}.\end{aligned}$$

Since $p \neq q$ there exist integers r, s such that $rp + sq = 1$; hence,

$$r\hat{f} + s\hat{h} = \delta_{\chi_0} \quad \text{on } G^\wedge$$

and $\mathbf{1}_{A\times B} \in \mathcal{R}[A\times B]$, as required. □

6. Transition from $\mathbf{S_n}$ to arbitrary groups

With all this in hand we are ready to prove the main result. If $\mathcal{N}(G)$ is the class of nilpotent subgroups, proper or not, we have shown that $I^{S_n}_*(\mathcal{N}(S_n)) = \mathcal{R}[S_n]$. An arbitrary G is of Brauer type if $\mathcal{R}[G] = I^G_*(\mathcal{N}(G))$. In fact, if $N \in \mathcal{N}(G)$ and $\pi \in N^\wedge$, then, by Proposition 4.1 and Theorem 5.1, $I^G_N(\pi)$ is a $\mathbb{Z}$-linear combination of characters

$$I^G_N(I^N_H(\phi)) = I^G_H(\phi),$$

where $H \in \mathcal{E}(N)$ and ϕ is a multiplicative character on H.

Theorem 6.1. *Every subgroup of S_n is of Brauer type.*

Proof. If $\pi \in S_n^\wedge$, then $\chi_\pi \in I^{S_n}_*(\mathcal{N}(S_n))$, so $\chi_\pi = \sum_i a_i I^{S_n}_{N_i}(\pi_i)$, with $\pi_i \in N_i^\wedge$. The restriction $\pi \mid G$ has trace character

$$\chi_\pi \mid G = \sum_i a_i \cdot (I^{S_n}_{N_i}(\pi_i) \mid G).$$

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By Mackey's subgroup theorem (see [**7**] and [**8**, pp. 138–142]), we have that

$$\mathrm{Res}^G_{S_n}(\mathrm{Ind}^{S_n}_{N_i}(\pi_i)) \cong \bigoplus_{x\in G\backslash S_n/N_i} \mathrm{Ind}^G_{G\cap x\cdot N_i}(x\cdot\pi_i \mid G\cap x\cdot N_i),$$

where $x\cdot\pi_i(g)=\pi_i(x^{-1}gx)$ on $x\cdot N = xN_ix^{-1}$, so the trace characters satisfy

$$(I^{S_n}_{N_i}(\pi_i)\mid G) = \sum_{x\in G\backslash S_n/N_i} I^G_{G\cap x\cdot N_i}(x\cdot\pi_i \mid G\cap x\cdot N_i)$$

on the group G. Replacing $x\cdot\pi_i \mid G\cap x\cdot N_i$ with its irreducible decomposition, we see that this is contained in $I^G_*(\mathcal{N}(G))$, so $\mathcal{R}[S_n]\mid G\subseteq I^G_*(\mathcal{N}(G))$, and the latter is an ideal in $\mathcal{R}[G]$, by Lemma 1.5.

Since $\mathbf{1}_{S_n}\in I^{S_n}_*(\mathcal{N}(S_n))$, we get that $\mathbf{1}_G=\mathbf{1}_{S_n}\mid G\in I^G_*(\mathcal{N}(G))$, which implies that $I^G_*(\mathcal{N}(G))=\mathcal{R}[G]$. Since nilpotent groups are of Brauer type, so is G. □

7. Further comments on $\mathcal{R}[S_n]$

The class of elementary groups, or even the class of p-groups, is quite large and would be hard to characterize. The preceding discussion suggests that the groups and multiplicative characters needed to produce a set of additive generators for $\mathcal{R}[S_n]$ can be narrowed to a class whose members can be described explicitly. Essentially, $\mathcal{R}[S_n]$ is generated by induced characters $I^{S_n}_M(\phi)$, where $M=M_1\times\cdots\times M_r\subseteq S_n$, $\phi=\phi_1\times\cdots\times\phi_r$ is a multiplicative character, and the factors (M_i,ϕ_i) are described by one of the following cases.

(15) (i) $M_i\cong \mathrm{Syl}_p(S_m)$, a Sylow subgroup in some symmetric group (p a prime divisor of $m!$) and $\phi=\mathbf{1}_H$, the trivial representation.

(ii) $M_i\cong(\mathbb{Z}_{p^\ell},+)$, with ϕ the canonical multiplicative character $\phi([s]_{p^\ell})=\mathrm{e}^{2\pi \mathrm{i}s/p}$.

The structure of p-Sylow subgroups in S_m for prime divisors of $m!$ is described explicitly in [**5**, pp. 81–83], and is particularly simple when m is a prime power p^k. (Then, it is a semi-direct wreath product of $\mathbb{Z}_p$ acting on a product of copies of p-Sylow subgroups in $S_{p^{k-1}}$.)

A slight modification to previous notation helps to frame the next results. If G is a group, let $\Gamma(G)$ be the set of pairs $(M,\phi)=(M_1\times\cdots\times M_r,\phi_1\times\cdots\times\phi_r)$, where each ϕ_i is one dimensional and $M\subseteq G$ is a direct product such that each factor (M_i,ϕ_i) is of one of the types listed in (15). We then define

$$I^G_*[\Gamma(G)]=\mathbb{Z}\text{-span}\{I^G_M(\phi)\colon (M,\phi)\in\Gamma(G)\}.$$

The groups G of interest are subgroups of S_n for a given n; clearly $G\subseteq S_n$ implies that $\Gamma(G)\subseteq\Gamma(S_n)$ and $I^{S_n}_*[\Gamma(G)]\subseteq I^{S_n}_*[\Gamma(S_n)]$.

In our discussion of $\mathcal{R}[S_n]$ we defined a trace character $\psi^{(\lambda)}=I^{S_n}_{S_\lambda}(\mathbf{1}_{S_\lambda})$ for each non-trivial partition $\lambda=(\lambda_1\geqslant\cdots\geqslant\lambda_n\geqslant 0)$ of n and the corresponding subgroups $S_\lambda=S_{\lambda_1}\times\cdots\times S_{\lambda_r}\subseteq S_n$, where $\lambda_r>\lambda_{r+1}=0$. These provide a $\mathbb{Z}$-basis for $\mathcal{R}[S_n]$

when we adjoin an 'extra character' $\psi^{(0)}$, whose value is ± 1 on the maximal conjugacy class C_n of long cycles. The next result exhibits *additive generators* (not necessarily a $\mathbb{Z}$-basis) obtained by induction from one-dimensional representations on direct products of groups of the types in (15).

Proposition 7.1. *For any $n \geqslant 2$, we have that*

$$\mathcal{R}[S_n] = I_*^{S_n}[\Gamma(S_n)] = \mathbb{Z}\text{-span}\{I_M^{S_n}(\phi)\colon (M,\phi) \in \Gamma(S_n)\}.$$

Proof. Arguing by induction on n, we first demonstrate that $\psi^{(\lambda)} = I_{S_\lambda}^{S_n}(\mathbf{1}_{S_\lambda})$ is in $I_*^{S_n}[\Gamma(S_n)]$ for $\lambda \neq \lambda_*$. Since $\lambda_i < n$ for each i, we have that $\mathbf{1}_{S_{\lambda_i}} \in I_*^{S(\lambda_i)}[\Gamma(S_{\lambda_i})]$. The trivial character on S_λ is a Kronecker product $\mathbf{1}_{S_\lambda} = \mathbf{1}_{S_{\lambda_1}} \times \cdots \times \mathbf{1}_{S_{\lambda_n}}$, and since

$$I_{M_1 \times M_2}^{G_1 \times G_2}(\pi_1 \times \pi_2) \cong I_{M_1}^{G_1}(\pi_1) \times I_{M_2}^{G_2}(\pi_2),$$

$\mathbf{1}_{S_\lambda}$ is a sum of functions $f(\sigma) = f(\sigma_1, \ldots, \sigma_n)$, with $\sigma_i \in S_{\lambda_i}$:

$$\begin{aligned} f(\sigma) &= I_{M_1}^{S_{\lambda_1}}(\phi_1)(\sigma_1) \times \cdots \times I_{M_n}^{S_{\lambda_n}}(\phi_n)(\sigma_n) \\ &= I_{M_1 \times \cdots \times M_n}^{S_{\lambda_1} \times \cdots \times S_{\lambda_n}}(\phi_1 \times \cdots \times \phi_n)(\sigma) \\ &= I_M^{S_\lambda}(\phi)(\sigma), \end{aligned}$$

where each $(M_i, \phi_i) \in \Gamma(S_{\lambda_i})$. Each factor $H_j^{(i)}$ in

$$M = M_1 \times \cdots \times M_n = (H_1^{(1)} \times \cdots \times H_{r(1)}^{(1)}) \times \cdots \times (H_1^{(n)} \times \cdots \times H_{r(n)}^{(n)})$$

has one of the two isomorphism types in (15), and, by the definition of $\Gamma(S_{\lambda_i})$, comes equipped with a character $\phi_i \mid H_j^{(i)}$ of the appropriate type. Then, since $M \subseteq S_\lambda \subseteq S_n$, we have $(M, \phi) \in \Gamma(S_n)$. By induction in stages, we get that $I_M^{S_n}(\phi) = I_{S_\lambda}^{S_n}(I_M^{S_\lambda}(\phi))$, so

$$I_M^{S_n}(\phi) \in I_*^{S_n}[\Gamma(S_n)].$$

Combining these last remarks, we see that $\psi^{(\lambda)} = I_{S_\lambda}^{S_n}(\mathbf{1}_{S_\lambda})$ is in $I_*^{S_n}[\Gamma(S_n)]$ for $\lambda \neq \lambda_*$.

To complete the proof we show that the extra character $\psi^{(0)}$ is in $I_*^{S_n}[\Gamma(S_n)]$. In our previous construction of $\psi^{(0)}$ there were just two possibilities.

Case 1 ($n = p^m$). In this case we showed that $\psi^{(0)} = r \cdot I_{M_1}^{S_n}(\phi_1) + s \cdot I_{M_2}^{S_n}(\phi_2)$, where $M_1 \cong \mathrm{Syl}_p(S_n)$ and $\phi_1 = \mathbf{1}_{M_1}$, $M_2 \cong \mathbb{Z}_{p^m}$, with ϕ_2 the canonical character, and $r, s \in \mathbb{Z}$ is chosen such that $\psi^{(0)} = 1$ on the maximal class $C_n \subseteq S_n$.

Case 2 (n is composite, with $n = n_1 n_2$ and $\gcd(n_1, n_2) = 1$). There exists a non-standard embedding of $S_{n_1} \times S_{n_2}$ as a subgroup $M = M_1 \times M_2 \subseteq S_n$, with $M_1 \cong S_{n_1}$, $M_2 \cong S_{n_2}$. In Proposition 2.8 we showed that M can be chosen such that $I_M^{S_n}(\mathbf{1}_M)$ is equal to 1 on C_n. Induction applies since $n_1, n_2 < n$; hence, $\mathbf{1}_{M_1} \in I_*^{M_1}[\Gamma(M_1)]$ and $\mathbf{1}_{M_2} \in I_*^{M_2}[\Gamma(M_2)]$. Applying previous arguments, we conclude that $\mathbf{1}_M$ is a $\mathbb{Z}$-linear combination of functions having the form

$$\begin{aligned} f(\sigma) = f(\sigma_1, \sigma_2) &= I_{H_1}^{M_1}(\mathbf{1}_{H_1})(\sigma_1) \cdot I_{H_2}^{M_2}(\mathbf{1}_{H_2})(\sigma_2) \\ &= I_{H_1 \times H_2}^{M}(\mathbf{1}_{H_1 \times H_2})(\sigma). \end{aligned}$$

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Each factor H_i is a product of groups $H_j^{(i)}$ as in (15), so $(H_1 \times H_2, \phi_1 \times \phi_2)$ is in $\Gamma(S_n)$. By induction in stages, $I_{H_1 \times H_2}^{S_n}(\mathbf{1}_{H_1 \times H_2})$ is in $I_*^{S_n}[\Gamma(S_n)]$. $\square$

Although characters induced from direct products of subgroups of the form in (15) provide additive generators for $\mathcal{R}[S_n]$, it is no longer clear how to pick out a set of free generators ($\mathbb{Z}$-basis) of the sort described in Corollary 2.11.

A variant of Proposition 7.1 exhibits a different class of subgroups whose trivial characters induce additive generators for $\mathcal{R}[S_n]$. Define $\mathcal{P}(S_n)$ to be the class of subgroups

$$M = M_1 \times \cdots \times M_r \subseteq S_n \quad \text{such that } M_i \cong S_{p_i^{m_i}}, \tag{7.1}$$

where the p_i are primes and $p_i^{m_i} \leqslant n$. This result, to a large extent, reduces the study of $\mathcal{R}[S_n]$ to the study of symmetric groups such that n is a prime power, to which one can apply Proposition 2.6.

We first observe that the arguments in Proposition 2.8 easily generalize to show that, if $n = \prod_{i=1}^r p_i^{m_i}$, we can produce a subgroup $M = M_1 \times \cdots \times M_r \subseteq S_n$ such that

(i) M contains the long cycle $\sigma_0 = (1, 2, \ldots, n)$,

(ii) $x\sigma_0 x^{-1} \subseteq M \Rightarrow x \in M$ for all $x \in S_n$,

(iii) $M_i \cong S_{p_i^{m_i}}$ for each i.

It follows by Lemma 2.9 that $I_M^{S_n}(\mathbf{1}_M)(\sigma_0) = 1$, so this induced character serves as the extra character $\psi^{(0)}$, but now all prime divisors of n play equal roles in determining M.

Fix an integer $n \geqslant 2$. For any finite sequence $\alpha = (\alpha_1, \ldots, \alpha_r)$ of integers such that $2 \leqslant \alpha_i \leqslant n$, we write M_α for any subgroup of the form

$$M_\alpha = M_{\alpha_1} \times \cdots \times M_{\alpha_r} \subseteq S_n \quad \text{such that } M_{\alpha_i} \cong S_{\alpha_i} \text{ for each } i,$$

and let $\mathcal{M}(S_n)$ be the class of all such subgroups. The subfamily $\mathcal{P}(S_n)$ consists of all subgroups $M \in \mathcal{M}(S_n)$ such that $\alpha_i = p_i^{k_i}$ (p_i prime, $p_i^{k_i} \leqslant n$) for all i. For any partition $\lambda_1 \geqslant \cdots \geqslant \lambda_n \geqslant 0$ of n, the subgroups $S_\lambda = S_{\lambda_1} \times \cdots \times S_{\lambda_n}$ defined earlier all lie in $\mathcal{M}(S_n)$, but $\mathcal{M}$ also includes the products M_α with non-standard embeddings (as in Proposition 2.8). Obviously, $\alpha_i \leqslant n$ if $S_\alpha \supseteq S_{\alpha_i}$ is to fit inside S_n, but we might not have $\sum_i \alpha_i = n$ for such embeddings, as we did for the subgroups S_λ associated with partitions of n.

Proposition 7.2. *For any $n \geqslant 2$, we have that*

$$\mathcal{R}[S_n] = I_*^{S_n}[\mathcal{P}(S_n)] = \mathbb{Z}\text{-span}\{I_M^{S_n}(\mathbf{1}_M) \colon M \in \mathcal{P}(S_n)\}.$$

Proof. We know that $\{\psi^{(0)}\} \cup \{\psi^{(\lambda)} \colon \lambda \neq \lambda_*\}$ is a $\mathbb{Z}$-basis for $\mathcal{R}[S_n]$. If $n = p^k$, then, trivially, $\mathbf{1}_{S_n} \in I_*^{S_n}[\mathcal{P}(S_n)]$; otherwise, $n = \prod_i p_i^{k_i}$ and there exists a product $S_\alpha = S_{\alpha_1} \times \cdots \times S_{\alpha_r}$ embedded in S_n such that $\alpha_i = p_i^{k_i}$ and $I_{S_\alpha}^{S_n}(\mathbf{1}_{S_\alpha}) = 1$ on C_n, so we may take this as the extra character $\psi^{(0)}$. In either case, there exists an $M \in \mathcal{P}(S_n)$ such that $I_M^{S_n}(\mathbf{1}_M) = 1$ on C_n and

$$\mathcal{R}[S_n] = \mathbb{Z}\text{-span}\{I_M^{S_n}(\mathbf{1}_M) \text{ and } I_{S_\lambda}^{S_n}(\mathbf{1}_{S_\lambda}),\ \lambda \neq \lambda_*\}.$$

We now argue by induction on n. For each $S_\lambda = S_{\lambda_1} \times \cdots \times S_{\lambda_r}$ corresponding to a partition $\lambda \neq \lambda_*$, we have that

$$\mathcal{R}[S_{\lambda_i}] = I_*^{S_{\lambda_i}}[\mathcal{P}(S_{\lambda_i})] = \mathbb{Z}\text{-span}\{I_{M_i}^{S_{\lambda_i}}(\mathbf{1}_{M_i})\colon M_i \in \mathcal{P}(S_{\lambda_i}) \subseteq \mathcal{P}(S_n)\},$$

so, if $\sigma = (\sigma_1, \dots, \sigma_r) \in S_\lambda$, the trivial character $\mathbf{1}_{S_\lambda}$ is a sum of products

$$I_{M_1}^{S_{\lambda_1}}(\mathbf{1}_{M_1})(\sigma_1) \times \cdots \times I_{M_r}^{S_{\lambda_r}}(\mathbf{1}_{M_r})(\sigma_r) = I_M^{S_\lambda}(\mathbf{1}_M)(\sigma),$$

where $M = M_1 \times \cdots \times M_r$. Each M_i is a direct product $M_1^{(i)} \times \cdots \times M_{n(i)}^{(i)} \subseteq S_{\lambda_i}$, with $M_j^{(i)} \cong S_m$ for some prime power m. Then,

$$M_1 \times \cdots \times M_r = (M_1^{(1)} \times \cdots \times M_{n(1)}^{(1)}) \times \cdots \times (M_1^{(r)} \times \cdots \times M_{n(r)}^{(r)}) \subseteq S_n$$

is also in $\mathcal{P}(S_n)$. By induction in stages, $I_{S_\lambda}^{S_n}(\mathbf{1}_{S_\lambda})$ is a sum of terms of the form

$$I_{S_\lambda}^{S_n}(I_M^{\S_\lambda}(\mathbf{1}_M)) = I_M^{S_n}(\mathbf{1}_M),$$

with $M \in \mathcal{P}(S_n)$.

The extra character $\psi^{(0)}$, defined above, and the characters $\psi^{(\lambda)}, \lambda \neq \lambda_*$, are all in $I_*^{S_n}[\mathcal{P}(S_n)]$ and, since these are a $\mathbb{Z}$-basis, we get that $\mathcal{R}[S_n] = I_*^{S_n}[\mathcal{P}(S_n)]$. $\square$

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