BULL. AUSTRAL. MATH. SOC. 10A45
VOL. 21 (1980), 321-328.

ON A CERTAIN KIND OF REDUCIBLE RATIONAL FRACTIONS

MORDECHAI LEWIN

The rational fraction

$$\left(1-u^{cpq}\right)\left(1-u^{a}\right)/\left(1-u^{p}\right)\left(1-u^{q}\right) ,$$

a, c, p, q positive integers, reduces to a polynomial under conditions specified in a result of Grosswald who also stated necessary and sufficient conditions for all the coefficients to be nonnegative.

This last result is given a different proof using lemmas interesting in themselves.

The method of proof is used in order to give necessary and sufficient conditions for the positive coefficients to be equal to one. For $a < 2pq$, $a = \alpha p + \beta q$, α, β nonnegative integers, $c > 1$, the exact positions of the nonzero coefficients are established. Also a necessary and sufficient condition for the number of vanishing coefficients to be minimal is given.

In a recent paper, Grosswald [2] poses and solves the following problem.

Let

$$f(u) = \prod_{i=1}^{k} \left(1-u^{g_i}\right)/\left(1-u^{h_i}\right) ,$$

Received 23 October 1979.

https://doi.org/10.1017/S0004972700006183 Published online by Cambridge University Press

where the g_i, h_i are positive integers.

Give necessary and sufficient conditions for $f(u)$ to reduce to a polynomial in u. Under what conditions will such a polynomial have only nonnegative coefficients?

Let a, c, p, q, be positive integers; a, p, q fixed and p, q relatively prime. For the particular case $k = 2$, $g_1 = cpq$, $g_2 = a$, $h_1 = p$, $h_2 = q$, it follows from [2, Theorem 1] that $f(u)$ reduces to a polynomial. Theorem 3 in [2] states necessary and sufficient conditions for the coefficients to be nonnegative. Corollary 1 in [2] treats the case $c = 1$, and shows that, if the coefficients are all nonnegative, then all the nonvanishing coefficients are equal to one.

In this note we wish to treat the case $c > 1$. We shall then determine the vanishing coefficients for any expansion for which all the nonvanishing coefficients are equal to one, without any restriction on c. We shall start with a few lemmas some of which may have interest of their own. We shall then give a different proof to a part of Theorem 3 in [2]. This proof will serve as a guide-line to the new results we are going to establish.

Let n be a positive integer. Put $\{p, q\} = R$. Let $P_R(n)$ denote the unrestricted number of partitions of n into parts p, q from R (see [2, p. 22]). Since R is assumed fixed we shall simply write $P(n)$. An integer m is *representable* by (p, q) if it is a nonnegative integral linear combination of p and q. Negative integers are thus not representable, while zero is representable. Let T be the set of all representable numbers and let T_0 denote the set of those integers whose least positive residue modulo pq is in T. Put $pq = b$, $cb + a - p - q = D$. We have

LEMMA 1. *For* $0 \le t \le b$ *we have* $P(t) = 1$ *for every* t *in* T.

Proof. Let $t \in T$ and $0 \le t \le b$. Put $t = xp + yq$. Then $x \le q$ and $y \le p$. Suppose that t has a different representation $t = x_1p + y_1q$. Let $x > x_1$. Then $y < y_1$. We have $(x-x_1)p = (y_1-y)q$ so that $q \mid x - x_1$. This is only possible if $x_1 = 0$ and $x = q$.

Likewise, $p \mid y_1 - y$ implies $y = 0$ and $y_1 = p$, but then the two representations are not distinct, contradicting our assumption. This proves Lemma 1.

LEMMA 2. *For every nonnegative integer* t *we have*

$$P(t+b) = P(t) + 1 .$$

Proof. We have:

Case 1. $t \notin T$. Then $t < b$ and hence $t + b < 2b$. Suppose $t + b$ has two distinct representations as in Lemma 1. Again we have $q \mid x - x_1$. Put $x - x_1 = mq$ where m is some positive integer. Then $x = mq + x_1$, so that $2b > t + b = xp + yq = mb + x_1 p + yq$.

It follows that $m = 1$ and hence $t = x_1 p + yq$, a contradiction, since t was not assumed in T .

Case 2. $t \in T$. Let x_1 be the smallest nonnegative integer for which there is a representation $t = x_1 p + y_1 q$. Let the other representations be $x_i p + y_i q$ with $x_1 < x_2 < \ldots < x_\lambda$ where $\lambda = P(t)$. It should be noted that $x_{i+1} - x_i = q$ and that (y_i) is a decreasing sequence with $y_i - y_{i+1} = p$. Consider now $kb + j$. Arranging the representations by ascending order of the x_i starting from the lowest we get

$$kb + j = x_1 p + (y_1 + p)q = \ldots = x_i p + (y_i + p)q = \ldots .$$

Since $y_i + p = y_{i-1}$ for $i > 1$ we get

$$kb + j = x_i p + y_{i-1} q$$

for $i = 2, 3, \ldots, n$. But then we have an additional representation $(x_n + q)p + y_n q$ yielding $n + 1$ representations. This proves Lemma 2.

Put $t = kb + j$, $k, j \geq 0$. We have

COROLLARY 1. $P(t) = P(kb+j) = k + P(j)$.

Proof. Apply Lemma 2 in succession.

LEMMA 3. $P(b-p-q-t) + P(t) = 1$ *for every integer* t ,

$0 \le t \le b-p-q$.

Proof. It is well known that there are exactly $(p-1)(q-1)/2$ positive integers which are not representable. They all are less than or equal to $pq - p - q$ (see for example [1, p. 299]). It follows that there are exactly $(p-1)(q-1)/2$ integers not exceeding $b - p - q$ which are representable. Since $b - p - q = b'$ is known to be irrepresentable [1], it follows that for every integer t exactly one of the two numbers t and $b' - t$ is representable. Applying now Lemma 1 we get Lemma 3.

LEMMA 4. *If* $a \in T$ *, then* $P(a+t) \ge P(t)$ *for every nonnegative integer* t .

Proof. Suppose $P(a+t) < P(t)$ for some t . Put $a = \lambda b + \alpha$, $t = \mu b + \beta$, $0 \le \alpha$, $\beta < b$. We now have $P(a+t) = \lambda + \mu + P(\alpha+\beta)$, $P(t) = \mu + P(\beta)$ by Lemma 3. By assumption we have $\mu + P(\beta) > \lambda + \mu + P(\alpha+\beta)$, so that $P(\beta) > \lambda + P(\alpha+\beta) \ge 0$. Since $\beta < b$, this implies $P(\beta) = 1$, $\lambda = 0$, $P(\alpha+\beta) = 0$. Then $a = \alpha$ and hence $P(\alpha) = 1$. Thus α and β are in T and hence so is $\alpha + \beta$, so that $P(\alpha+\beta) > 0$, a contradiction. This proves the lemma.

We now prove

THEOREM 1 [2, Theorem 3 (a)]. *The polynomial*

$$(1) \qquad g(u) = \left(1-u^{cb}\right)\left(1-u^{a}\right)/\left(1-u^{p}\right)\left(1-u^{q}\right) , \quad (p, q) = 1 , \quad c \in Z^{+} ,$$

has only nonnegative coefficients if and only if $a \in T$.

Proof. Let $\gamma(t)$ denote the coefficients of u^t in $g(u)$. By equating the coefficients of the formal expansion of $g(u)$ we obtain

$$(2) \qquad \gamma(t) = P(t) - P(t-a) - P(t-bc) + P(t-bc-a)$$

for every t , with the convention that P vanishes for negative arguments. Suppose a is not representable. Then clearly $a < b$. Choose $t = a$ in (2) so that $\gamma(t) = P(a) - P(0) = -1$. The condition $a \in T$ is therefore necessary and we assume it throughout the rest of the proof.

We may clearly assume $p, q > 1$, otherwise the theorem is trivial. We have

Case 1. $t < \min(a, bc)$. Then $\gamma(t) = P(t) \le c$ so that $\gamma(t) = 0$

if and only if $t \notin T$.

Case 2. $a \leq t < bc$. Then $\gamma(t) = P(t) - P(t-a) \leq c$. Lemma 4 implies that $\gamma(t)$ is nonnegative.

Case 3. $bc \leq t < a$. Then $\gamma(t) = P(t) - P(t-bc) = c > 0$, by Lemma 3.

Case 4. $\max(a, bc) \leq t \leq D$. Then

$$\gamma(t) = P(t) - P(t-a) - P(t-bc) = c - P(t-a) \leq c .$$

Since $t-a \leq D-a < bc$, it follows that $\gamma(t)$ is nonnegative.

It is clear that for $t > D$ we have $\gamma(t) = 0$ so that all the cases are covered and the theorem is proved.

The four cases in the proof of Theorem 1 will be made use of at later stages and will be referred to as Cases 1 to 4.

We now have

THEOREM 2. *For representable* a *the polynomial* (1) *has exactly* ca *coefficients equal to one, while all the other* $D + 1 - ca$ *coefficients are zero, if and only if the following condition is satisfied:* $c = 1$ *or* $a - p - q$ *is not representable.*

Proof. By [2, Corollary 1], we need only consider $c > 1$. Regarding the various cases in the proof of Theorem 1 we seek necessary and sufficient conditions for $\gamma(t) \leq 1$. If $a > b$, then $\gamma(b) = P(b) = 2$ and hence a necessary condition is $a \leq b$. For $t < a \leq b$ we have clearly $\gamma(t) = 0$ if and only if $t \notin T$, and $\gamma(t) = 1$ if and only if $t \in T$. It is equally clear that for $t < b$, $\gamma(t) \leq 1$.

For $b \leq t < bc$ we consider Case 2 requiring $P(t) - P(t-a) \leq 1$.

Define $\psi(t) = P(t) - P(t-a)$. The function ψ is periodic with period b and hence we may assume $0 \leq t-a < b$. In order that $\psi(t) \geq 2$ it is necessary and sufficient that $P(t) = 2$, $P(t-a) = 0$. Then $t-a \notin T$ and so $t - a = b - p - q - \tau_1$ for some $\tau_1 \in T$. Put $t = b + \tau_2$. Then, since $P(b+\tau_2) = 2$, it is necessary and sufficient that $\tau_2 \in T$. Using $t = b + \tau_2$ we arrive at

$$a = p + q + \tau_1 + \tau_2 = p + q + \tau ,$$

https://doi.org/10.1017/S0004972700006183 Published online by Cambridge University Press

with $\tau = \tau_1 + \tau_2 \in T$.

Case 3 is not applicable. Case 4, although easy to prove, need not be considered because $g(u)$ is known to be a reciprocal polynomial [2]. This completes the proof of Theorem 2.

We now wish to determine the vanishing coefficients of $g(u)$ satisfying the conditions of Theorem 2. We assume $t \le D$.

Let $a \ge 2b$. We seek the vanishing coefficients for $a \ge 2b$. Cases 1 and 3 need not be considered. Case 2 implies $P(t) > P(t-a)$ for every $t \ge a$, so that $\gamma(t) > 0$. Considering Case 4, let $\gamma(t) = 0$. Then $P(t-a) = c$ and hence $t-a \ge bc$, so that $t \ge bc+a > D$. We now assume $a < 2b$.

Let $t = \lambda b + \alpha$, $0 \le \alpha < b$. Define $B_\lambda = \{t \mid t = \lambda b+\alpha,\ 0 \le \alpha < b\}$ for λ a nonnegative integer. Let $t \in T_0$ if and only if $P(\alpha) = 1$. Consider the following three conditions:

I. t, $t - a$ are in the same class B_λ ;

II. $t \notin T_0$;

III. $t-a \in T_0$.

The case $c = 1$ is solved in [2]. If $a > b$ and $bc \le t < bc+a-b$, then applying Case 4 we have $\gamma(t) = P(t-a) \ne 0$. We now prove

THEOREM 3. *Let $a < 2b$, $c > 1$ and let $g(u)$ satisfy the conditions of Theorem 1. Then $\gamma(t) = 0$ if and only if the following hold:*

for $t < a$, $t \notin T$;

for $\max(bc, bc+a-b) \le t \le D$, condition III holds;

for $a \le t < bc$ precisely two of the three conditions I, II, III hold.

Proof. For $t < a$ the result is obvious.

Let $\max(bc, bc+a-b) \le t \le D$. The appropriate case is 4, so that $\gamma(t) = 0$ if and only if $P(t-a) = c$. Then either $t-a \in B_{c-1} \cap T_0$ or

https://doi.org/10.1017/S0004972700006183 Published online by Cambridge University Press

$t-a \in B_c \backslash T_0$. The second possibility is excluded since $t \leq D$ so that $t-a \in T_0$. The converse is derived by the same argument.

Now let $a \leq t < bc$. Consulting Case 2 we have $\gamma(t) = 0$ if and only if $\psi(t) = P(t) - P(t-a) = 0$. Since $\psi(t)$ is periodic with period b , we assume $t-a < b$. Then $P(t-a)$ is either 0 or 1 and we have the following cases.

Case A. $P(t-a) = 0$. Then $P(t) = 0$, so that $t < b$. It follows that t and $t - a$ are in the same class and both t and $t - a$ are not in T_0 . Conditions I and II are therefore satisfied but III is not. The proof goes of course both ways.

Case B. $P(t-a) = 1$. Then $P(t) = 1$. This is possible if and only if for the modified t we have $t \in B_0 \cap T_0$ or $t \in B_1 \backslash T_0$. Since $t-a \in B_0 \cap T_0$ it follows that III holds and either I or II, one of them excluding the other. This completes the proof of the theorem.

If $a < b$, we have $\gamma(a) = P(a) - 1 = 0$ for every representable a and $D-a \in T$ for $c > 1$. If $a = b + \delta$, $\delta \notin T$, then $\gamma(a) = P(a) - 1 = 0$ and again $D - a = (c-1)b - p - q - \delta \in T$, for $c > 1$. However we have the following

LEMMA 5. *Let* $a \in T_0$, $b \leq a \leq 2b$. *Then* $\gamma(t) = 0$ *if and only if either* $t \notin T$ *or* $D-t \notin T$.

Proof. We may assume $a < 2b$. Put $a = b + \tau$, $\tau \in T$. The numbers t and $t - a$ are clearly of different classes, so that I is not satisfied. We show that for $a \leq t \leq bc$ we have

$$P(t) - P(t-a) > 0 . \tag{3}$$

If t and $t - a$ are not in neighbouring classes, then (3) is proved. Assume t and $t - a$ in neighbouring classes. Let $t = kb + \delta$, $0 \leq \delta < b$. Condition I is not satisfied. We show that $t \notin T_0$ implies $t-a \notin T_0$. Let $t \notin T_0$. Then $\delta \notin T$, and $t - a = (k-1)b + \delta - \tau$. Since $t \in B_k$, we have $t-a \in B_{k-1}$, so that $\delta-\tau \geq 0$. $\delta - \tau = \sigma \in T$ implies $\delta = \tau + \sigma \in T$, a contradiction, so that $\delta-\tau \notin T$ and hence $t-a \notin T_0$. Thus two of the three conditions I, II, III cannot be satisfied and the result follows from Theorem 2.

Summing up we come to the following theorem assuming a representable.

THEOREM 4. *The* $(p-1)(q-1)$ *vanishing coefficients* $\gamma(t)$ *of the expansion into a polynomial for* $g(u)$ *, with* $t \notin T$ *or* $D-t \notin T$ *, are the only vanishing coefficients if and only if at least one of the following conditions is satisfied:*

1. $c = 1$;
2. $a \in T_0$, $b \leq a < 2b$;
3. $a \geq 2b$.

References

[1] Alfred Brauer, "On a problem of partitions", *Amer. J. Math.* 64 (1942), 299-312.

[2] Emil Grosswald, "Reducible rational fractions of the type of Gaussian polynomials with only non-negative coefficients", *Canad. Math. Bull.* 21 (1978), 21-30.

Department of Mathematics,
Israel Institute of Technology,
Technion,
Haifa,
Israel.