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UNITS IN GROUP RINGS OF THE INFINITE DIHEDRAL GROUP

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ABSTRACT. This paper studies the group of units $U(RD_\infty)$ of the group ring of the infinite dihedral group D_∞ over a commutative integral domain R. The structures of $U(\mathbb{Z}_2D_\infty)$ and $U(\mathbb{Z}_3D_\infty)$ are determined, and it is shown that $U(\mathbb{Z}D_\infty)$ is not finitely generated.

1. **Introduction.** One of the fundamental problems in group ring theory is the Isomorphism Problem: given a ring isomorphism $RG \approx RH$, can we claim that the groups G and H are isomorphic? In general the answer is no (see [1], Chapter 3) but the problem is still open when $R = Z$ —the ring of rational integers—the case most interesting for topologists.

Recently a substantial progress was made for finite groups due to works of K. Roggenkamp with L. Scott [1] and A. Weiss [3].

One of possible ways of the attack of the Isomorphism Problem is to study the group of invertible elements of the ring RH: any ring homomorphism $RG \longrightarrow RH$ maps G into this group. A lot is known about units in RG for finite groups ([2]). On the other hand, the domain of infinite groups remains still to be investigated.

In this paper we deal with invertible elements in group rings RD_∞ where R is a commutative domain with unity and D_∞ stands for the infinite dihedral group. In Section 3 we describe a certain subgroup of $U(RD_\infty)$ whose structure depends only on the structure of the additive group of R. This subgroup appears to be the whole group of units in cases $R = \mathbb{Z}_z$ and $R = \mathbb{Z}_3$. This fact enables us to describe in Section 4 the structure of groups $U(\mathbb{Z}_2D_\infty)$, $U(\mathbb{Z}_3D_\infty)$ and prove that the group $U(\mathbb{Z}D_\infty)$ is not finitely generated.

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2. **Notation.** In this paper R will be a commutative domain with unity. The infinite dihedral group can be defined as the semi-direct product of the infinite cyclic group C_∞ with the cyclic group of order two C_2:

$$D_\infty = C_\infty C_2 = \langle t \rangle \langle x \rangle$$

with a well-known presentation $D_\infty = \langle t, x | x^2 = 1, xt = t^{-1}x \rangle$. Since each element of the group D_∞ can be written as t^i or t^ix for some $i \in \mathbb{Z}$ we can write any element $\alpha \in RD_\infty$ in the form:

$$\alpha = \sum_{i\in\mathbb{Z}} \alpha_i t^i + \sum_{i\in\mathbb{Z}} b_i t^i x = a + bx \text{ where } a, b \in RC_\infty.$$

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Let $*: RC_\infty \longrightarrow RC_\infty$ be the involution of the group ring RC_∞ which comes from the non-trivial automorphism of the group C_∞, i.e., let $(\sum a_i t^i)^* := \sum a_i t^{-i}$. Using the relator $xt = t^{-1}x$ one can easily see that for any $a \in RC_\infty$ the relation $xa = a^*x$ holds. In particular: $(a+bx)(c+dx) = (ac+bd^*) + (ad+bc^*)x$. Applying this formula one can explicitly embed the group ring RD_∞ into a matrix ring.

REMARK 2.1. The function $i_R: RD_\infty \longrightarrow M_2(RC_\infty)$ defined as

$$i_R(a+bx) = \begin{pmatrix} a & b \\ b^* & a^* \end{pmatrix}$$

is a monomorphism of rings. ∎

From now on we identify elements of the ring RD_∞ with their images in $i_R(RD_\infty) \subseteq M_2(RC_\infty)$.

Let $U(P)$ denote the group of invertible elements (units) of a ring P.

If $\alpha = a + bx \in U(RD_\infty)$ then there exists an element $\beta \in RD_\infty$ such that $\alpha\beta = 1$. Since RC_∞ is a commutative domain, determinants make sense and $\det\alpha \cdot \det\beta = \det(\alpha \circ \beta) = \det 1 = 1$, which shows that $\det\alpha \in U(RC_\infty)$, i.e., it is a unit in the group ring RC_∞. But RC_∞ contains only trivial units (see [2]) so $\det\alpha = rt^i$ for some $i \in \mathbb{Z}, r \in U(R)$. Moreover $rt^{-i} = (\det\alpha)^* = (aa^* - bb^*)^* = aa^* - bb^* = \det\alpha = rt^{-i}$; thus $i = 0$ and $\det\alpha = r \in u(R)$. Conversely, if $\det\alpha = r \in U(R)$ then

$$\alpha^{-1} = \begin{pmatrix} a^*r^{-1} & -br^{-1} \\ -b^*r^{-1} & ar^{-1} \end{pmatrix} \in i_R(RD_\infty) \subseteq M_2(RC_\infty).$$

Thus we have proved the following:

REMARK 2.2. If R is a commutative domain then:

$$\alpha = a + bx \in U(RD_\infty) \iff \det\alpha = aa^* - bb^* \in U(R)$$

∎

For $0 \neq a \sum_{i\in\mathbb{Z}} \alpha_i t^i \in RC_\infty$ we set:

$$\begin{aligned} \max a &:= \max\{\, i \mid a_i \neq 0\} \\ \min b &:= \min\{\, i \mid a_i \neq 0\} \\ \deg a &:= \max a - \min a = \max aa^* \end{aligned}$$

If $\alpha = a + bx \in RD_\infty$ is a non-trivial unit then $a \neq 0, b \neq 0$. By Remark 2.2 we have $aa^* - bb^* \in U(R)$ hence $\deg a = \max aa^* = \max bb^* = \deg b > 0$. We define $\deg\alpha := \deg a = \deg b$. For trivial units α we extend this definition by setting $\deg\alpha := 0$.

We consider special nilpotents in the group ring RD_∞:

$$\begin{aligned} n_{ij} &= (1 + \operatorname{sgn}(i)t^j x)t^{|i|}(1 - \operatorname{sgn}(i)t^j x) \\ &= (-t^{-|i|} + t^{|i|}) + \operatorname{sgn}(i)t^j(t^{-|i|} - t^{|i|})x \quad \text{for } i,j \in \mathbb{Z}. \end{aligned}$$

where sgn(i) stands for the sign of i.

In fact $(n_{ij})^2 = 0$ because

$$(n_{ij})^2 = (\ldots)(1 \pm t^j x)(1 \mp t^j x)(\ldots) = (\ldots)(1 - (t^j x)^2)(\ldots) = 0.$$

For any $r \in \mathbb{R}, i, j \in \mathbb{Z}$ the element $1 + rn_{ij}$ is a unit in RD_∞ as

$$(1 + rn_{ij})(1 - rn_{ij}) = 1 - r^2(n_{ij})^2 = 1.$$

We will further consider the subgroup of $U(RD_\infty)$ generated by all units of the above form, so it is useful to introduce the following notation:

$$U = \langle 1 + rn_{ij} \rangle_{i,j \in \mathbb{Z}, r \in \mathbb{R}}$$

For all $k > 0, j \in \mathbb{Z}$:

$$V_j^k = \langle 1 + rn_{ij} \rangle_{0 < i \le k, r \in \mathbb{R}}$$

$$W_j^k = \begin{cases} \langle 1 + rn_{ij} \rangle_{0 > i \ge -k, r \in \mathbb{R}} & \text{if char} R \neq 2 \\ \{1\} & \text{if char} R = 2 \end{cases}$$

Obviously, the groups $\{V_j^k\}_{k=1}^\infty$ (respectively $\{W_j^k\}_{k=1}^\infty$) form an ascending system. We set:

$$V_j = \varinjlim_k V_j^k, \quad W_j = \varinjlim_k W_j^k$$

Natural inclusions induce homomorphisms from the free products:

$$\Phi_k \colon \star_j V_j^k * \star_j W_j^k \longrightarrow U \text{ for } k > 0 \text{ and}$$
$$\Phi = \varinjlim_k \Phi_k \colon \star_j V_j * \star_j W_j \longrightarrow U.$$

By $l(w)$ we will denote the length of word w in a corresponding free product.

3. **A subgroup of obvious units in RD_∞.** Let us start from the description of groups V_j^k and W_j^k (in any place where we consider groups W_j^k we assume that char $R \neq 2$). If sgn(i) = sgn(l) then

$$n_{ij} \cdot n_{lj} = (\ldots)(1 - \text{sgn}(i) t^j x)(1 + \text{sgn}(l) t^j x)(\ldots) = (\ldots) \cdot 0 \cdot (\ldots) = 0,$$

therefore the function $R^k \twoheadrightarrow V_j^k$ ($R^k \twoheadrightarrow W_j^k$) given by the formula :

$$\begin{aligned}(r_1, \ldots, r_k) &\longrightarrow 1 + r_1 n_{1j} + \cdots + r_k n_{kj} \\ &= (-r_k t^{-k} - \cdots - r_1 t^{-1} + 1 + r_1 t^1 + \cdots + r_k t^k) \\ &+ t^j (r_k t^{-k} + \cdots + r_1 t^{-1} - r_1 t - \cdots - r_k t^k) x\end{aligned}$$

respectively

$$\begin{aligned}(r_1, \ldots, r_k) &\longrightarrow 1 + r_1 n_{-1j} + \cdots + r_k n_{-kj} \\ &= (-r_k t^{-k} - \cdots - r_1 t^{-1} + 1 + r_1 t^1 + \cdots + r_k t^k) \\ &+ t^j (-r_k t^{-k} - \cdots - r_1 t^{-1} + r_1 t + \cdots + r_k t^k) x.\end{aligned}$$

is an isomorphism from the additive group of R^k onto the multiplicative group V_j^k (W_j^k). Therefore we obtain isomorphisms $V_j^k \cong R^k$ ($W_j^k \cong R^k$) and $V_j \cong \oplus_{i>0} R$ ($W_j \cong \oplus_{i>0} R$).

LEMMA 3.1. *Let* $k > 0$ *and let* $\omega \in \star_{j\in\mathbb{Z}} V_j^k * \star_{j\in\mathbb{Z}} W_j^k$ *be a non-empty, reduced word with the last letter* g *(i.e.,* $l(wg^{-1}) < l(w)$*). If* $\Phi_k w = a + bx \in U \subseteq U(RD_\infty)$*, then:*

(*i*) $\deg \Phi_k w > 0$ *(in particular* Φ_k *is a monomorphism)*

(*ii*) $g \in V_j^k \iff \max(t^{-j}b + a) < \max\{\max a, \max t^{-j}b\}$ *or* $\min(t^{-j}b + a) > \min\{\min a, \min t^{-j}b\}$

$g \in W_j^k \iff \operatorname{char} R \neq 2$ *and:* $\max(t^{-j}b - a) < \max\{\max a, \max t^{-j}b\}$ *or* $\min(t^{-j}b - a) > \min\{\min a, \min t^{-j}b\}$.

PROOF. Induction on the length of word w. From the explicit form of elements of V_j^k, W_j^k (that is words of length one) we conclude that for all words of length one our Lemma holds. We also observe that

(v) for $c + dx \in V_j^k$ we have $c^* = -c, d^* = -t^{-2j}d, c + t^{-j}d = 1$,

(w) for $e + fx \in W_j^k$ we have $e^* = -e, f^* = -t^{-2j}f, e - t^{-j}f = 1$.

Now let us assume that lemma holds for words of the length $n \geq 1$. Let w be a reduced word with $l(w) = n + 1, w = v \cdot g, l(v) = n$. There exists $j \in \mathbb{Z}$ such that $g \in V_j^k$ or $g \in W_j^k$. Consider the case $g \in V_j^k$. Let $\Phi_k(v) = y = zx, g = c + dx, \Phi_k(w) = a + bx = (y + zx)(c + dx) = (yc + zd^*) + (yd + zc^*)x$. Since w is a reduced word, so that last letter of v does not belong to V_j^k and by the inductive assumption (ii) we obtain the following inequalities:

$$(3.1.1)\qquad \begin{aligned} \max(t^{-j}z + y) &\geq \max\{\max y, \max t^{-j}z\} \\ \min(t^{-j}z + y) &\leq \min\{\min y, \min t^{-j}z\}. \end{aligned}$$

We calculate

$$\begin{aligned} a &= yc + zd^* = yc - t^{-2j}zd = yc - t^{-j}z(t^{-j}d) = yc - t^{-j}z(1-c) \\ &= c(y + t^{-j}z) - t^{-j}z. \end{aligned}$$

From (3.1.1) it follows that

$$\max(c(y + t^{-j}z)) = \max c + \max(y + t^{-j}z) > \max t^{-j}z$$

which implies the equality $\max a = \max(c(y + t^{-j}z) - t^{-j}z) = \max(c(y + t^{-j}z))$. On the other hand by (3.1.1) we have $\max(y + t^{-j}z) \geq \max y$, so $\max a = \max(c(y + t^{-j}z)) = \max c + \max(y + t^{-j}z) > \max y$. Replacing "max" by "min" and repeating the above calculations we obtain $\min a < \min y$. Thus $\deg \Phi_k(w) = \deg a = \max a - \min a > \max y - \min y = \deg \Phi_k(v) > 0$. Similarly we obtain $\deg \Phi_k(w) > \deg \Phi_k(v) > 0$ for $g \in W_j^k$, which completes the inductive step for (i).

Now, we will show that implications "$\Rightarrow$" in part (ii) are valid. Let $g \in V_j^k$; then

$$\begin{aligned} t^{-j}b + a &= t^{-j}(yd + zc^*) + (yc + zd^*) = t^{-j}yd - t^{-j}zc + yc - t^{-2j}zd \\ &= t^{-j}d(y - t^{-j}z) + c(y - t^{-j}z) = (y - t^{-j}z)(c + t^{-j}d) = y - t^{-j}z. \end{aligned}$$

Therefore $\max(t^{-j}b + a) = \max(y - t^{-j}z) \leq \max\{\max y, \max t^{-j}z\}$. But we have shown that $\max y < \max a$. Using similar calculations and applying inequality (3.1.1) we obtain

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$\max t^{-j}b = \max(y - c(y + t^{-j}z)) = \max c + \max(y + t^{-j}z) > \max t^{-j}z$, so $\max(t^{-j}b + a) \leq \max\{\max y, \max t^{-j}z\} < \max\{\max a, \max t^{-j}b\}$. In analogous way for $g \in W_j^k$ we obtain $\max(t^{-j}b - a) < \max\{\max a, \max t^{-j}b\}$. Considering "min" instead of "max" one can easily verify that:

$$g \in V_k^j \Rightarrow \min(t^{-j}b + a) > \min\{\min a, \min t^{-j}b\} \text{ and}$$
$$g \in W_k^j \Rightarrow \min(t^{-j}b + a) > \min\{\min a, \min t^{-j}b\}.$$

Hence to complete proof of (ii) it is enough to show that left sides of equivalences (ii) exclude one another. First let us notice that if $\max(t^jb \pm a) < \max\{\max a, \max t^jb\}$ then $\max a = \max t^jb$ and hence $\max(t^jb \pm a) < \max t^jb = j + \max b$. Therefore if $\max(t^jb + \varepsilon a) < \max\{\max a, \max t^jb\}$ $\max(t^lb + \delta a) < \max\{\max a, \max t^lb\}$ for $j, l \in \mathbb{Z}$, $\varepsilon, \delta \in \{\pm 1\}$ then $\max(t^jb + \varepsilon a - \varepsilon\delta^{-1}(t^lb + \delta a)) = \max(t^jb - \varepsilon\delta^{-1}t^lb) = \max b + \max(t^j - \varepsilon\delta^{-1}t^l) < \max\{j + \max b, l + \max b\}$. Eventually we have $\max(t^j - \varepsilon\delta^{-1}t^l) < \max(l, j)$. But if $t^j - \varepsilon\delta^{-1}t^l \neq 0$ then $\max(t^j - \varepsilon\delta^{-1}t^l) = \max\{l, j\}$ which would lead to a contradiction, therefore $t^j - \varepsilon\delta^{-1}t^l = 0$, i.e., $j = 1$ and $\varepsilon = \delta$—what was to be shown. ■

Let R be a commutative domain with unity. By D we denote the group of trivial units of the group ring RD_∞. We have an obvious isomorphism $D \cong D_\infty \times U(R)$. The groups U, V_j, W_j are defined as in Section 2.

THEOREM 3.2. *Let $G = \langle U, D \rangle$. Then:*

(i) $U \cong \star_{j \in \mathbb{Z}} V_j * \star_{j \in \mathbb{Z}} W_j \cong \star_{\mathbb{Z}} \oplus_{\mathbb{N}} R^+$, where R^+ denotes the additive group of the ring R.

(ii) $G = UD$.

PROOF. (i) We have shown in the proof of Lemma 3.1 that $V_j \cong \oplus_{\mathbb{N}} R^+$ (and for $\operatorname{char} R \neq 2$: $W_j \cong \oplus_{\mathbb{N}} R^+$). In order to prove (i) we should check that the homomorphism $\Phi = \varinjlim_k \Phi_k : \star V_j * \star W_j \longrightarrow U$ is an isomorphism. Φ is an epimorphism because each generator $1 + rn_{ij}$ lies in the image of Φ. Φ is a monomorphism because for $1 \neq w \in \star_j V_j * \star_j W_j$ there exists $k \in \mathbb{N}$ such that $w \in \star_j V_j^k * \star_j W_j^k$ and then by Lemma 3.1. (i) $\Phi(w) = \Phi_k(w) \neq 1$.

(ii) In order to prove (ii) it is enough to verify that:

$$1^\circ: U \cap D = \{1\} \qquad 2^\circ: U \text{is a normal subgroup of } G$$

1°: If $1 \neq \alpha \in U$ then by Lemma 3.1.(i) $\deg \alpha > 0$ so $\alpha \notin D$.

2°: $D = D \times U(R)$. $U(R)$ is contained in the centre of RD_∞ so it is sufficient to show that $t \circ U \circ t^{-1} \subseteq U$ and $x \circ U \circ x^{-1} \subseteq U$.

We calculate:

$$\text{(3.2.1)} \qquad \begin{aligned} t(1 + rn_{ij})t^{-1} &= 1 + rtn_{ij}t^{-1} = 1 + rn_{i(j+Z)} \in U \\ x(1 + rn_{ij})x^{-1} &= 1 + rxn_{ij}x^{-1} = 1 + rn_{i(-j)} \in U \end{aligned}$$

which completes the proof of the Theorem. ■

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COROLLARY 3.3.

$$\langle \operatorname{im}\Phi_k, D\rangle = \operatorname{im}\Phi_k D$$

PROOF. From Lemma 3.1. (i) it follows that for $1 \neq \alpha \in \operatorname{im}\Phi_k$ holds: $\deg\alpha > 0$ so $\operatorname{im}\Phi_k \cap D = \{1\}$. But by (3.2.1) $\operatorname{im}\Phi_k$ is a normal subgroup of $\langle \operatorname{im}\Phi_k, D\rangle$ which complete the proof. ■

PROPOSITION 3.4. *The groups U, G are not finitely generated.*

PROOF. If $\alpha_1,\ldots,\alpha_n \in U$ then there exist $k \in \mathbb{N}$ such that: $\alpha_1\ldots,\alpha_n \in \Phi(\star_j V_j^k * \star_j W_j^k)$. Then $\langle \alpha_1\ldots,\alpha_n\rangle \subseteq \operatorname{im}\Phi_k$. But $1 + n_{(k+1)j} \notin \operatorname{im}\Phi_k$ because Φ_{k+1} is a monomorphism. Therefore $\langle \alpha_1,\ldots,\alpha_n\rangle \neq U$. Similarly if $\beta_1,\ldots,\beta_n \in G$ then by Theorem 3.2 there exists $k \in \mathbb{N}$ such that $\langle \beta_1,\ldots,\beta_n\rangle \subseteq \operatorname{im}\Phi_k D$ but by the Corollary 3.3, $1 + n_{(k+1)j} \notin \operatorname{im}\Phi_k D$ so $\langle \beta_1,\ldots,\beta_n\rangle \neq G$. ■

4. **Description of groups** $U(Z_2D_\infty), U(Z_3D_\infty)$. In this section the group G defined in Section 3 for the rings $\mathbb{Z}_2D_\infty$ and $\mathbb{Z}_3D_\infty$ will be denoted by G_2 and G_3 respectively.

THEOREM 4.1.

$$U(\mathbb{Z}_2D_\infty) = G_2 \cong (\star_{\mathbb{Z}} \oplus_{\mathbb{N}} \mathbb{Z}_2)D_\infty$$
$$U(\mathbb{Z}_3D_\infty) = G_3 \cong (\star_{\mathbb{Z}} \oplus_{\mathbb{N}} \mathbb{Z}_3)D_\infty$$

PROOF. In both cases it is enough to prove that trivial units together with units of the form $1 \pm n_{ij}$ for $i,j \in \mathbb{Z}$ generate the whole group of units. We will prove this fact for the group $U(\mathbb{Z}_3D_\infty)$ only but it can be easily seen that the proof is valid also for the group $U(\mathbb{Z}_2D_\infty)$.

Let $\alpha \in U(\mathbb{Z}_3D_\infty), \alpha = a + bx$. If $\deg\alpha = 0$ then α is a trivial unit. Hence we can assume that $\deg\alpha > 0$. Let $j = \max a - \max b$,

$$\varepsilon = -a_{\max a} \circ (b_{\max b})^{-1} = \pm 1,$$
$$k = \min\{\min(a + \varepsilon t^j b) - \min a,\ \max a - \max(a + \varepsilon t^j b)\}$$

Let us notice that $aa^* - bb^* = \pm 1$ (Remark 2.2)$\Rightarrow aa^* \neq bb^* \Rightarrow a \neq \pm t^j b \Rightarrow a + \varepsilon t^j b \neq 0$; thus k is well-defined. Moreover, $k \geq 1$ because $a_{\max a} = -\varepsilon b_{\max b}$ and $a_{\min a} = -\varepsilon b_{\min b}$—the second equality follows from the equality $aa^* - bb^* = \pm 1$ and the assumption: $\deg\alpha = \deg a = \deg b > 0$. As $a_{\max a} a_{\min a} = aa^*_{\max aa^*} = bb^*_{\max bb^*} = b_{\max b} b_{\min b}$ hence $a_{\min a} = b_{\max b} \circ b_{\min b} \circ (a_{\max a})^{-1} = -\varepsilon b_{\min b}$. We will show that for $s = 1$ or $s = -1$ holds $\deg(\alpha \circ (1 + sn_{(\varepsilon k)j})) < \deg\alpha$.

$$\begin{aligned}\alpha \circ (1 + sn_{(\varepsilon k)j}) &= [a + s(-t^{-k} + t^k)a + s(-t^{-k} + t^k)\varepsilon t^j b] + [\ldots]x \\ &= [a + s(-t^{-k}(a + \varepsilon t^j b) + t^k(a + \varepsilon t^j b))] + [\ldots]x.\end{aligned}$$

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Let $h = -t^{-k}(a + \varepsilon t^j b) + t^k(a + \varepsilon t^j b)$.

Directly from the definition of k we have:

$$\max h = \max t^k(a + \varepsilon t^j b) = k + \max(a + \varepsilon t^j b) \leq \max a \tag{4.1.1}$$

$$\min h = \min t^{-k}(a + \varepsilon t^j b) = -k + \min(a + \varepsilon t^j b) \geq \min a \tag{4.1.2}$$

and at least one inequality is an equality. First let us assume that equality occurs in (4.1.1). The $h_{\max h} = h_{\max a} \in \{\pm 1\} = \mathbb{Z}_3 - \{0\}$. Also $a_{\max a} \in \{\pm 1\}$ thus we can choose $s \in \{\pm 1\}$ in such a way that $\max(a + sh) < \max a$ ($s = -h_{\max h} \circ a^{-1}_{\max a}$). Similarly if equality occurs in (4.1.2) then we can choose $s \in \{\pm 1\}$ ($s = -h_{\min h} \circ a^{-1}_{\min a}$) such that $\min(a + sh) > \min a$. As a result we obtain:

$$\begin{aligned} \deg(\alpha \circ (1 + sn_{(\varepsilon k)j}) &= \deg(a + sh) = \max(a + sh) - \min(a + sh) \\ &< \max a - \min a = \deg a = \deg \alpha. \end{aligned}$$

Simple inductive argument completes the proof. ∎

THEOREM 4.2. *Groups $U(\mathbb{Z} D_\infty)$, $U(\mathbb{Z}_3 D_\infty)$ and $U(\mathbb{Z}_2 D_\infty)$ are not finitely generated.*

PROOF. For groups $U(\mathbb{Z}_3 D_\infty)$, $U(\mathbb{Z}_2 D_\infty)$ the assertion follows directly from Theorem 4.1 and Proposition 3.4. The homomorphism $\mathbb{Z} \longrightarrow \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ induces a homomorphism of rings $\Psi\colon \mathbb{Z} D_\infty \longrightarrow \mathbb{Z}_2 D_\infty$ which restricted to $U(\mathbb{Z} D_\infty)$ gives us a homomorphism of groups $\Psi\colon U(\mathbb{Z} D_\infty) \longrightarrow U(\mathbb{Z}_2 D_\infty)$. Let us notice that for $1 \circ x, 1 \circ t, n_{ij} \in \mathbb{Z} D_\infty$ we have: $\Psi(1 \circ x) = 1 \circ x \in \mathbb{Z}_2 D_\infty$, $\Psi(1 \circ t \in \mathbb{Z}_2 D_\infty$, $\Psi(n_{ij}) = n_{ij} \in \mathbb{Z}_2 D_\infty$ so $G_2 \subseteq \operatorname{im} \Psi$. But by Theorem 4.1 $G_2 = U(\mathbb{Z}_2 D_\infty)$ thus $\Psi\colon U(\mathbb{Z} D_\infty) \twoheadrightarrow U(\mathbb{Z}_2 D_\infty)$ is an epimorphism. Therefore $U(\mathbb{Z} D_\infty)$ cannot be a finitely generated group. ∎

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