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EVENTUAL POSITIVITY AND ASYMPTOTIC BEHAVIOUR FOR HIGHER-ORDER EVOLUTION EQUATIONS

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Semigroups of linear operators have long been used to study evolution equations in an abstract functional analytical setting. In many applications in natural and social sciences, we are often interested in modelling the time evolution of a quantity that is naturally positive: the population density of a species within a domain, the concentration of a chemical in a diffusion process, the price of a commodity in an economic model and so on. For linear models, the appropriate functional analytic setting is therefore frequently an ordered function space, such as a *Banach lattice*, and the time evolution of the system is then governed by what is known as a one-parameter semigroup of positive linear operators, or a *positive semigroup* for short.

The study of positive semigroups on Banach lattices is by now a classic topic, and much is known about their stability and asymptotic behaviour (see for example [3]). By contrast, the study of the more subtle phenomenon of *eventual positivity*, in which, roughly speaking, positivity only occurs for sufficiently large times, is still relatively young. Eventual positivity for matrix semigroups was studied in [15] and since then has developed into a sub-field in its own right, as a natural extension of the classical Perron–Frobenius theory for positive matrices. Motivated by a case study of the Dirichlet-to-Neumann semigroup [4], a systematic theory of eventually positive semigroups and resolvents on infinite dimensional Banach lattices was initiated by Daners *et al.* in [5, 6]. In subsequent years, the theory has developed in many different directions, as described in a recent survey article [12]. In applications, the theory is especially relevant to higher-order evolution equations [1, 8, 13], for which positivity preserving principles are not generally valid. The work in this thesis represents contributions to both the abstract theory of eventual positivity as well as applications to concrete evolution equations.

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Chapter 2 begins with a 'crash course' in the theory of eventually positive semigroups, featuring a concise review of selected highlights. Although most of this material is already present in [5, 6], the presentation in the thesis also draws on more recent developments in the theory, and thus features more unified and streamlined proofs. The larger part of Chapter 2 features new results that are published in [14] and concerns the phenomenon of *local eventual positivity*, which was first discovered in the context of fourth-order evolution equations [10, 11]. An operator-theoretic approach to local eventual positivity was initiated by Arora in [2], building upon the Banach lattice techniques of [5, 6]. However, due to restrictive spectral assumptions, these results are often unsuitable to analyse PDE problems on unbounded domains. The work in this chapter is a first step in bridging this gap.

The material in Chapter 3 is a slightly expanded presentation of results obtained in [7]. We study the polyharmonic evolution equation $u_t + (-\Delta)^{\alpha} u = 0$ on Euclidean space $\mathbb{R}^N$, where $\alpha > 1$, and the biharmonic evolution equation $u_t + \Delta^2 u = 0$ on an infinite cylinder $\mathbb{R} \times \Omega$, where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary. In both problems, we obtain a result on local uniform convergence of solutions, which imply local eventual positivity under natural assumptions on the initial data. The analysis on $\mathbb{R}^N$ makes use of a simple Fourier transform technique, which forms the foundation of the much more technical analysis on the infinite cylinder. In the latter case, we require detailed estimates on solutions to a parametrised family of fourth-order elliptic boundary-value problems, which is interesting in its own right. In contrast to the existing literature on such equations (for example, [9–11]), our methods do not rely on explicit formulae for heat kernels of the polyharmonic operators, and instead are based on spectral theory and analytic perturbation theory.

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