Is "Physical Randomness" Just Indeterminism in Disguise?

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The topic of this session is "physical randomness". It might be doubted whether such a subject exists, for definitions of randomness have hitherto almost all been mathematical in nature. The only exceptions of which I am aware are the preceding paper by Benioff [1] and a paper by Wesley Salmon [8][2]. These attempts to inject some empirical content into randomness are highly desirable. But anyone attempting to formulate a physically based definition of randomness should at some point make clear what the connection is (if any) with a more traditional notion of disorder - that of indeterminism. Repeated reference to quantum mechanical examples whenever physical randomness is discussed indicates that a primary motivation for considering physical randomness to be important is because of the current belief that data sequences associated with quantum mechanical experiments are irreducibly random. (As an indication of this, in any situation in which physical randomness is discussed, translate the remarks about quantum phenomena into remarks about coin tossing, and they will lose much of their interest.) We already have a large and growing literature concerned with the question of whether quantum mechanics is an indeterministic theory or not. The small, but also growing literature on randomness looks to be a potential contribution to that question. And so we surely ought to inquire whether physical randomness has any connection with indeterminism, and if so, whether the two are in fact separable concepts. I hope to show here what that connection between randomness and indeterminism is.

First, some terminological matters. We speak of indeterministic theories, but not of random theories. Phenomena or systems can behave both indeterministically and randomly, but sequences are only elliptically indeterministic, although often random. To tidy up the terminology, it is best to take indeterministic theories as basic. Systems can then be represented by model-theoretic structures, some of which will be models of those theories. Elements of sequences will be time slices of structures. If the structure is $\langle U, F_1, \ldots, F_n \rangle$, an element

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of a sequence will be $<F_{i_1}(t_k),\ldots,F_{i_j}(t_k)>$ for $i_j \leq n$. It is worth noting that we can easily incorporate data other than outcome results of experiments in this format. The binary 0-1 representation for sequences usually discussed is merely for convenience and to avoid inessential technical complexities. One thing we do not want is to distinguish randomness and indeterminism on the grounds that the former ordinarily involves only values of observable quantities, while the latter uses values of unobservable quantities. The empiricist origins of randomness naturally led to a focus on experimental outcomes, but I shall consider here a more general type of randomness, and the above representation allows for that. Most importantly, the state variables of the system can be among the F_{ij}. Also, the n-tuples comprising the sequence elements could contain values of parameters representing environmental conditions. This will allow, for example, the use of statistical tests employing information about the physical surroundings of the system. It is obvious that a coding function from n-tuples into {0,1} will allow a transformation into the standard binary form.

Following a distinction I made in [2], we can find two principal reasons for having a philosophical interest in definitions of randomness. Our curiosity might be aroused because a satisfactory definition of randomness will help in solving difficulties in the foundations of probability, in particular with relative frequency interpretations and their problem of the single case. Such a definition would also aid in deciding what is to count as a model of the measure theoretic theory of probability, and also perhaps with models of statistical explanation and statistical laws.[3] This type of interest relates to the domain approach to randomness, which claims that random sequences constitute the correct domain of the theory of probability. Alternatively, investigation might be stimulated by the belief that an adequate definition of randomness will go some way towards formulating a suggestive definition of disorder, irregularity, and hence perhaps of indeterminism. This approach we can call the irregularity approach. It is from this second interest that a connection between indeterminism and randomness under some physical guise is likely to come. Yet the inferred connection is based, I think, on the following simple but erroneous argument. While this argument may rarely be voiced explicitly, it seems to underlie many of the beliefs that randomness of some sort is relevant to indeterminism. The argument is: (Laplacian) determinism implies predictability, hence unpredictability must imply indeterminism. Both the complexity definitions of randomness, and the place selection function approaches have formalized, in their different ways, the idea that random sequences are unpredictable. Hence at least certain types of randomness should imply indeterminism, and so determinism should entail non-randomness. This quick argument is wrong, and the reason it is wrong is instructive. The incorrectness of the argument can most easily be seen for the case of the von Mises/Church definition of randomness which uses the invariance of the limiting frequency under recursive place selection functions as the central tool. Because we are con-

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cerned here with lack of predictability, we need a definition of indeterminism which, like all definitions of randomness, is essentially linguistic in form. The definition must thus avoid causal talk, or talk of laws, or talk of anything but the logical and mathematical structure of the theory covering the data sequences in question. It may thus, of course, characterize classes of systems which are correctly described by the theory, but it will not use any metaphysical attributes of the systems. The obvious candidate for the job is Richard Montague's definition of a deterministic theory, given in his 1962 paper of the same name [5]. This, roughly speaking, formalizes the idea that a theory is deterministic if, given two models of the theory which are in the same state at some time, then they are in the same state at all times.[4] A system is deterministic if it is a model for a deterministic theory. Montague conjectured in the paper that a system which is covered by a deterministic theory can produce sequences which are random in the von Mises/Church sense, but did not prove it. The conjecture is indeed correct. We thus have, reformulating the claim slightly:

Theorem: There is a theory which is deterministic in the sense of Montague which has as a model a system which produces sequences which are random in the sense of von Mises/Church.

Proof: (See Appendix)

This result is much stronger than the usual claims that classical, and hence presumably deterministic, processes such as coin flipping can give us random sequences, for in the present case, even given complete information about the state variables, we still have random outputs. There is no lack of knowledge at the root of this randomness. What moral can we draw from this result? The evident conclusion is that exactly the same kind of decisions which have proved troublesome in formulating a satisfactory definition of randomness have also been centrally important in the attempt to find an adequate definition of a deterministic theory. The theory which was constructed in the proof of the above theorem is such that the sequences satisfying the theory are arithmetically definable, but not effectively computable. Whether or not the lack of effective computability is a deficiency in a scientific theory is a serious question, and not one which I wish to pursue here. (For a discussion of this question, see Kreisel [3]). The important point is that some decision about the level of complexity beyond which a functional relationship is no longer a candidate for a scientific theory has to be made. Ever since Russell [7] pointed out that an unrestricted form of Laplacian determinism can be trivially satisfied, efforts to avoid his criticism have run into the difficulty of stipulating the level of simplicity which distinguishes scientific from non-scientific deterministic theories. There seems to be no reason to assume that a deterministic universe would be simple. Yet this is the same difficulty with which workers in randomness have been faced when deciding which characterization of the sets of random sequences should be used. When the heuristic idea of gambling systems was available, this was perhaps not too difficult, but fortunately, gambling and

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science have little in common. The survey of attempts to find the "correct" characterization of the measure one sets in Benioff [1] indicates the difficulty of the task.

Returning to the theorem, we see that as a general procedure, inferring indeterminism from randomness is unwarranted. Inferences of any other implication relations are easily shown to be equally risky. It is trivial that systems described by deterministic theories can give non-random outputs. What of the connection between indeterministic systems and their output sequences? Here the special case of indeterministic theories which are used to generate probability distributions over the output sequences immediately provides the answer, for it is well known that with probability one, such systems will produce random sequences, and with probability zero will produce non-random sequences, where "random" can be taken in the sense of any of the approaches which use sets of measure zero as their basis. Hence we may conclude that neither randomness (in the senses specified) nor indeterminism (again in the senses specified) implies the other.

The above arguments involved particular definitions of indeterminism and randomness, some of which, such as the place selection function approach, are flawed. We can, however, provide a more general argument to show that indeterminism does not imply physical randomness. To formulate this argument, I shall use only those features of randomness and indeterminism which either occur in the majority of definitions of the two concepts, or in terms of which most of the remaining definitions can be recast. For indeterminism, the only feature we shall need is that a system is indeterministic if two replicas of that system which are in the same state at some time are in different states at some other time. For randomness, the relevant feature we need is that a sequence is random if it belongs to a specified class of sets of measure one. Most of the modern definitions of randomness can be put into this canonical form, with an accompanying class of measure zero sets containing the non-random sequences. For example, definitions of physical randomness will have empirical predicates occurring in the definition of the unit measure sets, whereas for mathematical randomness, no such empirical considerations will be used. We do not need any more specific characterization of the classes of random sequences than this. The important thing to bring out is that the set measure is not just any measure, but is a probability measure. Without, I think, putting a misleading interpretation on the enterprise, we can call any definition of randomness which is explicitly in the canonical form, or may be reduced to it, a "statistical definition of randomness". (See [2], pp.418-419 for examples of how to recast definitions which are not obviously canonical into the statistical form.) The point at issue is now simply this: if we wish to use randomness criteria as a guide to indeterminism, we must not conflate the properties of "arising from an indeterministic process" and "arising from a stochastic process". There are at least two reasons why such a conflation must not be made.

First, good sense can be made of the notion of indeterminism without having to resort to probability. Secondly, in the very area in which the theory of physical randomness is seen to be so important, (i.e., quantum mechanics) the classical Kolmogorovian theory of probability cannot be uncritically applied. I want to make just one point about the second reason here, which I think is extremely important. The modern propensity interpretation of the theory of probability was originally introduced with the aim of clarifying the role played by probability in quantum mechanical systems (See [6]). There is good reason to believe that in order to be different from other types of dispositional property, propensities of a genuinely chancy sort only occur in indeterministic systems. Further, some attempts have now been made to show that propensities are indeed probabilities, by showing that they satisfy the axioms of the theory of probability. But quantum mechanical propensities, if they exist, cannot be an interpretation of ordinary theories of probability, simply because the algebraic structure of quantum mechanical events is non-Boolean. Indeed, it would be surprising if propensities could give such an interpretation of classical probabilities, for it is evident that propensities do not give a satisfactory interpretation of inverse probability relations, such as Bayes' theorem. The causal force of propensity statements renders them directional, in a way that probability statements are not. Hence the challenge "Show that propensities are probabilities" is potentially misleading if we think there is a unique theory of probability to be interpreted. The significant feature of propensities may yet turn out to be the fact that they are not (ordinary) probabilities.

Let me now detail the first of these claims. Consider the following hypothetical situation: A new type of phenomenon has been observed, and disordered sequences of results from identically prepared systems are observed. Taking a cue from quantum mechanics, we claim all the systems are in the same "pure" state which is not an "eigenstate" of the observable in question, and make the further assumption that the preparation procedure for the different systems leaves the systems independent in some reasonable physical sense. Then in multiple repetitions of the above experiment, the following sequence of results is always obtained:

$$x_j = 1 \quad \text{if } m_i < j \leq m_{i+1} \quad \text{for even } i$$

$$x_j = 0 \quad \text{if } m_i < j \leq m_{i+1} \quad \text{for odd } i$$

$$x_j = 1 \quad \text{if } 1 \leq j \leq m_1$$

where $m_i = 2^{n_i}$ and $\{n_i\}_{i=1}^{\infty}$ is an infinite sequence which is not recursively enumerable, and neither is its complement.[5] The above outcome sequences are not random on any statistical definition of randomness, simply because the independent and identically distributed systems

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have not given rise to a convergent relative frequency. Faced with this, we might try these ploys:

1) the systems changed state during the course of the experiment, most likely at each point m_i.

2) the systems are not independent.

3) the systems are not covered by a probability distribution at all, even though they are indeterministic.

After further testing, we might well eliminate 1) and 2), and be forced to the conclusion that the class of indeterministic phenomena is not coextensive with the class of probabilistic phenomena. Hence any attempt to formulate a definition of physical randomness which is statistical in nature will not provide necessary conditions for indeterminism. Another, simpler, example illustrates why much less is required to conclude that a system is indeterministic than using an elaborate system of randomness tests. Suppose we observe the pattern 0000...010 ...0000...which is generated under conditions similar to those mentioned in the first example. Here a single deviant result stands out against a vast background of uniformity. Holding to the supposition that each measurement was made on a system prepared in the same state, we have to conclude that the systems are indeterministic, and we do this irrespective of how non-random the sequence is. Recall that it is sufficient for two systems in the same state to give different outcomes under identical measurement procedures, when they have been prepared in the same initial state. We count the result of a measurement on a system as a state of a system here, so that even if the initial state is statistical, the final state, that is the result of the measurement, will still be different. It is very tempting when considering sequences to look upon randomness tests as providing the basis for a kind of crude inductivism where the game is to guess the "law" covering the sequence. But we are not looking at the analogue of a planetary orbit observed at successive instants here. The relevant state for examining the determinism or indeterminism of the system is usually not contained in the previous experimental outcomes, but is rather a state which is a function of external variables. In most cases detailed knowledge of the internal structure of the sequence is not needed to decide whether the systems are indeterministic or not.

What role does randomness of a physical sort play in indeterministic contexts? Primarily, it acts as a potential falsifier for probability hypotheses covering the phenomena. The original intention of the relative frequentists was that the sequence of results should be generated under identical conditions, i.e., that the initial state of the systems should be the same on each repetition. If the systems are deterministic, the same result will be got each time. If they are statistical states, then the sequence should satisfy the theorems of probability theory for the particular distribution covering the systems. The point of trying to pick out a subsequence with different probability characteristics, for example, was to show that the probability distribution covering that subsequence was different from that covering the rest of the sequence, and since the distribution is a function of the statistical state, the initial states for that subsequence must have been different. Generalizing the framework, we can incorporate probability measures which allow different states on different trials into the statistical definitions of randomness. Then, if the sequence passes the tests for randomness, we infer that we have the correct probability distribution over the sequence. But this, of course, is a more detailed piece of information than we need for deciding that the systems are indeterministic. For that, it is sufficient that the sequence is covered by some probability distribution--we do not need to know which it is. So in general, physical randomness can act in an important way in taking us past the crude appellation of indeterminism, and providing distinctions among various types of indeterminism. Thus, we can consider whether we have probabilistic indeterminism or non-probabilistic indeterminism; whether it is a type of probabilistic indeterminism where the distribution has statistically relevant factors or whether it has an invariant distribution; whether there are necessary conditions in the environment for producing an outcome, or whether it is an indeterminism devoid of necessary conditions and hence truly spontaneous. Clearly there is a great variety of indeterministic systems, and one merit of physical randomness will be that it allows us to classify at least some of them via the probability hypotheses covering the systems.

As a final point, I do want to mention one area of philosophy of science which is indirectly affected by randomness considerations. This is the area of axiomatics. One of the most persuasive arguments offered in favour of adopting the set-theoretic approach to axiomatizing scientific theories by, for example, Suppes, is that using this type of axiomatization enables theories which contain probability apparatus to be formalized much more easily than by restricting ourselves to axiom systems in first order languages. This is simply because most of the set-theoretic apparatus is already built into the structures which are defined by the axiomatization. It was considered, however, that the use of naive set theory would be sufficient to adequately handle any mathematical aspects of a physical theory. As the examples cited at the beginning of Benioff's paper indicate, this is not at all an unjustified assumption. Nevertheless, the result of Solovay [10] to the effect that some models of set theory do not contain sequences which are random under certain "strong" definitions of randomness, raises doubt about this assumption. If the strong definitions of randomness mentioned are taken as having some use in physics, it is clear that more attention will need to be paid to the set-theoretic foundations of the "semantic" view of theories, simply because difficulties will arise in just that area (i.e., of probabilistic theories) which was originally pointed to as illustrating the advantages of the set-theoretic approach. This is not, of course, a criticism of that approach, but merely an indication that caution may be needed in certain cases.

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Appendix

Theorem: There is a theory which is deterministic in the sense of Montague and which has as a model a sequence which is random in the sense of von Mises/Church.

Proof: The construction of a von Mises Kollectiv is given overleaf, where the relevant notation will be found. We have to show two things here. First that the sequence is given by a theory, and second that the theory is deterministic in Montague's sense. The explicit description of the theory governing the generated Kollectiv is:

$$\forall t\, \{R^{+}(t) \rightarrow (\delta(t) = 0 \vee \delta(t) = 1) \wedge \delta(0) = 1 \wedge [N(t) \rightarrow$$

$$(\exists! y\{N(y) \wedge G(y,m(y),t) = 1 \wedge \sim\exists t'(t' < t \wedge G(y,m(y),t') = 1)\} \rightarrow$$

$$\delta(t) = 1] \vee [\exists t'(t' < t \wedge G(y,m(y),t') = 1 \wedge \sim\exists t''(t' < t'' < t \wedge$$

$$G(y,m(y),t'') = 1\} \rightarrow \delta(t) = 1 - \delta(t')] \vee [\sim N(t) \wedge \delta(t) = 0]$$

In order for a theory to be Montague deterministic, it must be definable in terms of R,N,t,., together with individual constants and state-variables for the theory. Further, it must have the property that for any two models of the theory M_1, M_2, if they are in the same state at some time t, then they are in the same state for all times t. Formally, this is given by: if D_1, D_2, D_3,....are the (single-place) state variables for the theory, then the state of a system s at time t is $st_s(t) = \langle D_1(t),\ldots,D_n(t)\rangle$. For the theory at hand we have only a single state variable, given in the theory by δ. So if s, s' are models of the theory, and $st_s(t_0) = st_{s'}(t_0)$ for some $t_0 > 0$, then $st_s(t) = st_{s'}(t)$ for all $t > 0$. It is clear that the above theory is deterministic in this sense. To show that the theory is definable in terms of R,N,+,., it is enough to check that each part of the theory (excluding the initial stipulation of $R^{+}(t)$) is in the arithmetic hierarchy. For then each part will be definable in elementary arithmetic. The only difficulty is with $G(y,m(y),t) = G^{m_y}(t)$. By examining the definition of G^{m_n} we see that m_n is recursive, multiplication and complementation are the only operations used in forming G_n, and n is a function of the enumeration of the recursive functions used. This enumeration is π^0_2. Hence $G(y,m(y),t)$ is arithmetically definable, and thus the theory itself is definable in terms of R,N,+,. . We let $\mathfrak{R}$ denote the total recursive functions.

To construct the random sequence, which satisfies the theory, we adopt a technique of Jean Ville ([11] pp. 58-63). Let $\mathfrak{R}$ be enumerated (non-effectively) as $\{F_1, F_2,\ldots\}$. We construct a new set of selection functions $\{G_1, G_2,\ldots\}$ and then construct a sequence x using these new functions. This sequence is first shown to have a limiting frequency of 1/2 which is invariant under selection by the G_i's. It is then shown to be invariant under selection by the F_i's.

We subscript the G_i by the binary expression of its index, i.e.,

$G_{a_1 \ldots a_n}$ where $a_i \varepsilon \{0,1\}$. The F_i are in fact represented by $F_i = \{f^i_j\}^\infty_{j=1}$ where f_j is a (recursive) function from $\{0,1\}^j \to \{0,1\}$. Informally, f_j operates on the initial segment of length j of a sequence. If $f_j(x(j))=1$, the element x_{j+1} is selected. If $f_j(x(j))=0$, x_{j+1} is rejected. Denote by aF the function F itself when a=1, and $\overline{F}$ when a=0, where $\overline{F}$ is the selection function such that for every $\overline{f}_j \varepsilon \overline{F}$, $\overline{f}_j(x(j))=0$ iff $f_j(x(j))=1$. Further, denote by $F^{(m)}$ the function which selects the first m members that F does and then is identically zero afterward, i.e., $F^{(m)} = \{g_i\}$ where

$$g_i = f_i \text{ if } \sum_{j=1}^{i} f_j \leq m$$
$$= 0 \text{ if } \sum_{j=1}^{i} f_j > m.$$

Then, for every finite sequence $a_1 \ldots a_n \varepsilon \{0,1\}^n$,

$$G_{a_1} = a_1 F_1$$

$$G_{a_1 \ldots a_n} = \overline{G^{(m_1)}_{a_1}}\ \overline{G^{(m_2)}_{a_1 a_2}} \cdots \overline{G^{(m_{n-1})}_{a_1 \cdots a_{n-1}}}\ a_1F_1 a_2F_2 \ldots a_nF_n \qquad (1)$$

$$= \overline{G^{(m_{n-1})}_{a_1 \ldots a_{n-1}}}\ a_n F_n G_{a_1 \ldots a_{n-1}}$$

where $m_i = 2^{2i+2}$.

Multiplication of selection functions is pointwise as follows

$$FG = \{f_i g_i\}^\infty_{i=1}.$$

Hence FG picks x_j if and only if both F and G do.

<u>Theorem 1</u>. For any infinite 0-1 sequence x, an element x_j is selected by at most one $G^{(m_n)}_{a_1 \ldots a_n}$.

<u>Proof</u>: Suppose not. There are two cases: either $G^{(m_n)}_{a_1 \ldots a_n}$, $G^{(m_n)}_{\alpha_1 \ldots \alpha_n}$ both pick x_j, or $G^{(m_n)}_{a_1 \ldots a_n}$, $G^{(m_{n+k})}_{a_1 \ldots a_n \alpha_{n+1} \ldots \alpha_{n+k}}$ pick x_j.

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First case: $a_i \ldots a_n$, $\alpha_1 \ldots \alpha_n$ must differ in at least one element, say $a_m \neq \alpha_m$. But $G^{(m_n)}_{a_1 \ldots a_n}$ picks x_j only if $G_{a_1 \ldots a_n}$ does. This occurs only if $a_m F_m$ picks x_j. But then $\alpha_m F_m$ cannot, and so $G_{\alpha_1 \ldots \alpha_n}$ and hence $G^{(m_n)}_{\alpha_1 \ldots \alpha_n}$ cannot. Contradiction. Second case: Again if $G^{(m_{n+k})}_{a_1 \ldots a_n \alpha_{n+1} \ldots \alpha_{n+k}}$ picks x_j, then $G_{a_1 \ldots a_n \alpha_{n+1} \ldots \alpha_{n+k}}$ does. But by (1), $G^{(m_n)}_{a_1 \ldots a_n}$ does not. Contradiction.

Theorem 2. Every element x_j of an infinite sequence x is selected by some $G^{(m_n)}_{a_1 \ldots a_n}$. Proof: Suppose x_j is picked by no $G^{(m_n)}_{a_1 \ldots a_n}$.

Every x_j is picked by either F_i or $\bar{F}_i$. Since each F_i is recursive, determine which is the case, and then acquire the sequences $a_1, a_1 a_2, \ldots, \prod_{i=1}^{n} a_i \ldots$ where $a_i = 0$ or 1, and the corresponding $G^{(m_n)}_{a_1 \ldots a_n}$. Then it is clear that all $G_{a_1 \ldots a_m}$ must pick x_j, since $\prod_{i=1}^{m} a_i F_i$ picks it, by definition of a_i, and none of the $G^{(m_{m-1})}_{a_i \ldots a_{m-1}}$ do, by hypothesis. But since the $m \to \infty$, the initial segment of some $G_{a_1 \ldots a_m}$ must choose x_j, i.e., some $G^{(m_m)}_{a_1 \ldots a_m}$. Contradiction.

Corollary. For every infinite sequence x, each element x_j is picked by one and only one $G^{(m_n)}_{a_1 \ldots a_n}$. With this, we can now construct the random sequence. Put $x_1 = 1$. The following method constructs the sequence.

To determine the value of x_{n+1}, first find the $G^{(m_r)}_{a_1 \ldots a_r}$ which selects it.

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If x_{n+1} is the first element selected by $G^{(m_r)}_{a_1 \ldots a_r}$, put $x_{n+1}=1$. Otherwise, inspect the previous selection of $G^{(m_r)}_{a_1 \ldots a_r}$: if it is 1, put $x_{n+1}=0$, otherwise put $x_{n+1}=1$. The construction continues in this way.

<u>Theorem</u> <u>3</u>. $\lim_{n \to \infty} \frac{\sum_{i=1}^{n} x(i)}{n} = 1/2$ in any subsequence picked by a G_i.

<u>Proof</u>: Consider an arbitrary selection function $G_{a_1 \ldots a_n}$. Denote by $G_{a_1 \ldots a_n}(x(m))$ the <u>sequence</u> extracted from $x(m)$ by $G_{a_1 \ldots a_n}$. Consider a selection of the form $G^{(m_{n+m})}_{a_1 \ldots a_n\ \beta_1 \ldots \beta_m}$. From the inductive definition (1), it is clear that any element selected by this must be selected by $G_{a_1 \ldots a_n}$ also. From the corollary, any element in $G_{a_1 \ldots a_n}(x(m))$ belongs to exactly one $G^{(m_{n+m})}_{a_1 \ldots a_n\ \beta_1 \ldots \beta_m}$. Since $G_{a_1 \ldots a_n}(x(m))$ is finite, let $G^{(m_{n+m_o})}_{a_1 \ldots a_n \beta_1 \ldots\ \beta_{r_o}}$ be the highest indexed such G which picks an element from $x(m)$. (Note that we don't need to look at $G^{(m_i)}_{a_1 \ldots a_i}$ for $i < n$.)

<u>Lemma</u>. If the sequence $G^{(m_n)}_{a_1 \ldots a_n}(x(m))$ is non-empty, each of the sequences $G^{(m_i)}_{a_1 \ldots a_i}(x(m))$ contains exactly m_i terms, where $1 < i < n$.

<u>Proof</u>: Take arbitrary $i < n$. Then if $G^{(m_n)}_{a_1 \ldots a_n}$ selects an element of $x(m)$ so does $G_{a_1 \ldots a_i}$. But also if $G^{(m_n)}_{a_1 \ldots a_n}$ does, $G^{(m_i)}_{a_1 \ldots a_i}$ does <u>not</u>. So since $G^{(m_i)}_{a_1 \ldots a_i}$ picks the first m_i choices of $G_{a_1 \ldots a_i}$, this means all m_i

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choices have already been made.

So, since each $G^{(m_{n+i})}_{a_1 \ldots a_n \beta_1 \ldots \beta_i}(x(m))$ contains m_{n+i} terms, we have, where t is the total number of elements selected from x(m) by $G_{a_1 \ldots a_n}$,

$$t > \sum_{i=0}^{m_o - 1} m_{n+i} = 2^{2(n+m_o+2)} - 1 \quad . \qquad \text{(A)}$$

There are 2^{m_o} sequences of type $G_{a_1 \ldots a_n \beta_1 \ldots \beta_{m_o}}(x(m))$ hence

$$t \leq \sum_{i=0}^{m_o} 2^i m_{n+i} \quad .$$

The number of non-empty sequences of type $G^{(m_{n+i})}_{a_1 \ldots a_n \beta_1 \ldots \beta_i}(x(m))$ is r, where

$$r \leq 1 + 2 + .. 2^{m_o} < 2^{m_o+1} \quad .$$

If each such sequence has $t_1 \ldots t_r$ terms respectively, of which $d_1 \ldots d_r$ are equal to 1,

$$t = t_1 + \ldots + t_r$$

$$d = d_1 + .. + d_r \quad .$$

By the way we constructed x, we have

$$\frac{t_i}{2} \leq d_i < \frac{t_{i+1}}{2} \quad .$$

So $\quad \frac{t}{2} \leq d < \frac{t}{2} + r \leq \frac{t}{2} + 2^{m_o+1} \quad .$

From (A) we have

$$2^{m_o+n+1} < \sqrt{t}$$

so $\quad \frac{t}{2} \leq d < \frac{t}{2} + \sqrt{t}\, 2^{-n} .$

Hence, as $t \to \infty$ $\lim\limits_{t\to\infty} \frac{d}{t} = 1/2$.

<u>Theorem</u>. The constructed sequence is invariant under selection by the

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F_i's.

Proof: Consider an arbitrary F_n. We show the limiting frequency of 1's in the subsequence selected by F_n is 1/2. First, we assume that F_n selects an infinite number of elements from x. Note that this entails that such functions as the identically zero function are excluded by this restriction. However, the notion of invariance under place selection is intended to apply only to selection functions which do select an infinite subsequence, so this is not a serious restriction.

By definition, $G_{a_1 \ldots a_n} = G_{a_1}^{\overline{(m_1)}} G_{a_1 a_2}^{\overline{(m_2)}} \ldots G_{a_1 \ldots a_{n-1}}^{\overline{(m_{n-1})}} a_1 F_1 \ldots a_n F_n$.

Thus, for fixed n, and each index $a_1 \ldots a_{n-1}$ the $G_{a_1 \ldots a_{n-1} 1}$ are disjoint, in the sense that no two $G_{a_1 \ldots a_{n-1} 1}$, $G_{a'_1 \ldots a'_{n-1} 1}$ pick the same element of x. Further, except for a finite number of elements of x, each element of x, say x_j, is chosen by some $G_{a_1 \ldots a_{n-1}} 1$. Why? Because for each F_i, either F_i picks x_j or $\overline{F}_i$ does. So for some index $\alpha_1 \ldots \alpha_{n-i} 1$

$\alpha_1 F_1 \ldots, \alpha_{n-1} F_{n-1}$ all pick x_j. Further, each $G_{\alpha_1 \ldots \alpha_k}^{(m_k)}$ picks only m_k elements from x. Thus after some finite point of the sequence,

$G_{\alpha_1}^{\overline{(m_1)}}(x(j-1))\ G_{\alpha_1 \alpha_2}^{\overline{(m_2)}}(x(j-1)) . G_{\alpha_1 \ldots \alpha_{n-1}}^{\overline{(m_{n-1})}}(x(j-1)$ must all be equal to 1.

Let the number of elements not picked by any $G_{\alpha_1 \ldots \alpha_{n-1} 1}$ be N. Hence we can represent F_n as $\Sigma\, G_{\alpha_1 \ldots \alpha_{n-1} 1} + N$ where the summation is over all values of $\alpha_1 \ldots \alpha_{n-1}$, and the sum is over disjoint elements. Note that none of the N elements are picked by a $G_{\alpha_1 \ldots \alpha_{n-1} 1}$. Now, let $t_i(k)$ be the number of elements picked by G_i, where $1 \leq i \leq 2^{n-1}$ from x(k), and let $d_i(k)$ be the number of 1's amongst the $t_i(k)$. If $F_n(x(k))$ contains t (k) terms, and d(k) 1's, then

$$t(k) = \sum_{i=1}^{2^{n-1}} t_i(k) + N$$

and $d(k) = \sum_{i=1}^{2^{n-1}} d_i(k) + V$ where V is the number of 1's amongst the N. We must have $N/2 \leq V+N/2 \leq N+N/2$.
Now, by the construction of x for each i,

$$t_i(k)/2 \leq d_i(k) < t_i(k)/2 + \sqrt{t_i}\,2^{-n}.$$

Hence summing, we have

$$\sum_{i=1}^{2^{n-1}} \frac{t_i(k)}{2} \leq \sum_{i=1}^{2^{n-1}} d_i(k) < \sum_{i=1}^{2^{n-1}} \frac{t_i(k)}{2} + \sum_{i=1}^{2^{n-1}} \sqrt{t_i}\,2^{-n}.$$

$$\therefore \quad \frac{t(k) - N}{2} \leq d(k) - V < \frac{t(k) - N}{2} + \sum_{i=1}^{2^{n-1}} \sqrt{t_i}\,2^{-n}.$$

Then if $t(k) \to \infty$ as $k \to \infty$, we also have $t_i(k) \to \infty$, and

$$\frac{1}{2} - \frac{N}{2t(k)} \leq \frac{d(k)}{t(k)} - \frac{V}{t(k)} < \frac{1}{2} - \frac{N}{2t(k)} + \sum_{i=1}^{2^{n-1}} 2^{-n}/\sqrt{t_i(k)}$$

and hence $\lim_{k\to\infty} \frac{d(k)}{t(k)} = 1/2$.

Since n was arbitrary, the sequence x has limiting frequency 1/2 for any subsequence selected by any F_n. Note, however, that x converges to 1/2 from above.

Notes

[1]The research for this paper was partially supported by NSF grant # SOC77-08837. I should also like to thank Leonard Monk and Zeno Swijtink for helpful discussions.

[2]Salmon has now formulated an improved version of this paper.

[3]See for example Mackie [4] Chapter 9 and Salmon [9]. Many of Mackie's difficulties with the notion of "distributiveness" of statistical laws would be resolved if a satisfactory definition of randomness were available.

[4]Montague's definition refers to cases where the domain of the models is fixed. He later proves a result connecting this situation with one

where different domains are allowed. Although I retain his approach in the proof of the theorem, it seems preferable when considering these kinds of definitions of indeterminism to take the different domains idea as central, in order to avoid having to consider possible worlds. When I use an unformalized version of Montague's approach later in the paper, it is to be taken in that sense, so replicas of systems can be considered. Further, some elementary assumptions about the invariance of theories under linear time transformations seems desirable so that these repetitions need not be done simultaneously.

[5]This example is based on one suggested by Zeno Swijtink.

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