The optimal tennis serve: a mathematical model

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Mathematical modelling has several valuable properties that address a number of pedagogical issues. As such, it makes for an excellent classroom exercise. First and foremost, it reinforces the insight that mathematics is nearly ubiquitous. It is all around us, hiding in plain sight. Secondly, in contrast to textbook problems which are often highly artificial, mathematical modelling is decidedly real. Next, it serves to un-silo the curriculum. As mathematics students advance, they populate their toolbox with more and more tools. However, these tools are generally course specific. Algebra tools in algebra class, calculus tools in calculus class, and so on. In attacking a modelling problem, the student needs to decide what tool to pick up and how to apply it. Finally, modelling is empowering in that it allows, indeed it requires, the student to decide for herself what the salient features of a problem are and which features are extraneous and can safely be suppressed.

These last two characteristics are interrelated, of course. A student's mathematical limitations might require the suppression of certain details. Conversely, if the student feels that this suppression renders the model unrealistic, she might feel compelled to learn some mathematics with which she had previously been unfamiliar.

Below, we present an example that illustrates each of these characteristics.

Consider the tennis serve. The server stands just behind the baseline and slightly to one side of its midpoint and attempts to hit the ball into the service box on the other side of the net and diagonally opposite her position. There are two main decisions that the server must make: in what direction should she aim the ball and with how much force should she hit it. For any choice of direction, there are two extremes regarding force. The minimal amount of force that produces a legal serve has the ball just clear the net and land in the service box. We shall call this the short serve. The maximal amount of force will have the ball just make it into the service box without going past the far boundary of the box. We shall call this the long serve. For any given direction, we examine the distance between the points of landfall for the short serve and the long serve. The greater this distance is, the more leeway the server has between hitting the ball with too much force or too little. It is a measure of how forgiving the serve is. Below, we find the direction for which this distance is maximal and we deem it the optimal serve.

We begin by giving the dimensions of the regulation tennis court for singles and establishing the coordinate system that we shall be using. The court is 78 feet long and 27 feet wide. The net straddles the width of the court and at its midpoint is 3 feet high. The service box is 13.5 feet wide (half the width of the court) and 21 feet deep (see [1]). In Figure 1, below,

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we have labelled points and assigned them coordinates. We have also added the server, represented here by the thick black line, orthogonal to the plane of the court, with her outstretched arm holding her racquet directly above her head. We are taking the top of this line to be H feet above the ground. It is the point where the racquet hits the ball. The service box is outlined in bold.

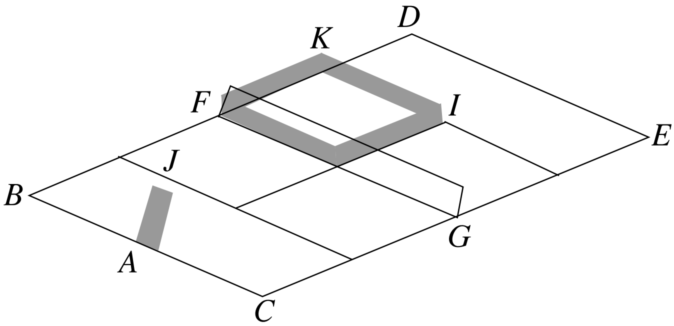

FIGURE 1

Here are the coordinates of the labelled points:
$A(0, 0, 0)$; $B(0, 13.5, 0)$; $C(0, -13.5, 0)$; $D(78, 13.5, 0)$; $E(78, -13.5, 0)$; $F(39, 13.5, 0)$; $G(39, -13.5, 0)$; $K(60, 13.5, 0)$; $I = (60, 0, 0)$; $J(0, 0, H)$.

The four corners of the court are labelled B, D, E and C, F, K and I represent three of the four corners of the service box.

Now that we have established dimensions and coordinates, let us create our model. We make several simplifying assumptions. First and foremost, we are eliminating the opponent. Our focus is entirely on the serve. Next, there is no friction. Consequently, the trajectory of the ball, which is acted upon solely by the initial impetus and gravity, will be a parabolic arc. We take the point of impact of the ball and the racquet to be the vertex of this parabola. Next, we shall view the ball as a single point and the lines that bound regions on the court will be considered to have no thickness. This last assumption makes the ball hitting a line and being within bounds a distinction without a difference. Likewise, the requirement that the server's feet be just behind the baseline and slightly to one side of the midpoint is ignored by making this assumption. As such, we are placing the server's feet at the origin, the point labelled A in Figure 1. The point of impact will be $(0, 0, H)$.

Some remarks regarding these choices are in order. Considering the high speed with which the served ball is travelling, one might consider removing gravity from the model and taking our trajectories to be straight lines. Doing so would simplify matters considerably. With linear trajectories, one could compute the distance from the server's feet to the point of landfall for the short serve directly from similar triangles.

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On the other hand, we might want to consider both gravity and friction. This would complicate matters considerably. First, we would need to use the real dimensions of the ball, instead of taking it to be a point. (Regulations require its diameter to be between 2.57 and 2.70 inches.) Next, we would have to consider the drag coefficient of the fuzzy exterior in the atmosphere whose viscosity is affected by temperature, humidity and altitude. Lastly, a truly realistic model would necessitate considering the direction and velocity of the wind.

All things considered, we have chosen the middle ground which includes gravity but suppresses friction. Such trade-offs between verisimilitude and simplicity are the hallmarks of mathematical modelling.

As we stated at the outset, our goal is to find the direction in which the distance between the points of landfall for the short and long serves is maximal. This distance is depicted by the dotted line in Figure 2, below. It connects the two points of landfall rendered as spheres. The short serve is rendered by a solid line and the long serve is depicted by a dashed line. The server with outstretched arm is shown as a thick vertical line, as it was in Figure 1.

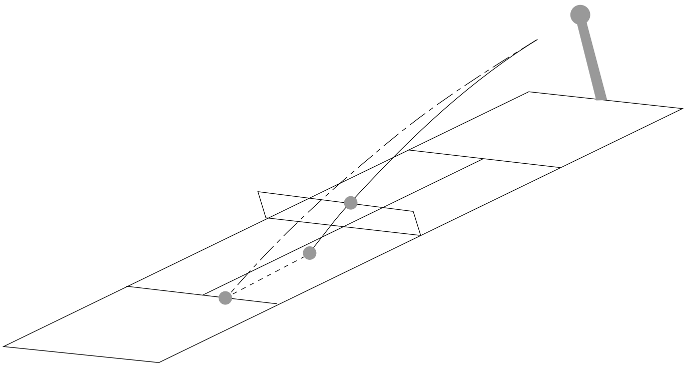

FIGURE 2

Let us now determine the point of landfall for the short serve in a given direction. Let us suppose that our server rotates her body through angle t in the anti-clockwise direction with $0 < t < \arctan\left(\frac{13.5}{60}\right) \approx 12.68°$. The reason for this upper bound for t is evident from Figure 3.

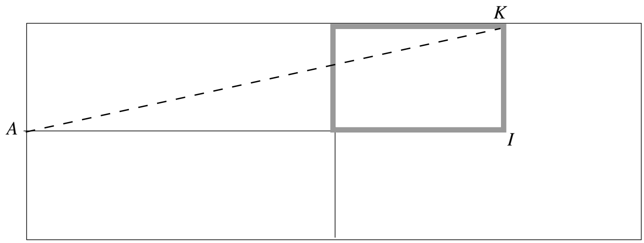

FIGURE 3

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Here we are viewing the court from above. The point labelled A is the origin in the xy-plane. This is where the server's feet are located. The service box is outlined in bold. The distance from A to I is 60 feet and the distance from I to K is 13.5 feet.

Having rotated through $t°$, the distance from the server's feet to the base of the net is $\frac{39}{\cos t°}$ in this direction. Now let us return to the xz-plane and imagine that a 'phantom net' was $\frac{39}{\cos t°}$ feet away. We seek the equation for the parabolic arc in the xz-plane that passes through the points $(0, H)$ and $\left(\frac{39}{\cos t°}, 3\right)$. Ordinarily, one would need three points to determine the coefficients of $z = ax^2 + bx + c$ but, as we have taken the point $(0, H)$ to be the vertex of the parabola, these two points are sufficient. We get

$$z = \frac{\cos^2 t° (3 - H)}{39^2} x^2 + H.$$

This is the equation for the trajectory of the ball in the xz-plane. However, we want the trajectory in the plane, $P(t)$, that is orthogonal to the xy-plane and makes an angle of $t°$ with the xz-plane. To this end, we rotate the parabola computed above about the z-axis, generating a paraboloid. The intersection of this paraboloid and the plane, $P(t)$, will be the curve we seek. We do this as follows:

We solve the quadratic equation for the trajectory for x in terms of z to get

$$x = \sqrt{\frac{(z - H) 39^2}{\cos^2 t° (3 - H)}}.$$

This will serve as our radius function allowing us to rotate our parabola about the z-axis, generating a paraboloid with the equation:

$$x^2 + y^2 = \frac{(z - H) 39^2}{\cos^2 t° (3 - H)}.$$

The plane, $P(t)$, will have equation: $\cos(t + 90)° x + \sin(t + 90)° y = 0$. This is equivalent to $(-\sin t°)x + (\cos t°)y = 0$ which simplifies to $y = (\tan t°)x$.

Finally, we intersect the plane, $P(t)$, with the paraboloid by solving for x in terms of z and y in terms of z to yield a parametric representation of the ball's trajectory in this plane. We get

$$x = 39\sqrt{\frac{z - H}{3 - H}}; \qquad y = 39 \tan t° \sqrt{\frac{z - H}{3 - H}}; \qquad z = z.$$

The point of landfall of the short serve in the direction of angle $t°$ is obtained by taking $z = 0$. This point will have coordinates:

$$\left(39\sqrt{\frac{-H}{3 - H}},\ 39 \tan t° \sqrt{\frac{-H}{3 - H}},\ 0\right).$$

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The distance from this point to the server's feet (the origin) is:

$$\frac{39}{\cos t^\circ}\sqrt{\frac{H}{H - 3}}.$$

(As an aside, we note that this value would have been $\dfrac{39H}{\cos t^\circ (H - 3)}$ had we chosen to use linear trajectories.)

The paraboloid, the plane, $P(t)$, and the curve of intersection, which is the trajectory of the ball of the short serve in the chosen direction, are all depicted in Figure 4, below. Here, we have distorted the scale in the interest of clarity.

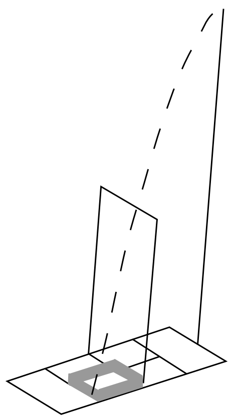

FIGURE 4

We now turn our attention to the point of landfall of the long serve. Unlike the case of the short serve, these calculations are trivial. For t° in the range $0 < t^\circ < \arctan \frac{13,5}{60}$, the coordinates of the point of landfall will be $(60, 60 \tan t^\circ, 0)$ and the distance of this point to the origin is $\frac{60}{\cos t^\circ}$. We note that this is entirely independent of the ball's trajectory.

Subtracting the distance from the origin to the point of landfall of the short serve from the distance we have just computed yields

$$D = \frac{1}{\cos t^\circ}\left(60 - 39\sqrt{\frac{H}{H - 3}}\right).$$

This distance is clearly minimal when $t = 0$ and increases as t increases. The maximal value that t can take on and still yield a legal serve is the arctangent of (13.5/60), as we saw from Figure 3. Consequently, the server should rotate her body through $t^\circ = \arctan \frac{13,5}{60}$ (approximately 12.68°) to achieve the most forgiving and, hence, optimal serve. In short, she must aim for the far corner of the service box.

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One final observation: We have intentionally kept H, the height above the court where the ball is hit by the racquet, as a parameter to gauge its effect on the results. For a fixed value of t, D is an increasing function of H. Therefore, the taller the player, the more leeway she will have in the degree of force she uses in her serve.

Reference

1. https://www.harrodsport.com/advice-and-guides/tennis-court-dimensions

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