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On finite groups admitting automorphisms with nilpotent fixed-point group

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Let p denote a prime, G a finite p'-group and A an elementary abelian group of operators on G. Suppose that A has order p^3 and that if $\omega \in A^\#$ then $C_G(\omega)$ is nilpotent. It is proved that G is nilpotent.

It is of considerable interest in the theory of groups to relate the structure of a group to the structure of the fixed-point groups of automorphisms acting on the group. For example, a result of Thompson ([1], 10.2.1) has proved that if an automorphism of prime order acts without non-trivial fixed-points on a finite group then the group is nilpotent. A result of Martineau [2] shows that if G is a finite group admitting an elementary abelian fixed-point-free group of automorphisms A of order r^2, r a prime, then G has a normal subgroup F such that F and G/F are nilpotent. Thompson's Theorem implies that the fixed-point group of any automorphism in A is nilpotent.

The purpose of this note is to prove another result of this kind. We use the notation of [1].

THEOREM. *Let p denote a prime, G a finite p'-group and A an elementary abelian group of operators on G. Suppose that A has order p^3 and that if $\omega \in A^\#$ then $C_G(\omega)$ is nilpotent. Then G is nilpotent.*

Proof. Assume by way of contradiction that the theorem is false and let G denote a counterexample of smallest possible order.

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First, let us suppose that G is soluble. By a well known argument, the Fitting subgroup $F = F(G)$ is the unique minimal normal A-subgroup of G. If N is any proper normal A-subgroup of G then N is nilpotent. Hence G/F is an elementary abelian r-group for some prime r and is irreducible under the action of A. Let R denote an A-invariant Sylow r-subgroup of G. R is elementary abelian, irreducible under the action of A, and complements F in G. Since A is abelian, if $\omega \in A$ then $C_R(\omega)$ is a subgroup of R which is normalised by A. Hence, for each $\omega \in A$, $C_R(\omega) = 1$ or $C_R(\omega) = R$. Since A is elementary abelian, $R = \left\langle C_R(\omega) \mid \omega \in A^{\#} \right\rangle$. Thus if $B = C_A(R)$ then $|A : B|$ is at most p. B normalises F, so $F = \left\langle C_F(\omega) \mid \omega \in B^{\#} \right\rangle$. For each $\omega \in B^{\#}$, $C_G(\omega) = RC_F(\omega)$. Since $C_G(\omega)$ is nilpotent if $\omega \in B^{\#}$, we can conclude that R centralises F. Thus G is nilpotent, contrary to our definition of G.

Thus we may suppose that G is not soluble. It is clear that G must be characteristically simple and so is a direct product of isomorphic, non cyclic, simple groups. Let q denote any odd prime divisor of $|G|$, and let Q denote an A-invariant Sylow q-subgroup of G. Then $N_G\left(Z\left(J(Q)\right)\right)$ is a proper A-subgroup of G and hence is nilpotent. In particular, $N_G\left(Z\left(J(Q)\right)\right)$ has a normal q-complement. By the Glauberman-Thompson Theorem ([1], 8.3.1), G has a normal q-complement. This is a contradiction and so completes the proof.

References

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[2] R.P. Martineau, "On groups admitting a fixed point free automorphism group II", (to appear).

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