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THE RECIPROCITY LAW FOR THE TWISTED SECOND MOMENT OF DIRICHLET *L*-FUNCTIONS OVER RATIONAL FUNCTION FIELDS

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Abstract

We prove the reciprocity law for the twisted second moments of Dirichlet L-functions over rational function fields, corresponding to two irreducible polynomials. This formula is the analogue of the formulas for Dirichlet L-functions over $\mathbb{Q}$ obtained by Conrey ['The mean-square of Dirichlet L-functions', arXiv:0708.2699 [math.NT] (2007)] and Young ['The reciprocity law for the twisted second moment of Dirichlet L-functions', *Forum Math.* **23**(6) (2011), 1323–1337].

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1. Introduction

The study of mean values of L-functions has a central role in analytic number theory, with many applications, such as the proportion of zeros satisfying the Riemann hypothesis, nonvanishing at the central points and subconvexity estimates. For all these applications, it is crucial to understand twisted moments, since their asymptotic behaviour is a key technical ingredient in most proofs.

In his investigations of the second moment of primitive Dirichlet L-functions [4], Conrey discovered an approximate reciprocity relation between twisted second moments for families of Dirichlet L-functions corresponding to two different prime moduli, p and h. More precisely, if we write

$$\mathcal{S}(p,h) := \sideset{}{^*}\sum_{\chi \pmod p} \left| L\left(\frac{1}{2},\chi\right)\right|^2 \chi(h),$$

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where the $*$ indicates the summation over all primitive characters modulo p, then Conrey showed [4, Theorem 10] that for primes p, h, with $2 \le h < p$,

$$\mathcal{S}(p,h) = \frac{p^{1/2}}{h^{1/2}}\mathcal{S}(h,-p) + \frac{p}{h^{1/2}}\Big(\log\frac{p}{h} + \gamma - \log(8\pi)\Big) + \zeta\Big(\frac{1}{2}\Big)^2 p^{1/2} + O\Big(h + \log p + \frac{p^{1/2}}{h^{1/2}}\log p\Big).$$

Here, the second term on the right dominates if $h = O(p^{1/2})$, but for larger h the first term can be larger. In any case this formula determines the asymptotic behaviour of $\mathcal{S}(p,h) - (p/h)^{1/2}\mathcal{S}(h,-p)$ uniformly for $h < p^{2/3}$.

Young [6] extended the range of uniformity of Conrey's formula, obtaining the following asymptotic formula which holds uniformly for primes p and h with $h < p^{1-\varepsilon}$:

$$\frac{p^{1/2}}{\varphi(p)}\mathcal{S}(p,h) - \frac{h^{1/2}}{\varphi(h)}\mathcal{S}(h,-p) = \frac{p^{1/2}}{h^{1/2}}\Big(\log\frac{p}{h} + \gamma - \log(8\pi)\Big) + \zeta\Big(\frac{1}{2}\Big)^2\Big(1 - 2\frac{p^{1/2}}{\varphi(p)}(1-p^{-1/2}) + 2\frac{h^{1/2}}{\varphi(h)}(1-h^{-1/2})\Big) + \mathcal{E}(p,h), \tag{1.1}$$

where the error term is bounded by

$$\mathcal{E}(p,h) \ll hp^{-1+\varepsilon} + h^{-C},$$

for all fixed $\varepsilon, C > 0$. Bettin [1] has made an extensive study of this error term. One particularly interesting feature discovered in [1, Theorem 1] is that $\mathcal{E}(h/p) := \mathcal{E}(p,h)$ can be extended to a continuous function $\mathcal{E}(x)$ with respect to the real topology.

Very recently, in [3] and [2], these reciprocity formulas are generalised for certain families of L-functions associated to automorphic forms. These 'spectral' reciprocity formulas reveal deep symmetries in those families and lead to new nonvanishing and subconvexity results.

The goal of this note is to study the reciprocity phenomenon in the context of Dirichlet L-functions over rational function fields. Let $\mathbb{F}_q[t]$ be the ring of polynomials over the finite field $\mathbb{F}_q$ of cardinality q, where q is a power of odd prime. For two irreducible polynomials $P, H \in \mathbb{F}_q[t]$, we are interested in the analogous twisted second moment

$$\mathcal{S}(P,H) = \sum_{\chi\,(\mathrm{mod}\,P)}^{*}\Big|L\Big(\frac{1}{2},\chi\Big)\Big|^2\chi(H),$$

where the $*$ again indicates the summation is over all primitive Dirichlet characters modulo the polynomial P.

Theorem 1.1. *For any two monic irreducible polynomials $P, H \in \mathbb{F}_q[t]$, with $H \ne P$ and $\deg(H) \le \deg(P)$, we have the following reciprocity formula between the twisted second moments:*

$$\frac{|P|^{1/2}}{\varphi(P)}\mathcal{S}(P,H) - \frac{|H|^{1/2}}{\varphi(H)}\mathcal{S}(H,-P) = \frac{|P|^{1/2}}{|H|^{1/2}}\Big(\deg(P) - \deg(H) - \mathcal{Z}\Big(\frac{1}{2}\Big)^2\Big) + \mathcal{Z}\Big(\frac{1}{2}\Big)^2\Big(1 - 2\frac{|P|^{1/2}}{\varphi(P)}(1-|P|^{-1/2}) + 2\frac{|H|^{1/2}}{\varphi(H)}(1-|H|^{-1/2})\Big), \tag{1.2}$$

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where $\mathcal{Z}(s)$ is the zeta-function for $\mathbb{F}_q[t]$, $\varphi(P)$ is Euler's totient function on $\mathbb{F}_q[t]$ and the norm $|P| = q^{\deg(P)}$.

Since $\deg(P)$ is the function field analogue of $\log p$, the formula (1.2) is in complete analogy with the formula (1.1) obtained for Dirichlet L-functions over $\mathbb{Q}$, except that in our case, there is no error term $\mathcal{E}(p, h)$, that is, the formula is an exact reciprocity formula. In particular, if P and H are two different irreducible polynomials of the *same* degree, we get the nice identity $\mathcal{S}(P, H) = \mathcal{S}(H, -P)$, that is,

$$\sum_{\chi \,(\mathrm{mod}\, P)}^{*} \left|L\left(\frac{1}{2}, \chi\right)\right|^2 \chi(H) = \sum_{\chi \,(\mathrm{mod}\, H)}^{*} \left|L\left(\frac{1}{2}, \chi\right)\right|^2 \chi(-P).$$

2. Background on Dirichlet L-functions over function fields

Throughout the paper we will mostly be working with monic polynomials. We denote by $\mathcal{M} \subseteq \mathbb{F}_q[t]$ the set of all monic polynomials. Additionally, for any integer $n \geq 0$, $\mathcal{M}_n \subseteq \mathcal{M}$ denotes the set of all monic polynomials of degree n. The cardinality of this set is $|\mathcal{M}_n| = q^n$.

The norm $|Q| = q^{\deg(Q)}$ of a nonzero polynomial $Q \in \mathbb{F}_q[t]$ is equal to the number of elements in the quotient ring $\mathbb{F}_q[t]/Q\mathbb{F}_q[t]$. Euler's totient function φ is defined by

$$\varphi(Q) := \#\{M \in \mathbb{F}_q[t]/Q\mathbb{F}_q[t] : (M, Q) = 1\} = |Q| \prod_{\substack{P \mid Q \\ P \in \mathcal{M}, \text{irreducible}}} \left(1 - \frac{1}{|P|}\right).$$

Recall that a Dirichlet character modulo a nonzero polynomial Q is a function $\chi : \mathbb{F}_q[t] \to \mathbb{C}$ satisfying $\chi(A + BQ) = \chi(A)$ and $\chi(AB) = \chi(A)\chi(B)$ for all $A, B \in \mathbb{F}_q[t]$ and such that $\chi(A) \neq 0$ if and only if $(A, Q) = 1$. The Dirichlet character χ_0 such that $\chi_0(A) = 1$ for all $(A, Q) = 1$ is called the trivial (or principal) character modulo Q.

The number of Dirichlet characters modulo Q is $\varphi(Q)$. A character χ is *primitive* modulo Q if there is no proper divisor $\tilde{Q}$ of Q so that $\chi(A) = 1$ whenever A is coprime to Q and $A \equiv 1 \pmod{\tilde{Q}}$.

Dirichlet characters satisfy the usual orthogonality relations: for any two Dirichlet characters $\chi, \psi \pmod{Q}$,

$$\frac{1}{\varphi(Q)} \sum_{A \,(\mathrm{mod}\, Q)} \chi(A)\overline{\psi}(A) = \begin{cases} 1 & \text{if } \chi = \psi, \\ 0 & \text{otherwise,} \end{cases}$$

where the sum is over any set of representatives for $\mathbb{F}_q[t]/Q\mathbb{F}_q[t]$, and for any two polynomials $A, B \in \mathbb{F}_q[t]$, coprime to Q,

$$\frac{1}{\varphi(Q)} \sum_{\chi \,(\mathrm{mod}\, Q)} \chi(A)\overline{\chi}(B) = \begin{cases} 1 & \text{if } A \equiv B \pmod{Q}, \\ 0 & \text{otherwise,} \end{cases}$$

where the sum is over all Dirichlet characters modulo Q. In particular, for an irreducible modulus P and polynomials A, B such that $(AB, P) = 1$, and for the

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summation over primitive (therefore nontrivial) characters modulo P,

$$\frac{1}{\varphi(P)}\sideset{}{^*}\sum_{\chi \,(\mathrm{mod}\, P)} \chi(A)\overline{\chi}(B) = \begin{cases} 1 - 1/\varphi(P) & \text{if } A \equiv B \pmod{Q}, \\ -1/\varphi(P) & \text{otherwise.} \end{cases} \tag{2.1}$$

The zeta function of $\mathbb{F}_q[t]$ is defined by the infinite series

$$\mathcal{Z}(s) := \sum_{N \in \mathcal{M}} \frac{1}{|N|^s} = \prod_{\substack{P \in \mathcal{M} \\ P \text{ irreducible}}} \left(1 - \frac{1}{|P|^s}\right)^{-1}, \quad \Re(s) > 1.$$

Since the number of monic polynomials of degree n is q^n, it follows that

$$\mathcal{Z}(s) = \frac{1}{1 - q^{1-s}},$$

which provides an analytic continuation of the zeta-function to the whole complex plane, with a simple pole at $s = 1$. In particular we will need the following special value:

$$\mathcal{Z}\left(\frac{1}{2}\right) = -\frac{1}{q^{1/2} - 1}. \tag{2.2}$$

The L-function corresponding to the Dirichlet character χ modulo Q is defined by

$$L(s, \chi) = \sum_{N \in \mathcal{M}} \frac{\chi(N)}{|N|^s} = \prod_{\substack{P \in \mathcal{M} \\ P \text{ irreducible}}} \left(1 - \frac{\chi(P)}{|P|^s}\right)^{-1}, \quad \Re(s) > 1.$$

But since

$$L(s, \chi) = \sum_{n=0}^{\infty} \frac{1}{q^{ns}} \sum_{N \in \mathcal{M}_n} \chi(N),$$

if we write $N = KQ + R$, with $\deg(R) < \deg(Q)$ or $R = 0$, for any nontrivial Dirichlet character $\chi \pmod{Q}$ and for all $n \geq \deg(Q)$,

$$\sum_{N \in \mathcal{M}_n} \chi(N) = \sum_{K \in \mathcal{M}_{n-\deg(Q)}} \sum_{R \,(\mathrm{mod}\, Q)} \chi(KQ + R) = \sum_{K \in \mathcal{M}_{n-\deg(Q)}} \sum_{R \,(\mathrm{mod}\, Q)} \chi(R) = 0.$$

In particular, for any nontrivial $\chi \pmod{Q}$, the Dirichlet L-function $L(s, \chi)$ is a *polynomial* in q^{-s}, of degree at most $\deg(Q) - 1$ [5, Proposition 4.3]:

$$L(s, \chi) = \sum_{n=0}^{\deg(Q)-1} \frac{1}{q^{ns}} \sum_{N \in \mathcal{M}_n} \chi(N).$$

Lastly, for any integer $m \geq 1$,

$$\sum_{\substack{M \in \mathcal{M} \\ \deg(M) \leq m}} \frac{1}{|M|^{1/2}} = \sum_{j=0}^{m} \sum_{M \in \mathcal{M}_j} \frac{1}{|M|^{1/2}} = \sum_{j=0}^{m} \frac{|\mathcal{M}_j|}{q^{j/2}} = \sum_{j=0}^{m} q^{j/2} = \frac{q^{(m+1)/2} - 1}{q^{1/2} - 1}. \tag{2.3}$$

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3. Proof of Theorem 1.1

We introduce the notation $\mathbf{p} := \deg(P) - 1$ and $\mathbf{h} := \deg(H) - 1$. Using the orthogonality relation (2.1)

$$\begin{aligned}
\frac{1}{\varphi(P)}\mathcal{S}(P,H) &= \frac{1}{\varphi(P)}\sideset{}{^*}\sum_{\chi \,(\mathrm{mod}\, P)} \Big|L\Big(\frac{1}{2},\chi\Big)\Big|^2 \chi(H) \\
&= \frac{1}{\varphi(P)}\sideset{}{^*}\sum_{\chi \,(\mathrm{mod}\, P)} \sum_{\substack{M\in\mathcal{M}\\ \deg(M)\le\mathbf{p}}} \frac{\chi(M)}{|M|^{1/2}} \sum_{\substack{N\in\mathcal{M}\\ \deg(N)\le\mathbf{p}}} \frac{\overline{\chi}(N)}{|N|^{1/2}}\chi(H) \\
&= \sum_{\substack{M,N\in\mathcal{M}\\ \deg(M),\deg(N)\le\mathbf{p}\\ MH\equiv N\,(\mathrm{mod}\,P)}} \frac{1}{|MN|^{1/2}} - \frac{1}{\varphi(P)} \sum_{\substack{M,N\in\mathcal{M}\\ \deg(M),\deg(N)\le\mathbf{p}}} \frac{1}{|MN|^{1/2}} \qquad (3.1)\\
&= \sum_{\substack{M,N\in\mathcal{M}\\ \deg(M),\deg(N)\le\mathbf{p}\\ MH\equiv N\,(\mathrm{mod}\,P)}} \frac{1}{|MN|^{1/2}} - \frac{1}{\varphi(P)}\Big(\frac{q^{(\mathbf{p}+1)/2}-1}{q^{1/2}-1}\Big)^2. \qquad (3.2)
\end{aligned}$$

Here, the second sum in (3.1) is evaluated by (2.3). Similarly,

$$\begin{aligned}
\frac{1}{\varphi(H)}\mathcal{S}(H,-P) &= \frac{1}{\varphi(H)}\sideset{}{^*}\sum_{\chi \,(\mathrm{mod}\, H)} \Big|L\Big(\frac{1}{2},\chi\Big)\Big|^2 \chi(-P) \\
&= \sum_{\substack{L,N\in\mathcal{M}\\ \deg(L),\deg(N)\le\mathbf{h}\\ -PL\equiv N\,(\mathrm{mod}\,H)}} \frac{1}{|LN|^{1/2}} - \frac{1}{\varphi(H)}\Big(\frac{q^{(\mathbf{h}+1)/2}-1}{q^{1/2}-1}\Big)^2. \qquad (3.3)
\end{aligned}$$

Returning to the evaluation of the twisted moment $\mathcal{S}(P,H)$, the contribution to the double summation in (3.2) of the terms for which $MH = N$ is equal

$$\frac{1}{|H|^{1/2}} \sum_{\substack{M\in\mathcal{M}\\ \deg(M)\le\mathbf{p}-\deg(H)}} \frac{1}{|M|} = \frac{1}{|H|^{1/2}} \sum_{j=0}^{\mathbf{p}-\deg(H)} \frac{|\mathcal{M}_j|}{q^j} = \frac{\mathbf{p}-\mathbf{h}}{|H|^{1/2}}.$$

For the off-diagonal contribution, that is, for the terms for which $MH \neq N$, we write $MH = N + PL$, where the polynomial $L \neq 0$. Since $\deg(N) < \deg(P)$ and since M, H, P are monic, we conclude that L also must be monic. Moreover, since $0 \le \deg(L) = \deg(M) + \deg(H) - \deg(P) = \deg(M) + \mathbf{h} - \mathbf{p} \le \mathbf{h}$, it follows that $\mathbf{p} - \mathbf{h} \le \deg(M) \le \mathbf{p}$. Therefore, after changing the variable M into L,

$$\sum_{\substack{M,N\in\mathcal{M}\\ \deg(M),\deg(N)\le\mathbf{p}\\ MH\equiv N\,(\mathrm{mod}\,P)\\ MH\neq N}} \frac{1}{|MN|^{1/2}} = \sum_{\substack{L,N\in\mathcal{M}\\ \deg(L)\le\mathbf{h},\deg(N)\le\mathbf{p}\\ PL\equiv -N\,(\mathrm{mod}\,H)}} \frac{1}{\big|\frac{N+PL}{H}N\big|^{1/2}} = \frac{|H|^{1/2}}{|P|^{1/2}} \sum_{\substack{L,N\in\mathcal{M}\\ \deg(L)\le\mathbf{h},\deg(N)\le\mathbf{p}\\ -PL\equiv N\,(\mathrm{mod}\,H)}} \frac{1}{|LN|^{1/2}},$$

since $|N + PL| = |P|\,|L|$. In the last sum, we split the summation into two parts, according to whether $\deg(N) \le \mathbf{h}$ or $\mathbf{h} < \deg(N) \le \mathbf{p}$. The first contribution is exactly

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the sum that appears on the right-hand side in (3.3). On the other hand, the second sum is

$$\sum_{\substack{L,N\in\mathcal{M}\\ \deg(L)\le\mathbf{h}\\ \mathbf{h}<\deg(N)\le\mathbf{p}\\ -PL\equiv N \,(\mathrm{mod}\, H)}} \frac{1}{|LN|^{1/2}} = \sum_{\substack{L\in\mathcal{M}\\ \deg(L)\le\mathbf{h}}} \frac{1}{|L|^{1/2}} \sum_{\substack{N\in\mathcal{M}\\ \mathbf{h}<\deg(N)\le\mathbf{p}\\ N\equiv -PL \,(\mathrm{mod}\, H)}} \frac{1}{|N|^{1/2}}. \tag{3.4}$$

For fixed L, let R be the unique polynomial with $\deg(R) < \deg(H)$ and such that $-PL \equiv R \pmod{H}$. Then the set of polynomials N appearing in the inner sum is

$$\{HS + R \mid S \in \mathcal{M}_{\le\mathbf{p}-\mathbf{h}-1}\}.$$

For all polynomials in this set, $|N| = |HS + R| = |HS|$, so the contribution of (3.4) is equal to

$$\frac{1}{|H|^{1/2}} \sum_{\substack{L\in\mathcal{M}\\ \deg(L)\le\mathbf{h}}} \frac{1}{|L|^{1/2}} \sum_{\substack{S\in\mathcal{M}\\ \deg(S)\le\mathbf{p}-\mathbf{h}-1}} \frac{1}{|S|^{1/2}} = \frac{1}{|H|^{1/2}} \frac{q^{(\mathbf{h}+1)/2}-1}{q^{1/2}-1} \frac{q^{(\mathbf{p}-\mathbf{h})/2}-1}{q^{1/2}-1},$$

by (2.3).

Therefore, after collecting everything together,

$$\begin{aligned}\frac{1}{\varphi(P)}\mathcal{S}(P,H) &= \frac{\mathbf{p}-\mathbf{h}}{|H|^{1/2}}\\ &+ \frac{|H|^{1/2}}{|P|^{1/2}}\bigg(\frac{1}{\varphi(H)}\mathcal{S}(H,-P) + \frac{1}{\varphi(H)}\bigg(\frac{q^{(\mathbf{h}+1)/2}-1}{q^{1/2}-1}\bigg)^2 + \frac{1}{|H|^{1/2}} \frac{q^{(\mathbf{h}+1)/2}-1}{q^{1/2}-1} \frac{q^{(\mathbf{p}-\mathbf{h})/2}-1}{q^{1/2}-1}\bigg)\\ &- \frac{1}{\varphi(P)}\bigg(\frac{q^{(\mathbf{p}+1)/2}-1}{q^{1/2}-1}\bigg)^2.\end{aligned}$$

Here, we recall (2.2) and after some elementary transformations, we arrive at the required formula (1.2).

References

[1] S. Bettin, 'On the reciprocity law for the twisted second moment of Dirichlet L-functions', *Trans. Amer. Math. Soc.* **368**(10) (2016), 6887–6914.

[2] V. Blomer and R. Khan, 'Twisted moments of L-functions and spectral reciprocity', arXiv:1706.01245 (2017).

[3] V. Blomer, X. Li and S. D. Miller, 'A spectral reciprocity formula and non-vanishing for L-functions on GL(4) × GL(2)', arXiv:1705.04344 (2017).

[4] J. B. Conrey, 'The mean-square of Dirichlet L-functions', arXiv:0708.2699 [math.NT] (2007).

[5] M. Rosen, *Number Theory in Function Fields*, Graduate Texts in Mathematics, 210 (Springer, New York, NY, 2000).

[6] M. P. Young, 'The reciprocity law for the twisted second moment of Dirichlet L-functions', *Forum Math.* **23**(6) (2011), 1323–1337.

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