Science in Context (2022), **35**, 49–80
doi:10.1017/S0269889723000145

RESEARCH ARTICLE

Euclid's Fourth Postulate: Its authenticity and significance for the foundations of Greek mathematics

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Argument
The Fourth Postulate of Euclid's *Elements* states that all right angles are equal. This principle has always been considered problematic in the deductive economy of the treatise, and even the ancient interpreters were confused about its mathematical role and its epistemological status. The present essay reconsiders the ancient testimonies on the Fourth Postulate, showing that there is no certain evidence for its authenticity, nor for its spuriousness. The paper also considers modern mathematical interpretations of this postulate, pointing out various anachronisms. It further discusses the validity of the ancient proof by superposition of the Fourth Postulate. Finally, the article proposes an interpretation of the history of the concept of angle in Greek geometry between Euclid and Apollonius, and puts forward a conjecture on the interpolation of the Fourth Postulate in the Hellenistic age. The essay contributes to a general reassessment of the axiomatic foundations of ancient mathematics.

Keywords: Euclid; axioms; history of mathematics; Greek mathematics; ancient philosophy; foundations of mathematics

The theory of angles in Euclid's *Elements* (third century BCE) is grounded on three definitions and a postulate. The definitions state that an *angle* (γωνία) is an inclination between lines, a *rectilinear angle* (εὐθύγραμμος γωνία) is an angle formed by straight lines, and a *right angle* (ὀρθὴ γωνία) is a rectilinear angle formed by two straight lines making equal adjacent angles.[1]

The definitions are complemented by a postulate stating that:

> All right angles are equal to one another.
>
> ᾐτήσθω … πάσας τὰς ὀρθὰς γωνίας ἴσας ἀλλήλαις εἶναι.

This Fourth Postulate (henceforth P4) has elicited mixed reactions over the centuries. On the one hand, modern mathematicians recognize in it an important principle of geometry, and praise Euclid highly for having correctly pinpointed a fundamental assumption underlying elementary geometry. On the other hand, philosophers are confused by its ascription to the list of the five postulates: the others seem to be constructive principles (e.g. "to draw a straight line from any point to any point"), whereas P4 spells out a state of affairs.

[1] These are Definitions 8, 9 and 10 in Book I: ἐπίπεδος δὲ γωνία ἐστὶν ἡ ἐν ἐπιπέδῳ δύο γραμμῶν ἁπτομένων ἀλλήλων καὶ μὴ ἐπ᾽ εὐθείας κειμένων πρὸς ἀλλήλας τῶν γραμμῶν κλίσις. ὅταν δὲ αἱ περιέχουσαι τὴν γωνίαν γραμμαὶ εὐθεῖαι ὦσιν, εὐθύγραμμος καλεῖται ἡ γωνία. ὅταν δὲ εὐθεῖα ἐπ᾽ εὐθεῖαν σταθεῖσα τὰς ἐφεξῆς γωνίας ἴσας ἀλλήλαις ποιῇ, ὀρθὴ ἑκατέρα τῶν ἴσων γωνιῶν ἐστι, καὶ ἡ ἐφεστηκυῖα εὐθεῖα κάθετος καλεῖται, ἐφ᾽ ἣν ἐφέστηκεν. It may be worth mentioning that the definitions in the manuscripts are not numbered and appear as a single block of text. In particular, we have an ancient papyrus containing these three definitions which presents some small textual variants, showing that definitions 8 and 9 were conceived as a single sentence, and definition 10 as a separate sentence (Turner et al. 1985).

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https://doi.org/10.1017/S0269889723000145 Published online by Cambridge University Press

This philosophical perplexity is not new, and as early as the first century BCE Geminus suggested that this principle was ill-placed among the postulates and that it should have been repositioned among Euclid's *common notions* (κοιναὶ ἔννοιαι): a different set of principles that, according to Geminus, included assertions rather than constructive principles (e.g. "if equals be added to equals, the wholes are equal"). A further reason for perplexity was occasioned by the fact that Geminus also reported a *proof* of the Fourth Postulate, as if this alleged fundamental statement had no real foundational import and could easily be taken as a theorem in the *Elements*.

Geminus' opinions were often debated in the following Euclidean tradition. The early modern editions of the *Elements*, for instance, unconcerned with philological accurateness, listed P4 among the common notions, or proved it among the theorems, or even accepted it as a common notion equipped with a demonstration. Present-day historians struggle with it, making attempts to show that the postulate is constructive after all, or arguing why Euclid could not accept its proof. The mathematical meaning of the principle is also debated, as is its foundational role in the deductive structure of the *Elements*. The outcome of this research is a general puzzlement on P4, and most of today's interpreters simply settle on a *non liquet*.

In the present paper, I attempt a general reassessment of the significance of the Fourth Postulate in Greek mathematics and address the question of its authenticity. I begin with a textual analysis in section 1, where I offer a discussion on the embedding of P4 in the text of the *Elements* and present ancient testimonies on it. I show that, contrary to what happens with the other principles in the *Elements*, we have no positive evidence for either the authenticity or the spuriousness of P4. The two following sections are devoted to mathematical analyses: in section 2, I challenge various modern interpretations of the meaning of P4, and in section 3 I further examine the ancient proof of P4 and its alleged shortcomings. The conclusions of these three sections are largely aporetic, but nevertheless establish a few important *negative results* regarding what the Fourth Postulate cannot be, and what we do not know about it. I consider these negative results as the main outcome of the present essay, but I also complement them with a *positive conjecture* in section 4, according to which P4 was not spelled out by Euclid himself and was rather added in a further Hellenistic stratification of the text of the *Elements*. I argue that this may have happened in the generations immediately after Euclid, taking a cue from Apollonius' new definition of angles. While this conjecture is destined to remain unproven, it may solve several historical puzzles surrounding the Fourth Postulate, including the *vexata quaestio* of its incongruous ascription to the list of postulates.

The present essay may be seen as a particular contribution to a broader study of the historical foundations of Euclidean geometry. The conjecture advanced in the last section, indeed, would allow for a unitary reading of what I argue are Euclid's *four* original postulates (the first three, and the fifth). It is not my intention to advance here a general interpretation of the epistemological meaning of Euclid's postulates. I would like to stress, nevertheless, that by conjecturing that Euclid himself did not state P4, I do not intend to suggest that he had *overlooked* an important principle that was later recognized as essential for the foundations of geometry. Rather, I would argue that Euclid's "ideal of proof" was not a gapless sequence of propositional inferences licensed by axioms and previous theorems. In a context that allowed for diagrammatic reasoning and other unaxiomatized demonstrative techniques (such as superposition), postulates did not have the modern function of licensing inferences. As a consequence, Euclid had no need to spell out the Fourth Postulate. He did acknowledge that all right angles are equal, and he did prove theorems requiring this assumption, but he did not believe that it had to be made explicit as a postulate or a common notion—these principles simply played a different role in his geometry.

1. Textual evidence for the Fourth Postulate

The authenticity of the Fourth Postulate may seem, at first, to be beyond question. No ancient source ever doubted its authenticity, and all the manuscripts of the *Elements* that we have—be

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they in Greek, Arabic or Latin—as well as all the indirect sources mentioning Euclid's principles, include P4 in the list of postulates. The reconstruction of the textual traditions of the *Elements*, therefore, has to include P4 in all the lines of transmission of the ancient text, and there is no question of there being an archetype of the *Elements* which lacks this principle. The philological question is easily settled.

Yet, it must also be acknowledged that the tools of philology do not allow us to go back to the early stratifications of the text: the earliest extant manuscripts containing the *Elements* date to the ninth century CE, some twelve centuries after Euclid. Assuming that Proclus' reference to Geminus is correct, we can state that P4 was listed among the postulates in the first century BCE, and it may have been added to the *Elements* in the two centuries intervening between Euclid and Geminus—during the flourishing of Hellenistic mathematics. Such an early interpolation is very difficult to ascertain, since the first centuries of the textual tradition of the *Elements* are shrouded in darkness and beyond the possibility of philological restoration.

A few interpreters in the past have indeed suggested that P4 was interpolated during this period. They did not advance any textual argument to this effect, though, merely using this possibility as a desperate escape route in order to account either for the odd presence of a non-constructive principle among the postulates or for the fact that P4 had an ancient proof.[2] I do not think that such preconceived views about the nature of postulates and proofs should guide historical research and count as real arguments for the excision of a principle from a text that otherwise offers no further reason for doubt.

At the same time, the textual situation is not as clear as one would like. While it is true that we have no textual reasons to put the authenticity of P4 in question, it is also true that we have *no positive evidence whatsoever* that it was counted among the principles in the *Elements* before the times of Geminus. This marks a sharp difference with the other principles in the *Elements*.

Johan Ludvig Heiberg's critical edition of the *Elements*, based on Byzantine manuscripts, lists some nine common notions and six postulates among Euclid's principles (their exact number changes from manuscript to manuscript). Among these, Heiberg himself identified four common notions and one postulate as spurious. Heiberg offered convincing, positive evidence for his claim: these principles were either explicitly reported as interpolations by ancient sources such as Proclus and Simplicius, or were missing from several other lines of transmission of the text of the *Elements* (such as Boethius, Censorinus, or the Aristotelian commentators). I myself have argued that two more common notions (on superposition and on the whole and the part), which had already been regarded as spurious by some other commentators, are indeed so, and I have provided positive evidence for their late interpolation in the text.[3]

The remaining three common notions are, by contrast, not just "possibly non-spurious" but positively authentic. The first common notion ("things which are equal to the same thing are also equal to one another") is quoted three times in the course of the demonstrations in the *Elements*, and is referred to by Hellenistic authors. The second and the third (on the addition and subtraction of equals) are quoted explicitly in Euclid's *Data*, and the third is mentioned several times by Aristotle as the main example of a mathematical axiom.[4]

[2]Among earlier interpreters who made claims for excision, I can mention here Paul Tannery, who simply thought that such a postulate was not needed in order to ground geometry, and therefore concluded that it seems "plus glorieux pour Euclide de n'avoir formulé que les trois postulats de construction" (Tannery 1884, 173). See also below in note 36 for Seidenberg's stance.

[3]See De Risi 2021b.

[4]The first common notion appears in *Elements* I, 1, 2 and 13; the second in *Data* 3; the third in *Data* 4 and 12. The evidence of their authenticity is even stronger than this, since some proofs (such as *Elements* I, 13) are formulated in a lengthier way in order to justify each deductive step by one of these common notions. The first common notion is mentioned in connection to Apollonius in Proclus, *In Euclidis* 194–95. Carneades' reference to it is to be found in Galen, *De optima doctrina*, § 2 (ed. Kuhn, vol. 1, 45). Aristotle's references to the third common notion are numerous: see for instance *An. post.* A 10, 76^b21 and 76^a41–42; A 11, 77^a26–31; *An. pr.* A 24, 41^b22–23; *Metaph.* K 4, 1061^b20.

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Similar supporting evidence exists for four postulates out of five. The Parallel Postulate is quoted by Euclid four times in the course of the demonstrations, and discussions on the theory of parallels were referred to by Aristotle and other ancient scholars.[5] The first three constructive postulates, licensing ruler-and-compass constructions, are seldom explicitly mentioned by Euclid (they are so pervasive in the *Elements* that it would be highly impractical to refer to them every time a construction is performed), but they are nonetheless mentioned time and again.[6] Note that we have no reason to doubt these internal quotations: they are present in all the lines of transmission of the text of the *Elements* and we have no clues that they may have been interpolated. In sum, we do have *positive evidence* for the spuriousness of a certain number of principles, and we do have *positive evidence* for the authenticity of the remaining principles.

The only exception is the Fourth Postulate. In all the demonstrations in the thirteen books of the *Elements*, P4 is never quoted as such, and Euclid never states that all right angles are equal—except in the list of principles.

The equality of all right angles is implicitly assumed innumerable times in the course of the *Elements*. It is very common, indeed, that in a demonstration Euclid assumes the equality of right angles in proving the congruence of two triangles having further sides or angles that are equal, or that he adds equal angles to right angles to obtain equal angles or, again, that he states that an angle sum of two right angles (e.g. the interior angle sum of a triangle) is indeed equal to the sum of another pair of right angles. Sometimes, these inferences do not require P4. There are occasions where, for instance, the compared right angles are adjacent to one another: in these cases, a simple appeal to the definition of a right angle (i.e. an angle which is equal to its adjacent angle) would establish their equality (a similar case is in the proof of *Elements* I, 13). But in many other cases the definition would not suffice, and Euclid would really need a stronger principle, such as P4, in order to justify the inference he makes.

In most cases in which a proof needs the equality of certain right angles, Euclid tacitly assumes their equality, making no explicit reference to the fact that the said angles are equal insofar as they are right. Euclid simply writes that "since angle ABC is equal to angle DEF"—both being right angles—then some conclusion follows. The definition of a right angle is also never mentioned in the cases in which it would suffice, and this fact may show that Euclid did not attach any special importance to the assumptions licensing the inference.

There are only a handful of instances (e.g. *Elements* III, 1, 3 and 4; *Elements* IV, 4 and 13), in which Euclid says something to the effect that a certain angle, which is explicitly referred to as a right angle, is equal to another right angle. Even these formulations do not seem to be allusions to P4: in these cases, too, the equality of the angles is not explicitly grounded on their being right angles, and the reference to the fact that these angles are right seems to be an addition and an explanation of something otherwise obvious. A slightly more explicit (but not decisive) stance is in *Elements* III, 18, in which Euclid proves, as a *reductio* argument, that both an angle and a proper part of it are right, concluding that this is impossible since two right angles are equal to one another.

[5]The Parallel Postulate is quoted in *Elements* I, 29 and 44; *Elements* II, 10; *Elements* VI, 4. For some famous passages in which Aristotle discussed the theory of parallels, see Heath 1949.

[6]The terms employed for expressing constructions in the course of the propositions are quite informal, and do not match the statements of the postulates perfectly. For instance, while the First Postulate allows one to *conduct* (ἀγαγεῖν) a straight line from point to point, in the propositions Euclid normally writes about *joining* (ἐπιζευγνύσθαι) straight lines. Similarly, the continuous extension of a straight line in a straight line in the Second Postulate is simply mentioned in the propositions as the extension of a line, without bothering every time to mention the conditions of continuity and collinearity that would explicitly refer the procedure back to the postulate. Nonetheless, these are *sometimes* expressed: collinearity, for instance, is mentioned in *Elements* I, 2 and elsewhere; continuity (extension κατὰ τὸ συνεχές without collinearity) in *Elements* IV, 16 and *Elements* XII, 16; both continuity and collinearity in *Elements* XI, 1. The drawing of circles is stated throughout the *Elements* with the very same words used in the Third Postulate (cf. for instance *Elements* I, 1). The strong mathematical cohesion of this group of principles makes their authenticity more likely: they seem to stand or fall together. For a few textual analyses on the constructive postulates, see Einarson 1936 and Vitrac 1990–2001, vol. 1, 195.

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There are, however, five instances in the text of the *Elements*, in which the equality of two right angles is inferred *by reason of their being right angles*. These are *Elements* I, 47 (Pythagoras' Theorem); *Elements* III, 25; *Elements* III, 30; *Elements* VI, 8; and *Elements* XI, 8.[7] In all these demonstrations, the received text states that two angles are equal and adds "indeed, each of them is right" (ὀρθὴ γὰρ ἑκατέρα). The Greek formula recurs identically in the five passages. Such a statement is not a straightforward reference to P4 (it does not say that *all* right angles are equal), but it may represent a textual marker pointing back to an explicit postulate licensing the inference. The ὀρθὴ γὰρ ἑκατέρα clause may be the only trace of P4 in the text of the *Elements*.[8]

Even this textual clue, however, can be seriously questioned. An inference followed by a short, independent clause explaining the grounds for it may easily be suspected of interpolation. That this is indeed the case seems to be confirmed by the Latin-Arabic tradition of the *Elements*, which is partially based on sources different from the extant Greek manuscripts. An important part of this tradition was clearly a translation of a Greek text that did not have the ὀρθὴ γὰρ ἑκατέρα clause. In particular, the Euclidean commentary by Al-Nayrīzī (tenth century) shows a translation of the *Elements* that does not include the clause in these five demonstrations. Similarly, the Latin translations (made from different Arabic sources) by Adelard of Bath (twelfth century) and Campano da Novara (thirteenth century) also lack the clause. Other translations made from the Arabic, such as those by Gerardo da Cremona and John of Tynemouth (twelfth century) have some scanty references to the fact that the angles are right in some (but not all) of these five propositions. In any case, these Latin texts do not translate an Arabic version of the ὀρθὴ γὰρ ἑκατέρα clause.[9]

[7]One might also include a less explicit case in *Elements* XII, 12, where the text has an incidental ὀρθαὶ γάρ referring to right angles. The passage has no match in the Latin-Arabic tradition (see below), in which the demonstration is generally abridged. The Greek expression, in any case, strikes me as an interpolation: see, for the same expression, the Greek scholia mentioned in note 10.

[8]It is worth noting here that the reference to right angles in *Elements* III, 25 and 30 (i.e. two of the five occurrences of the ὀρθὴ γὰρ ἑκατέρα clause) is mathematically improper. In the course of these demonstrations, in fact, the angles compared with one another are adjacent right angles made by the dropping of a perpendicular onto a straight line. Yet, we know that such angles are equal to one another thanks to *Elements* I, 12, and since they are equal to one another they are also right (thanks to the definition of right angles). The demonstrations in *Elements* III, 25 and 30, therefore do not establish the equality of the two right angles by means of P4, and in fact such proofs do not even need to spell out the fact that the two angles in question are right. Arguably, the ὀρθὴ γὰρ ἑκατέρα clause was added, in these cases, by someone who did not realize its uselessness in this connection. I might note in passing that the different clause ὀρθὴ ἄρα ἑκατέρα is also recurrent in the *Elements*, and it often signals the converse implication: since two angles are equal (for instance, those formed by a perpendicular), *then* they are both right angles (by *Elements* I, def. 10). See *Elements* III, 9 and 31, and *Elements* XI, 4 (and cf. e.g. *Elements* XI, 6, 15 and 26 for different contexts).

[9]It is possible that the reference to the ὀρθὴ γὰρ ἑκατέρα clause was missing in the early Arabic translation by al-Hajjāj and was rather to be found in the translation by Ishāq ibn Ḥunain. But this is very difficult to ascertain, since there is substantial evidence that we only know hybrid versions of the two translations (see for instance Brentjes 2018). In particular, the clause does not appear in Al-Nayrīzī's version (possibly based on al-Hajjāj's translation) of the extant propositions (*Elements* I, 47, *Elements* III, 25, and *Elements* III, 30). It never appears in the five theorems in the Latin translations of the *Elements* (possibly also based on al-Hajjāj's translation) by Adelard of Bath, Robert of Chester (which is, however, an abridged version), Hermann of Carinthia (also abridged), or Campano da Novara. The only exception is a thirteenth-century reworking by Chester, which has the clause "*quia uterque rectus*" in *Elements* I, 47, but in none of the other instances. On the contrary, Gerardo da Cremona's translation, which was probably (partially) based on the different Arabic version by Ishāq, includes a reference to the equality of right angles in the proofs of *Elements* I, 47 and *Elements* VI, 8 (but not in the other three instances). The same happens with John of Tynemouth's translation (which is, however, more often connected with al-Hajjāj's version), stating in the proof of *Elements* I, 47 that the two angles are equal, *scilicet recti*, and in the proof of *Elements* VI, 8 that the angles are equal, *quia recti*. It is difficult to ascertain whether these remarks in Gerardo's and Tynemouth's versions were in an Arabic source or were interpolated by the Latin translators themselves. The fact that similar remarks were not added in the proofs of *Elements* III, 25 and 30, which do not need P4 (see above, note 8), may suggest that they did not come from a Greek source translated into Arabic, but were rather additions by an Arabic or Latin mathematician. The above-mentioned medieval editions of the *Elements* can be read in: Besthorn and Heiberg 1893–1932; Busard 1967, 1977, 1983, 1984, 1996, 2001, 2005; and Busard and Folkerts 1992.

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While the details of the Arabic-Latin textual tradition of the *Elements* are still very obscure, the fact that some of these manuscripts report a translation of the five demonstrations that skip the clause on right angles seems to be sufficient evidence that a Greek text without such a clause was circulating in Late Antiquity and grounded a branch of the transmission of the *Elements*. It is very unlikely that the clause was erased from the text by a scribe, and we may conclude that the ὀρθὴ γὰρ ἑκατέρα formula was probably added as an explanatory scholium and later interpolated.

This conclusion finds further confirmation in the tradition of Book XIV of the *Elements*. This was a treatise on regular solids written by Hypsicles in the second century BCE, and later (possibly in the sixth century CE) appended to Euclid's *Elements*. The textual history of Book XIV is partially different from the rest of the *Elements*, and in the extant Greek manuscripts of it we find two examples of a similar clause on the equality of right angles (in the form "indeed, they are right", ὀρθαὶ γάρ) *as a marginal scholium* rather than incorporated into the main text. I would suggest that the manuscripts of Hypsicles' book witness an early stage of the process of interpolation of the clause, and shed some light on the making of the text of the above-mentioned five theorems of the *Elements*.[10]

The situation does not change if we look at other works by either Euclid or other ancient authors.

A survey of Euclid's *Data* and other minor works shows that in none of their demonstrations is any reference made to P4, or even to the fact that two angles are equal insofar as they are both right. The only exceptions come from Euclid's *Optics*. We have two rather different Greek versions of this work, and also different Arabic translations of it. According to all modern interpreters, these texts are quite composite and full of modifications coming from the subsequent tradition. In some passages (Proposition 47 of the so-called A version of the Greek text, and Propositions 19 and 47 of the so-called B version), we find the standard clause on the equality of right angles, ὀρθὴ γὰρ ἐστιν ἑκατέρα. The main Arabic text of the *Optics* offers, in the proof of Proposition 47 (here numbered as 55), a reference to right angles in order to explain their equality, but does not translate the Greek clause as such. It omits any reference to right angles in Proposition 19 (here Proposition 20). Moreover, the Greek text has a very explicit reference to the equality of right angles (αἱ δὲ ὀρθαὶ ἴσαι) in Proposition 34 of the A version only, which is echoed in a scanty reference to right angles in the proofs of the Greek B version (Proposition 35) and the Arabic version (Proposition 37). Since we do not know the exact relationships between these versions, it is difficult even to fathom a story about these variants. But the external interventions on these texts are so numerous that nothing can be drawn from the reference to right angles in the proofs, which are very likely post-Euclidean revisions made by someone well aware of P4.[11] I think that, given the situation of the texts, and even more so given that P4 was not quoted in the proofs in the *Elements*, we may safely conclude that these other references were interpolated as well.

Other Greek mathematicians predating Geminus do not seem to be aware of the Fourth Postulate. The extant works by Autolycus, Aristarchus and Theodosius do not refer to P4 or the equality of right angles. Archimedes did employ, in the course of his demonstrations, the fact that all right angles are equal, just as Euclid did; and like him, he never bothered to suggest that such an equality might be grounded on a particular principle. Still later, Apollonius too did not mention P4, and he usually tacitly assumed the equality of right angles throughout the *Conics*. This notwithstanding, two isolated passages in Book V and Book VI of this work validate an inference on angular equality by means of a reference to right angles: " . . . but the angles ZBΔ and AΛE are equal because they are right" (*Conics* V, 43); " . . . and the angles at τ and γ are equal because they

[10]The Greek and Arabic critical edition of the text of Book XIV, including the scholia, may be found in Vitrac and Djebbar 2011–2012. The two ὀρθαὶ γάρ clauses are found in scholia 14 and 15 to Prop. 1, on pp. 124–25 of the first part of the essay. I note that the same explicative clause appears in Proclus' commentary on *Elements* I, 26 (*In Euclidis* 351).

[11]The two Greek versions of the *Optics* were edited by Heiberg. Important studies on their relations, besides Heiberg's own reflections, may be read in Jones 1994 and Knorr 1994. The Arabic version is in Kheirandish 1999. See also Rashed 1997.

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are right"(*Conics* VI, 18).[12] Such inferences might suggest a grounding on a principle such as P4. These two books of the *Conics*, however, have only survived in Arabic, and it is hard to say whether their original text contained any reference to P4. Since no mention of P4 is found in any of Apollonius' books extant in Greek, it remains perfectly possible that the Arabic translator, knowing Euclid's work, took the liberty of adding a reference to P4.

Non-mathematical works written in the Classical or Hellenistic period also did not refer to P4. Philosophers who discussed mathematical definitions and principles, such as Plato or Aristotle, never mentioned the equality of all right angles.[13]

In conclusion, if we leave aside the likely interpolations, it appears that the Fourth Postulate was never referenced in the course of the demonstrations in the *Elements*, and there is no explicit reference to it in any extant Greek text predating Geminus. This cannot count, of course, as a mark of spuriousness for this postulate. It is possible that Euclid, who was bringing together several works written by previous mathematicians, added P4 among the principles of geometry and did not bother to add the references to it in the course of the demonstrations. It is also quite possible that authors such as Archimedes, Apollonius and other Hellenistic mathematicians did not quote P4 just because they did not find it useful, or appropriate, to refer to the basic assumptions of another treatise.[14] The surviving evidence is just too thin to make any positive statement on the inauthenticity of P4.

We have, however, reached an important *negative* result. Euclid's works and the other early sources provide no evidence whatsoever that P4 was originally included among the postulates in the *Elements*. This textual situation is unique as far as Euclid's postulates and common notions are concerned, and it opens the possibility of conceiving different stories about the epistemology underlying Euclid's principles or the reasons that may have brought Euclid, or someone after him, to include P4 among the postulates.

2. Mathematical significance of the Fourth Postulate

In this section, I consider four standard modern interpretations of the significance of the Fourth Postulate in the foundations of geometry, which are related to the works of Klein, Clifford, Hilbert and Zeuthen respectively. While these interpretations were devised at the end of the nineteenth century and were integral to the foundational programs of the time, they are still widespread in the current literature. Taken as historical accounts of Euclid's own ideas, however, they are grossly anachronistic. I will argue, indeed, that the importance of P4 in the foundations of geometry is *eminently modern*, and it depends on a conception of the nature and aims of geometry that was alien to Euclid and Greek mathematics in general. I will also offer, at the end of this section, a brief account of the historical developments that produced such modern interpretations.

The present section may be seen as a criticism of the use of the notion of "mathematical equivalence" in historical research: the equality of all right angles may well be mathematically equivalent to the isotropy of space or angular transportation according to modern standards, but its historical meaning in Greek mathematics could not have been this.[15] A byproduct of this line of research is to unpick the argument that P4 ought to be authentic since its importance in the foundations of mathematics is such that a system of geometrical axioms cannot dispense with it.

[12]Cf. Rashed 2008–2010, vol. 3, 339–40; vol. 4, 141–42.

[13]Bartel van der Waerden (1977–1978) attributes to Aristotle a reference to the Fourth Postulate which is found in Proclus (*In Euclidis* 76). But this is clearly a scholastic example made up by late Aristotelian commentators (it can also be found in Philop. *In an. post.* 36), and there is no trace of it in Aristotle's extant works.

[14]We see this happening in Pappus, for instance, in a period in which P4 was surely counted among the principles of the *Elements*. Pappus mentions from time to time that two right angles are equal (e.g. *Collectiones* E 42, Prop. 20; E 55, Prop. 30; Z 83–84, Prop. 44; H 115, Prop. 61; H 243, Prop. 174; H 287, Prop. 215), but he does this in passing and without referring to P4 in any form. A general assessment on the practices of cross-reference in Greek mathematical texts may be read in the section on the "toolbox" of Netz 1999.

[15]The notion of mathematical equivalence in historical studies has been famously attacked by Unguru 1975.

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The present section complements the negative results obtained in the previous section, by showing that we have neither textual nor *mathematical* arguments in favor of the authenticity of P4.

Modern interpretations of P4 generally connect it with the notion of *space*. According to these interpretations, the meaning of the postulate is that all right angles are equal *in any spatial position whatsoever*. Accordingly, Euclid's concern would be to assume that a right angle constructed *here* must necessarily be equal to a right angle constructed *there*. Therefore, by means of this postulate Euclid would exclude from the models of his geometry many spatial structures in which such equality fails. These interpretations of P4 come in two variants, which are seldom distinguished in the literature.

A first interpretation of P4 was advanced (among others) by Felix Klein, who equated this principle with an axiom on the isotropy of space.[16] The idea is that P4 would state that a right angle does not change its shape when displaced in space, and continues to stay "right" at any point. It is not difficult to see that this property is equivalent to isotropy, so that P4 would indeed state, in modern terms, that the space of Euclid's geometry has constant curvature.[17] This interpretation of P4 may be paired with the further interpretation of the Parallel Postulate as a principle stating the flatness of space. Euclid's five postulates, therefore, would first license ruler-and-compass constructions (the first three), and then add two more "spatial principles" to the effect that space has constant curvature (the Fourth Postulate) and, more specifically, a vanishing curvature (the Fifth Postulate). In these nineteenth-century foundational programs, P4 thus became *the most important axiom of geometry* as a whole, since it expressed the very condition that a space must satisfy in order to become the object of a geometry.[18]

A second, related interpretation of P4 was advanced by William Clifford. Rather than focusing on the preservation of angles through motions, Clifford equated P4 with the statement that one may construct equal right angles at any point (and direction) in space. This may fail in spaces with non-differentiable points. In particular, the geometry on a (semi)cone would be a perspicuous counterexample to P4. This geometry is locally Euclidean almost everywhere (since the cone has zero Gaussian curvature), but has a singular point at its vertex. An angle centered at the vertex would be right, according to Euclid's definition, if it is equal to its adjacent angle. Such a right angle centered at the vertex, however, will not in general be congruent with a "Euclidean" right angle on the lateral surface of the cone, and therefore P4 fails in this geometry. Since cones were widely studied in ancient geometry (and indeed more studied than surfaces with variable curvature), it has been claimed that Euclid may have introduced P4 in order to make explicit that the background plane in which his geometry is performed is indeed a "standard" plane rather than a conical surface.[19]

[16]See Klein's famous book (Klein 1925, vol. 2, 213–14), which was however very cautious on this point. A similar opinion is also to be found in Lindemann's edition of Clebsch's lectures on geometry, and Lindemann remarked that these ideas depended on Klein's views (Clebsch 1891, vi and 548–49). In the same period, Clifford (1901, vol. 1, 372–74) also interpreted P4 as expressing the "elementary flatness" of space. This reading of P4 was later adopted in the highly successful English edition of the *Elements* by Heath (1925, vol. 1, 200). For a recent mathematical reference to the meaning of P4 in the foundations of geometry, see Ratcliffe 2006, 335.

[17]In fact, the preservation of angles would give rise to a conformal mapping, but in Riemannian geometry this may easily be connected with the group of isometries (rigid motions). See for instance the results in Tanaka 1959. It is true, however, that this reading offers a certain semantic latitude on the meaning of P4, and for instance Ratcliffe, mentioned in the above note, takes P4 to mean the homogeneity (rather than the isotropy) of space.

[18]This was true for Klein's original program, but in a sense it remains true in some ("neo-Kleinian") branches of contemporary geometry, which still regard homogeneity (i.e. the transitivity of the group of transformation on the manifold) as the main feature needed to say that a certain structure on a topological space is a *geometry* (Thurston 1997).

[19]The classic treatment of the surface of the cone in relation to the Fourth Postulate is the above-mentioned Clifford 1901: "I can make two lines cross at the vertex of a cone so that the four adjacent angles shall be equal, and yet not one of them equal to a right angle" (vol. 1, 373–74). This example has been expounded in popular mathematical textbooks such as Henderson and Taimina 2005, who concluded that P4 "could be rephrased: *There are no cone points*" (366). I note that this interpretation refers to the construction of right angles at various positions and it is therefore especially fitting with constructivist readings of the *Elements* (see below, note 27). See the comprehensive discussion in Blåsjö 2022.

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These two interpretations of P4 are badly anachronistic. We have no hint whatsoever that Euclid or any other Greek mathematician conceived postulates (or other principles) as statements ruling out certain spatial structures. We have no hint, for instance, that a denial of the Parallel Postulate would have been regarded by Greek mathematicians as the foundation of a geometry on the hyperbolic plane. Spherical geometry (the only kind of "non-Euclidean" geometry that the Greek world produced) was not axiomatized in this way, and it was not regarded as non-Euclidean at all. No mention of the geometry on the surface of a cone, or of right angles on the cone, is found in ancient texts.[20] No Greek source ever suggested that a geometry in which P4 is false could be envisaged, and much less that it could be represented as the geometry on a non-flat surface.

In fact, the modern interpretation of P4 only makes sense insofar as geometry is regarded as *the science of space*: a theory describing the mathematical structure of space and its properties. This conception of geometry was first advanced in the early modern age and came to be fully developed (and widely accepted by the mathematical community) only in the nineteenth century. Greek geometry had nothing to do with space, and one would look in vain for any occurrence of the word τόπος in the thirteen books of Euclid's *Elements*. Space and spatial properties were not discussed in ancient geometry, and the philosophy of mathematics (such as Plato's or Aristotle's) also excluded any relation whatsoever between geometrical figures and space. Τὰ μαθηματικὰ οὐ πού: *mathematical objects are nowhere*.[21] Geometry was regarded by Euclid, and indeed by all Greek mathematicians and philosophers, as a *science of figures* with the aim of investigating the properties of circles, triangles, squares, conic sections, and other similar objects. These were not conceived as being placed in any background space and, what is more important, surely not in a space possessing geometrical properties, curvature, or singular points. I have argued at length elsewhere that this is the most important divide between ancient and modern geometry, and tracing the distinction between a geometry of figures and a geometry of space is crucial for any historical understanding of mathematics.[22]

The idea of the isotropy of space simply makes no sense in this epistemic context, and the original foundational significance of P4 could not be the one that modern geometry ascribes to the equality of all right angles. The meaning of P4 was not concerned with the *position* of different right angles, but rather with their bare *plurality*. A right angle is considered to be an individual figure, and P4 plainly states that all these figures are equal to one another, in the same way in which, say, triangles with equal sides are equal to one another. Their positions are never at stake, since they were not conceived as being embedded in space.

There are further modern interpretations of P4 that, although still conceived in the framework of a geometry of space, do not explicitly refer to space in order to stress the foundational relevance of this postulate. By avoiding the reference to space, these interpretations may have at first a greater plausibility, but they are nonetheless historically problematic.

The Fourth Postulate was read as an axiom grounding the comparison of angles. In the *Grundlagen der Geometrie*, indeed, David Hilbert assumed an axiom to the effect that any angle may be uniquely reproduced in any other point and on any ray. This is a weaker axiom than those on the isotropy of space or the absence of singular points, and permits the comparison of angles that are at a distance by allowing one of them to be re-created in another place. Hilbert employed this axiom to prove that all right angles are equal. Since Euclid lacks an axiom on angle

[20]Proclus, *In Euclidis* 123, offers a discussion on the nature of angles, which is in fact based on the behavior of angles at the vertex of a cone (angles on surfaces are mentioned again in *In Euclidis* 126). This is, as far as I know, the only ancient text connecting together angles and cones at a foundational level. The gist of the argument, however, is that an angle at the vertex of a cone may be considered either as the angle made by the two straight lines in space (i.e., considered as sides of a triangle cutting the cone), or as the angle made by the same straight lines on the surface of the cone, and therefore delimiting a curved surface. While this discussion seems, indeed, to differentiate between intrinsic and extrinsic properties of a surface, no conclusion is drawn on right angles or on the equality of angles as such, and Proclus did not connect it with P4 at all.

[21]Arist. *Metaph.* N 5, 1092^{a}19–20.

[22]For a first survey of the topic, see De Risi 2015.

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transportation, it may be claimed that P4 (together with the postulates allowing ruler-and-compass constructions) may be a good substitute for it. The assumption of the equality of all instances of a standard angle, stated by P4, would indeed allow to compare angles constructed in different positions.[23]

The problem with this interpretation is that angle transportation is not at all related to P4 in the deductive structure of the *Elements*. Euclid's theorem on angle transportation is *Elements* I, 23, teaching how "on a given straight line and at a point on it, to construct a rectilinear angle equal to a given rectilinear angle." The latter proposition, however, is itself based on the transportation of segments (*Elements* I, 2-3), and the transportation of triangles (i.e. the criteria of congruence). The transportation of segments is simply based on the constructive postulates on straight lines and circles (and the definition of a circle). The transportation of triangles (here, *Elements* I, 8) is proven by superposition, which is a technique not grounded in any more fundamental principle. In particular, P4 is not employed by Euclid for proving either the transportation of segments or the transportation of triangles (or the technique of superposition), and therefore the transportation of angles (*Elements* I, 23) does not really depend on it. Euclid could not have devised P4 in order to compare angles at a distance.[24]

A last important interpretation of the foundational meaning of P4 was offered by Hieronymus Zeuthen. He realized that the most important theorem in the *Elements* employing P4 at a foundational level is *Elements* I, 14. This proposition states that if any two adjacent angles are equal to two right angles, then they lie on a straight line. It is also the main theorem of the sequence of *Elements* I, 13–15 dealing with the elementary properties of angles. Of these, *Elements* I, 13 does not require P4 and only relies on the definition of a right angle; but *Elements* I, 14 does require the assumption of the equality of all right angles (as *Elements* I, 15 also does), and it is in fact the *first* proposition of the *Elements* that needs P4.[25] The importance of *Elements* I, 14 is

[23]Hilbert 1968, 23–24. The Fourth Postulate is here proven as Theorem 21 of the first chapter, depending on Axiom III, 4 stating the uniqueness of angle transportation. It may be noted that the proof of Theorem 21 makes use of Theorem 14 (see below, note 32), which is in turn grounded on Axiom III, 4 (again), but also on Axiom III, 5 on the congruence of triangles (just as Euclid's *Elements* I, 23 depends on *Elements* I, 8). The proof was still missing in the 1898–1899 lectures on the foundations of geometry on which the text of the *Grundlagen* was based. In these lectures, Hilbert accepted an axiom to the effect that all straight angles are equal, but was uncertain whether it could be proven from other axioms (Hallett and Majer 2004, 246 and 324). He stated that Euclid and Lindemann were mistaken in assuming P4 as a principle (Hallett and Majer 2004, 325; cf. Hilbert 1968, 23), and in a note even suggested that P4 might not belong to Euclid's original *Elements* (Hallett and Majer 2004, 246). Hilbert did not claim himself that P4 may be read as a principle of angle transportation.

[24]Proclus (*In Euclidis* 283 and 333), ascribes the solution of the problem of angle transportation (*Elements* I, 23), together with the construction of a right angle (*Elements* I, 12) to an astronomical treatise by Oenopides of Chios (fifth century BCE). Interpreters still debate the exact meaning of this attribution, since both problems are so simple that it is doubtful that before Oenopides' time anyone was able to solve them. Árpád Szabó (1969, 369–373) argued that Oenopides may have been the first to formulate the first three constructive postulates of the *Elements* that allow these constructions to be performed (Szabó did not mention P4), but his view has been criticized by Wilbur Knorr (1981, 150) and others. Knorr believed that Oenopides was the oldest source on this subject available to Eudemus (Proclus' source on the matter). Thomas Heath (1925, vol. 1, 295, relying on previous work by Bretschneider) claimed that Oenopides had first advanced the proofs of these problems in the form that is actually presented in the *Elements*. Attilio Frajese (1967) suggested that Oenopides may have proven *Elements* I, 12 and 22 (the latter used in the proof of *Elements* I, 23), both of which revolve around the conditions of intersection between straight lines and circles. Regardless, it seems very unlikely that Oenopides, in a work on astronomy, could have related the transport of angles to the equality of all right angles at a foundational level, thus contributing in any way to the formulation of P4.

[25]Some earlier propositions in the *Elements* would be false if P4 were false. For instance, the side-angle-side criterion of equality of triangles (*Elements* I, 4) could not work. This and other similar results preceding *Elements* I, 14, however, are grounded in the *Elements* on other unaxiomatized principles, such as the technique of superposition. Proclus (*In Euclidis* 297–298) reports an addition to the proof of *Elements* I, 14, given by Porphyry (third century CE). The proof of *Elements* I, 14 also had further amendments in the Arabic tradition (Brentjes 1997–1998, 67). None of these additions brought anything important to Euclid's demonstration. Much more interesting is an alternative proof offered in al-Nayrīzī's commentary, which is grounded on superposition. Albert the Great, generally drawing on Al-Nayrīzī, states that such a proof is Heron's, but the Arabic manuscript of Al-Nayrīzī employed by Besthorn and Heiberg for their edition of the text does not make this ascription, and the editors ungenerously remarked: "*hae ambages Arabibus relinquendae*" (Besthorn and Heiberg 1893–1932, vol. 1,

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apparent by its application in such strategic places as *Elements* I, 45, concluding the theory of equivalence for polygons (possibly the most important result of Book I) and the related theorems in Book VI (e.g. *Elements* VI, 25), as well as *Elements* I, 47 (Pythagoras' Theorem).[26]

Zeuthen made a further step in appreciating the foundational significance of *Elements* I, 14, and interpreted this proposition as stating the uniqueness of a perpendicular to a straight line from a point, and even as a theorem stating that two straight lines cannot have a segment in common without coinciding (these results follow, indeed, from *Elements* I, 14). Zeuthen took the cue from the latter statement and attributed to P4 the foundational meaning of stating the uniqueness of the extension of a straight segment. In Zeuthen's view, the Second Postulate licensed the extendibility of straight segments, and the Fourth Postulate complemented it by asserting the uniqueness of such an extension. Zeuthen added that this interpretation has the consequence that P4 is itself constructive like the other postulates.[27]

There is no trace of any of this, however, in historical sources. The uniqueness of the perpendicular or the uniqueness of the straight extension are never mentioned by Euclid, who seems not to have regarded these statements as problematic or in need of proof. Even more, as we have seen, the Fourth Postulate is never mentioned in the course of the *Elements*, and in particular is not mentioned in the important proof of *Elements* I, 14—as if Euclid had not appreciated any strong connection between the postulate and this proposition.

While Zeuthen did not refer to the episode, we are informed by Proclus of a famous dispute between the Epicurean Zeno of Sidon and the Stoic Posidonius (second-first century BCE). In this occasion, Zeno asserted that Euclid lacked a proposition on the uniqueness of the straight extension of a segment, offered a proof of it which is a slight variation on *Elements* I, 13 and 14, and further criticized his own proof in order to show that it (like any other proof of the same statement, we may guess) relies on an unavoidable *petitio principii*. In short, the uniqueness of extension cannot be proven. Posidonius replied that the demonstration offered by Zeno was his own fault, and that no such proof had ever appeared in a treatise of elementary geometry.[28] This episode proves that there was indeed, as Zeuthen envisaged, an ancient discussion on the uniqueness of extension that revolved around the proof of *Elements* I, 14. But it also shows that, according to Posidonius, no one before Zeno (and, in particular, not Euclid) had argued that *Elements* I, 14 might be interpreted in this way. We may conclude that Zeuthen's interpretation cannot account for the meaning of P4 in the *Elements*.

The previous remarks open a more general case concerning the significance of P4 in the foundations of ancient geometry. The lack of explicit references to this postulate in the proofs of

80–81; Tummers 1984, vol. 2, 4; Lo Bello 2003, 65). I finally note that the diagram of *Elements* I, 14 in Heiberg's edition shows a straight line on which two other lines converge forming three acute angles (of about 60° each). Saito's edition of this diagram in the manuscripts, however, shows a straight line and a perpendicular (thus, a right angle) plus a further line forming two acute angles (Saito 2006).

[26]The importance of *Elements* I, 14 in the deductive structure of the *Elements* is missed in the useful tables by Mueller and Vitrac, which simply ignore this one proposition. See Mueller 1981, 18–20, in which *Elements* I, 14 appears as a deductive dead end, and Vitrac 1990–2001, vol. 1, 518 (but cf. 514). See by contrast the table in (Dodgson 1879), which grounds *Elements* I, 47 on *Elements* I, 14.

[27]Zeuthen 1896, 124. I may add that a few recent historians, without endorsing Zeuthen's interpretation, still attempt to show that P4 is constructive after all. According to these thoroughly constructivist interpretations, every geometrical figure mentioned in the *Elements* should be regarded as the result of a previous act of construction. As a consequence, P4 would itself be constructive, stating that all constructions of right angles have the same outcomes and therefore produce equal angles. I am not entirely convinced, however, by this line of reasoning. If P4 does express that every construction of right angles has the same result, one may ask why similar statements are not required for other constructions as well, such as those of a straight line, a circle, an equilateral triangle, or a parallel line. Euclid seems never to question the universal replicability of his constructions, and many scholars have advanced hypotheses on the role of constructive schemata in the process of universal generalization of the theorems. The most refined and subtle attempts towards a constructivist reading of the *Elements* have been made by Ferreirós (2016, 130), Sidoli (2018b, 434), and Blåsjö (2022).

[28]Proclus, *In Euclidis* 217–18.

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the *Elements* obscures the mathematical goals for which it was devised. This situation remained unchanged in Late Antiquity, as if the commentators were not so sure about its foundational aims. Proclus, who dedicated many philosophical remarks to P4 in the section of his commentary dealing with principles, never quoted it again while explicating the propositions. His commentary on *Elements* I, 14 is particularly telling: Proclus pedantically mentions all the principles needed to prove this proposition, but completely overlooks P4—the only essential assumption on which the entire demonstration rests:

> In the construction he uses one postulate, the second (that a finite straight line can be extended in a straight line), just as in the proof he uses the preceding theorem and two axioms (things equal to the same thing are equal to one another, and if equals be subtracted from equals the remainders are equal); and for the reduction to impossibility he uses the axiom that the whole is greater than the part, for when the one common angle had been subtracted, the whole was equal to the part, which is impossible.[29]

The importance of P4 in the deductive structure of the *Elements*, and the relevance of the equality of all right angles in the foundations of geometry in general, only began to be appreciated in the early modern age. In Christoph Clavius' edition of the *Elements* from 1574, for instance, we find explicit references to P4 in the proof of *Elements* I, 14.[30] In the following Euclidean tradition we see a mounting attention to this principle, which is quoted more and more frequently in the course of the demonstrations that make use of it. In the same period, the idea of a geometry of space began to supplant the Greek geometry of figures, and P4 explicitly assumed the meaning that right angles are equal to one another *irrespective of their position in space*. Mathematicians such as Claude Richard (1645) axiomatized the possibility of rigid motions of figures, Gerolamo Saccheri (1733) stressed the connection between the uniqueness of the extension of a segment with P4, while Thomas Simpson (1747) provided principles on the transportations of segments and angles that were not grounded on superposition.[31] The foundations of the theory of angles were developed widely, and Giovanni Alfonso Borelli (1658) stated new important theorems that became standard after Hilbert's reworking of the theory.[32] A few years later, Nicolaus Mercator (1678) suggested a different foundational approach to the cluster of theorems of *Elements* I, 13-15.[33]

The increasing significance of P4 in modern mathematics reached a high point in the celebrated *Éléments de géométrie* by Adrien-Marie Legendre (1794), the last great reworking of Euclid's book that still regarded it as a living mathematical treatise. Legendre's book explicitly defines geometry as a theory of extension rather than figures. It begins with a proof of P4 (the first theorem in the volume), followed by proofs of *Elements* I, 13, then the statement that two straight lines do not enclose a space, then *Elements* I, 14 and *Elements* I, 15, thus grounding the whole of

[29]Proclus, *In Euclidis* 296 (transl. Morrow).

[30]Clavius 1574, 25v. Cf. for instance also Giordano 1680, 33–34. On the axiomatizations of the early modern editions of the *Elements* see De Risi 2016.

[31]More specifically, see Richard 1645, 18, Saccheri 1733, 84–85 (it is Lemma 5 after Proposition 32, depending itself on Lemma 2), and Simpson 1747, 6. An English translation and commentary of Saccheri's masterwork is in De Risi 2014.

[32]Borelli's main theorem on angles is the one stating that if an angle (which he defined as *quantitas inclinationis*) is congruent to another angle then the supplementary angle of the former is congruent to the supplementary angle of the latter. Borelli proved it through superposition (he did not accept P4 as an axiom) and later grounded on it *Elements* I, 15, 13 and 14 in this order (Borelli 1658, 23–24, Prop. 5). Borelli's theorem served as a foundation for many geometrical results by David Hilbert (it is Theorem 14 of the *Grundlagen*; see Hilbert 1968, 17), and was also discussed in Hartshorne's important textbook (Hartshorne 2000, 93). I disagree with Hartshorne, however, who compared this theorem with Euclid's *Elements* I, 13. The latter has no strong foundational significance in *modern* mathematics (see below, note 68), and P4 would be needed to deduce Hilbert's Theorem 14 from *Elements* I, 13. On Borelli's work on the foundations of geometry, see De Risi 2022.

[33]It is especially remarkable that Mercator (1678) began his geometrical treatise by proving *Elements* I, 13, 14 and 15, just as Legendre did a century later.

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geometry on P4 and the Euclidean theory of angles depending on it.[34] Only after presenting many theorems did Legendre introduce the Parallel Postulate (that he famously attempted to prove), thus clearly distinguishing a part of geometry grounded on this latter principle ("Euclidean" geometry) from another part, which was still undetermined in terms of the theory of parallels. The "absolute" part of geometry, however, was grounded on the Fourth Postulate just as the Euclidean part was grounded on the Fifth Postulate: Legendre appears to have thought that P4 was the most basic condition necessary to have any geometry whatsoever. It was this idea that, after the developments of non-Euclidean and Riemannian geometry, resulted in the nineteenth-century interpretations of P4 as a principle establishing isotropy and constant curvature as conditions for rigid motions and shape constancy.

Between the sixteenth and the nineteenth centuries, therefore, we witness the gradual ascent of the Fourth Postulate from being a neglected principle the utility of which no one could tell, to one of the core axioms in the foundations of geometry. This very history, however, hints that the important interpretations of P4 inspired by Klein, Clifford, Hilbert, and Zeuthen were indeed peculiarly modern, and that in Antiquity this principle may have been considered much less significant. We have no mathematical reasons to suppose that it ought to be assumed as an original postulate in Euclid's *Elements*.

3. The proof of the Fourth Postulate

The idea that P4 played no important role in the *Elements* is strengthened by the fact, which sounded amazing to many modern interpreters, that Greek mathematicians claimed to have a proof of this statement. This is incompatible, of course, with any strong foundational reading of P4. It comes as no surprise that the ancient demonstration was harshly criticized by modern mathematicians, who praised Euclid for not having indulged in such a faulty proof, thus clearly understanding (they argued) the unprovability of the Fourth Postulate.[35] A minority position was taken by other interpreters, who took the ancient proof as valid and claimed, as a consequence, that P4 must be spurious: for Euclid would have not accepted among the postulates a principle that could so easily be proven.[36] I agree that the proof is valid according to Greek standards, and it contributes to the many puzzles raised by the Fourth Postulate.

The proof is reported by Proclus in the same context of Geminus' epistemological objection to the inclusion of P4 among the postulates. Geminus, apparently, reported such a proof (which was already well known in the first century BCE) in order to argue that P4 was, after all, a theorem and not a postulate. Proclus tersely added to this that the proof "was given by other commentators and requires no great study." The same demonstration is also mentioned by Simplicius, and everything seems to suggest that it was widely circulating and accepted in Antiquity. Since then, it has become a standard proof of the postulate in the Middle Ages and in the Early Modern Age, when it featured in most editions of the *Elements*. It was reworked and employed in the *Grundlagen der Geometrie* by Hilbert himself—who grounded it, of course, on his own axiom system. Only more recently has it been replaced, in some formal systems for elementary geometry, by different and more complex demonstrations.[37]

[34]Legendre 1794. I further note that even George David Birkhoff's celebrated system of elementary geometry has as its first and most basic theorem the statement of *Elements* I, 13 and 14 (Birkhoff 1932).

[35]See, for instance, Heath 1925, vol. 1, 200, but this view is quite widespread in the literature.

[36]This was the opinion of Seidenberg (1975).

[37]The proof is first recorded by Proclus, *In Euclidis*, 188–89. The version given by Simplicius probably draws on the latter, and may be read *apud* al-Nayrīzī (Besthorn and Heiberg 1893–1932, vol. 1, 21–24). This Arabic commentary influenced all further (Arabic, Hebrew and Latin) medieval interpreters, who consistently referred to it. I have only found a simpler, different proof in ibn al-Haytham, which is however still very close to the original; cf. Hooper Sude 1974, 92–93. The proof continued to be accepted by many, but not all, editions in the early modern age (among those quoted in the last section, it was employed by Clavius, Richard, Saccheri and Legendre). For Hilbert's proof, see the references in note 23. At some point, Hilbert apparently

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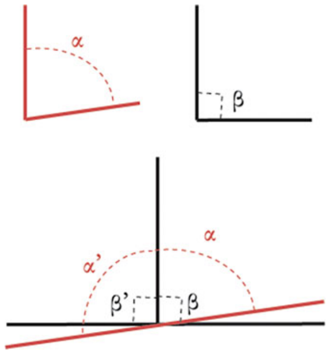

Figure 1. The ancient proof of the Fourth Postulate.

The ancient demonstration works through *reductio*: should right angle α be smaller than right angle β, then it would be possible to superpose them in such a way that angle α falls inside angle β, and then by extending their sides one should find two other adjacent right angles, α' and β', respectively equal (by the definition of a right angle) to α and β. Therefore, angle α' should be smaller than angle β', whereas it is clear from the diagram that the opposite relation is true (see figure 1).[38]

From a modern point of view, the proof is defective in at least two respects. First, it employs superposition, which is not a procedure that was ever explained or axiomatized in Greek mathematics. Second, it is entirely grounded on a diagrammatic inference, which merely shows that a straight line happens to cross another straight line—without spelling out explicit principles explaining this fact.

The issue of the use and abuse of diagrammatic inferences in mathematical proofs may only be discussed in the context of a general appraisal of these techniques in Greek geometry, a topic on which we have today a thriving literature.[39] There is some consensus that ancient geometers regarded the equality of magnitudes as a quantitative (say, "exact") relation among figures that cannot be established by inspecting the diagram. According to Greek mathematicians, namely, one cannot see in the diagram that the square drawn on the hypotenuse is equal to the sum of the squares on the other sides, but has rather to prove this fact through a chain of inferences. Similarly, one cannot read out from the diagram the equality of the magnitude of two angles, and needs a propositional way of ascertaining this. Greek geometry, however, permitted the drawing of diagrammatic inferences concerning mereological containment: e.g. one may see in the diagram that an angle is a part of another angle and therefore lesser than it (a "co-exact" attribution).

The diagrammatic inference in the proof reported by Geminus nicely exemplifies this strategy. We cannot tell from looking at the first diagram whether angles α and β are equal—this is an exact

thought that the proof had been first offered by Legendre; cf. for instance the fourth edition of the *Grundlagen* (Hilbert 1913, 17). The first different modern demonstration of P4 was probably offered by Tarski in the 1940s, in a formal system that (contrary to what happens in Hilbert's *Grundlagen*) did not assume angles as a primitive notion (Schwabhäuser, Szmielew and Tarski 1983). For a recent approach, see Beeson Narboux and Wiedijk 2019; this modern proof of P4 still employs angle transportation (*Elements* I, 23).

[38] I note here that Proclus' proof has a different diagram than the one found in the Leiden manuscript of al-Nayrīzī. The Latin translation of al-Nayrīzī's commentary by Gerardo da Cremona, however, has Proclus' diagram (Curtze 1899, 33; Tummers 1994, 30). The same proof is also found in Al-Kūhī's commentary on the *Elements* (tenth century), also drawing on al-Nayrīzī, with yet another different diagram (Berggren and Van Brummelen 2005). The proof seems to make use of the common notion stating that things that may be superposed are equal. In fact, the *reductio* hypothesis that the two right angles are not equal implies, by contraposition, that they cannot be superposed upon one another, and therefore that lines α and β do not coincide.

[39] The starting point for this interpretation of the role of diagrams in ancient proofs is the celebrated paper by Manders (2008). Among others, I have advanced some refinements of Manders' view in my De Risi 2021a.

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attribution. However, after we superimpose them and extend their sides, we can infer, by inspecting the diagram, that angle β', being contained in angle α' (being *a part* thereof), is smaller than the latter—this is a co-exact attribution. This gives rise to a contradiction with the opposite statement we had propositionally established from the *reductio* hypothesis. The demonstration of P4, then, is not entirely trivial. Its construction and argument are used to transform a statement about properties subject to exact measurements (all right angles are equal) into an equivalent statement about properties that may be diagrammatically established (angle β' is contained in angle α'). According to Greek standards, a diagrammatic inference may legitimately play a role in establishing P4.

The issue of superposition can only be formulated in the modern framework of a geometry of space. We have seen, indeed, that many modern interpretations of P4 connect it with principles on isotropy or angle transportation, but these modern axioms are simply pleonastic if one accepts, as Euclid and other Greek mathematicians did, the technique of superposition. Consequently, modern mathematicians rejected the proof of P4 as a *petitio principii.* Since P4 states the equality of right angles in different positions in space, so their reading goes, we cannot prove this statement by bringing the angles to the same position in space through the technique of superposition.

All difficulties vanish, however, as soon as we realize that the ancient proof was not concerned with the position of the right angles at all, and that space was never under consideration. We may usefully compare this proof with the other demonstrations employing superposition. In *Elements* I, 8, for instance, superposition is used to prove that any two triangles with equal sides are equal. Their position is irrelevant, and for this very reason they may be freely superposed in order to prove the theorem. In the modern interpretation, the proof of P4 would represent an essentially different use of superposition, since the very aim of the proof would be to show the equality of right angles in different positions, and the latter result cannot be attained by moving and superposing them. I claim we should, by contrast, conceive of the ancient proof of P4 on the same footing as the proof in *Elements* I, 8 (and others): they both achieve their goal since they are not concerned with position at all. Once we refrain from considering Greek geometry as a science of space, the ancient proof of P4 is perfectly fine, and we easily understand why no ancient commentator reported any concerns about it.

I will further belabor the latter point, since Nathan Sidoli has recently advanced an important interpretation of P4 and the technique of superposition. According to Sidoli, this technique was only considered viable, in Greek geometry, insofar as it was applied to objects (triangles, right angles, etc.) that are not *given in position* (τῇ θέσει δεδόσθαι). This last expression, coming from Euclid's *Data*, means that the object in question has a fixed position in relation to other objects in a certain configuration. I agree with this view, which expresses an important foundational stance employing Euclid's own mathematical terminology. Sidoli maintains, however, that the ancient proof of P4 has, for this reason, only limited scope: it works when the right angles under consideration are not "given in position", but it cannot work in all those cases in which the right angles are part of a larger configuration, and therefore are in a fixed position relative to one another. Thus, for instance, the equality of the right angles in the proof of *Elements* I, 46 could not be proven by superposition insofar as these angles are given in position as the four angles of a square. For this reason, Sidoli concludes, even though the ancient proof of P4 is sound, it does not apply to all possible cases, and a theorem such as *Elements* I, 46 would still need the assumption of P4 as a further principle. This would explain Euclid's choice of putting it among the postulates.[40]

Sidoli's argument is fascinating, but I think it proves too much. Take again *Elements* I, 8: in this theorem, two triangles may be superposed since they are not given in position, and Euclid may

[40]The thesis was argued in Sidoli's important paper (Sidoli 2018b, 422–23), which also offers in footnote 45 a subtle argument on how to transform the proof of *Elements* I, 46 in order to make P4 unnecessary in it, even though the procedure cannot be generalized (the footnote also claims, I think wrongly, that *Elements* I, 14 and 15 do not need P4). For a fuller treatment of the notion of givenness, see also Sidoli 2018a.

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conclude that two triangles having equal sides are equal. When Euclid further applies this theorem (for instance in *Elements* I, 9), the triangles under consideration are given in position in a certain configuration, and nonetheless Euclid does not hesitate to make use of *Elements* I, 8 to conclude the demonstration—and similarly in all other cases. The point seems to be that once Euclid has proven by superposition that any two triangles with equal sides (which are not given in position) are equal, he assumes that all triangles with equal sides are equal—even those which are given in position. The equality of triangles does not depend on the way in which this is proven, and in particular it does not depend on whether the triangles are given, or not, in position. Similarly, then, if one accepts the ancient proof of P4, in which superposition may be employed since the right angles are not given in position, the same result (according to Euclid) would have universal validity and apply to all right angles—even in those cases in which they are given in position (such as in *Elements* I, 46). To assume that this generalization does not work in the special case of P4 is tantamount to saying that P4 states the invariance of right angles *by position* and accepting the modern reading of it as an axiom of space.

I would claim, therefore, that the ancient proof of the Fourth Postulate was sound according to Euclid's standards and that it had universal applicability. Its early formulation and general acceptance by the Greek mathematical community (that we know of), may explain the minor and ancillary role that P4 played in the foundations of geometry in the ancient Euclidean tradition.

I would like to further advance the conjecture that the proof of P4 reported by Geminus was originally devised by his main source on matters of mathematics: the philosopher Posidonius. We know that the Parallel Postulate had been discussed at the time of Posidonius and that he had probably offered a proof of it. It seems very possible that Posidonius supplemented his proof of the Parallel Postulate with a proof of P4. We know, after all, that Geminus considered neither P4 nor the Parallel Postulate to be constructive in the sense of the first three postulates, and that he therefore wanted to delete both of them from the list of principles. If this was the opinion of his source as well, Posidonius may have attempted to prove both statements in order to adhere to Euclid's three constructive principles as the only postulates.[41]

Some interpreters, such as Thomas Heath and Oskar Becker, regarded P4 as a prerequisite for the Parallel Postulate, insofar as the very enunciation of the latter makes reference to right angles. This thesis has been used as an argument in favor of the authenticity of P4. The latter conclusion is, however, very weak, and not supported by any ancient text. I doubt we may establish that Euclid conceived P4 as a premise for the Parallel Postulate. On the other hand, should we accept that the proof of P4 reported by Proclus and Simplicius was actually conceived of by Posidonius in his essay proving the Parallel Postulate, then Posidonius may have himself pointed out a connection between the two principles and considered P4 (and his proof thereof) as a premise for the Parallel Postulate (and his proof thereof). Should this be the case, Heath's and Becker's opinion might be partially vindicated by applying it to Posidonius rather than Euclid.[42]

We have a few more clues that the P4 demonstration may have been produced by Posidonius. We have seen that Zeno and Posidonius discussed the statement that two straight lines cannot

[41]Proclus, *In Euclidis* 176. This text only discusses a definition of parallel lines offered by Posidonius, but since there is an easy demonstration of the Parallel Postulate (also offered by Proclus, *In Euclidis* 371–72) based on this definition, I consider it quite evident that Posidonius himself had advanced such a demonstration. The proofs of P4 and the Parallel Postulate are associated in *In Euclidis* (183–84) with a reference to Geminus, and it may easily be the case that both proofs were to be found in the same tract by Posidonius.

[42]See Heath 1925, vol. 1, 201; and more extensively Becker 1955. In particular, Becker suggested: (1) that in an early version of the *Elements* P4 may have been expressed as a genitive absolute, being in fact not a postulate itself but rather a clause of the Parallel Postulate, and that, accordingly, Euclid only had four postulates; (2) that a still older statement of the Parallel Postulate did not refer to angles, but just to the mutual inclination of straight lines, and did not need any formulation of P4 at all (which was added by Euclid himself). Both claims are, also according to Becker, highly speculative. I barely need to note that Heath's and Becker's views were historically grounded versions of Klein's mathematical interpretation of P4 (isotropy) as a precondition for the Parallel Postulate (flatness). Their claims have been criticized by Ian Mueller (1981, 30).

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have a common segment, and connected this to the equality of right angles. In this connection, it is likely that the role and validity of P4 was also under consideration. We further know that Zeno's arguments were probably based on Epicurean conceptions of *minima* in geometry. Zeno's belief in the existence of *minima* resulted, for instance, in the objection that a straight line might not be susceptible to being bisected into two equal parts, since it may happen to be composed of an *odd number* of *minima*.[43] We are also aware that minimal angles were discussed in Antiquity, even though the exact contexts of such discussions are quite obscure.[44] Proclus tells us that Zeno and Posidonius also debated whether Euclid presupposed, or not, that any two straight lines may always be put at a right angle. Proclus does not mention here any issue about *minima*, but it may be fathomed that atomistic objections against Euclid's definition of equal adjacent right angles may have been raised in the same vein as those denying the possibility of bisecting a segment: an odd number of minimal angles may prevent the possibility of having equal angles at the meeting of two lines. Posidonius' debate with Zeno may have been the background for the ancient proof of P4.[45]

The ascription of the proof to Posidonius remains conjectural. Nonetheless, it may provide a foundational reason for either (1) the addition of P4 to the *Elements* in the centuries following Euclid, when the atomistic objections to geometry were first raised; or (2) the proof of such a postulate by Posidonius in case that P4 had been already formulated (either by Euclid or someone in between Euclid and Zeno) and later attacked by the Epicureans. I consider the second option more probable, since skeptical doubts would not be assuaged by bluntly stating a postulate, but in the lack of any further textual evidence I will not belabor the point, and will instead conclude with a wider historical conjecture.

4. A Conjecture: Euclid and Apollonius on angles

At this point, we must address the question of the authenticity of the Fourth Postulate. However legitimate, this question must nonetheless be carefully formulated and contextualized, since the *Elements* may be considered as a multi-authored work.[46] They are composed of thirteen different essays, probably written by different mathematicians in different times, all of them in turn drawing on further, possibly scattered, older results. At some point these essays were collected together into the *Elements*, and a mathematician called Euclid is credited for having first created this unitary work. We do not know whether this collection already contained all thirteen books, or

[43]Proclus, *In Euclidis* 278; Sext. Emp. *Adv. Math.* Γ 108–16.

[44]Proclus, *In Euclidis* 125. According to Proclus, Plutarch of Athens claimed that the angle is "the first interval under the point," to which Proclus himself objected that there is no such first, minimal interval. We find an explicit definition of an angle as a "*minimum* under the inclination of two straight lines" in Sext. Emp. *Adv. Math.* Γ 100, with no reference to its author, and further references to "the first interval under the point" in *Adv. Math.* Γ 104 and 106. It is possible, then, that such an Epicurean definition was already circulating in the age of Sextus, and Plutarch had taken it over in the fourth century. A further reference to an argument against Euclid's notion of an angle, relying on Epicurean *minima*, seems to be found in a fragment by Demetrius Lacon discussing *Elements* I, 9 (how to bisect an angle): for the text, see Angeli Dorandi 1987. Epicurus had written a treatise *On the angle in an atom* (περὶ τῆς ἐν τῇ ἀτόμῳ γωνίας; Diog. Laert. K 28), and Francesco Verde aptly refers to three verses of Lucretius in which the Epicurean *clinamen* of atoms is described as an inclination under *a minimal angle* ("quare etiam atque etiam paulum inclinare necessest/corpora; nec plus quam minimum, ne fingere motus/obliquos videamur et id res vera refutet", *De rerum natura*, II, 243–45; Verde 2013, 296). On Epicurean mathematical atomism, see especially Vlastos 1965 and the important Netz 2015.

[45]The most refined version of the ancient proof of P4 is to be found, in the early modern age, in Saccheri's *Euclides vindicatus* (1733). Here Saccheri is very careful in spelling out all the hidden assumptions of this proof, and aims to show not only that all right angles are equal, but also that they are equal "without any deviation even infinitely small" (84–85). Saccheri has in mind the early modern theories of indivisibles and infinitesimals, as well as the possibility of asymptotic straight lines making "curvilinear" angles of sorts with perpendiculars, but it cannot be excluded that he also referred to Zeno's objections.

[46]On the notion of multiple authorship, see for instance Brentjes 2020. On the changing boundaries between primary texts and commentaries, see the important debate between Reviel Netz and Karine Chemla (Netz 1998, Chemla 1999, Netz 2005).

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the extent of Euclid's editorial work and adaptation of previous material. We do not know if several versions of the same treatise were produced by Euclid himself during his lifetime, and we can easily imagine that his students and followers continued to rewrite the *Elements* for many years after his death. The treatise was probably not considered to be an "authorial" work in the first period of its diffusion, and Archimedes, for instance, refers to some theorems that we find in the *Elements* by attributing them to Eudoxus rather than Euclid.[47]

Later authors commented and further modified the text of the *Elements*. Heron's commentary (first century CE), for instance, may itself have been considered an edition of the text in Antiquity. Later on, Theon of Alexandria (fourth century) apparently systematized the book, and added some demonstrations. The relation between authorship and commentary changed in the course of the centuries, and Theon, for instance, while he felt authorized to modify the received text, also marked out (albeit in another work) some of the interventions he made—thus distinguishing the work of an author (Euclid) and that of an editor (himself).[48] This was clearly not the case, however, in the first centuries after Euclid, and in this respect the very notion of authorship may be questioned. As far as we understand, indeed, the Fourth Postulate belongs to the original, opaque and multi-authored core of the *Elements* that was formed in the first three centuries BCE: this is as much as we can say about the "authenticity" of P4.

We may nonetheless try to dissect the *Elements* in order to outline different stratifications of the text, and advance conjectures on how the various pieces stay together or were added to one another. In particular, we may conceive of many different stories about the motives and occasions of the interpolation of the Fourth Postulate in the years between Euclid and Geminus. I have already mentioned that Skeptic and Epicurean challenges to the equality of right angles may have provided reasons either to include P4 among the principles of the *Elements*, or to offer a proof of it. Simpler explanations are also possible. In the course of the proofs, Euclid happens to add together equal angles and right angles, concluding that the results are equal. The inference is drawn through the common notion on the additivity of measure ("if equals be added to equals, the wholes are equal"), but Euclid did not spell out that right angles are equal. It is possible that a commentator, a scholiast, or just a mathematician reworking the text of the *Elements*, introduced P4 as an explicit condition for the applicability of the common notion on addition. Should this be the case, the introduction of P4 may have been motivated, for instance, by the needs of teaching.

Philosophical disputes and pedagogical concerns were not, however, the only reasons to modify the *Elements*. We may conceive, indeed, also purely mathematical grounds for the addition of P4 to the list of postulates. In this section, I explore this possibility, and venture the conjecture that the Fourth Postulate was added to the text of the *Elements* in the second century BCE, following a transformation of the definition of angles initiated by Apollonius. The advantages of such a conjecture over the others (including the hypothesis that Euclid himself formulated P4) is its ability to cope with the many textual and mathematical difficulties that we have seen so far. The conjecture is based on an interpretation of the theories of angles in Greek mathematics, and exploits a tension between the Fourth Postulate and Euclid's definitions of angles—two different *strata*, I claim, of the text of the *Elements*.

I begin by presenting the theory of angles that emerges from the definitions of the *Elements* and other ancient sources, and argue that it can do without P4 (section 4.1). I then discuss an objection and introduce a second conjecture to explain some puzzles about the treatment of angles in the *Elements* (section 4.2). Finally, I consider Apollonius' theory of angles as a possible source of the interpolation of P4 in the *Elements* (section 4.3), before presenting some concluding remarks.

[47]Cf. for instance Gardies 1997. Scholars disagree on Archimedes' knowledge of Euclid and his attitude towards the *Elements*.

[48]Theon's reference to a corollary of *Elements* VI, 33 as his own addition to Euclid's text is to be read in Theon's commentary on Ptolemy's Almagest (Heiberg and Menge 1883–1899, vol. 1, 183).

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4.1. Euclid's definitions of angles

In *Elements* I, def. 8, Euclid defines an angle as an *inclination* (κλίσις) between lines. The concept of "inclination" is taken as primitive, and it does not appear again in the *Elements*.[49] The term κλίσις also recurs in relation to angles in the works of Autolycus of Pitane (fourth or third century BCE). Therefore, a definition of angles as inclinations may have been common in pre-Euclidean mathematics.[50] More or less in the same period as Autolycus, Aristotle seems to have understood an angle as a *quality* of lines, and accordingly he called "similar" (ὅμοιαι) angles that Euclid would later call "equal" (ἴσαι) angles. Aristotle's student Eudemus, who is credited by Proclus with an entire book on angles, also claimed that angles are qualities of broken lines.[51]

While the details underlying these notions are quite obscure, all these early definitions of angles as qualities (Aristotle, Eudemus) or inclinations (Autolycus, Euclid) of lines concurred in taking angles to be *shapes* and *configurations*. This point of view is sharply contrasted by Proclus with subsequent definitions of angles as quantities or magnitudes. According to Proclus and other ancient commentators, Euclid's definition of angles was *essentially non-quantitative*.

I maintain that we should take this idea seriously. By defining angles as inclinations between lines, Euclid was not thinking about the part of the plane delimited by such lines, nor the length of an arc between these lines, nor the width of their "opening", but about a more abstract *relation* between two meeting lines. This relation between lines makes the configuration, or shape, of an angle. No measure is involved, but only the positional arrangement of a pair of intersecting lines. In the same vein, parallelism was not defined by Euclid as a determinate distance between straight lines (a metric notion that would become customary in the following Euclidean tradition), but rather as a positional arrangement of non-intersecting lines.[52]

If this is true, for Euclid a *right* angle was a unique, symmetrical configuration of lines. When the inclination of two meeting straight lines is the same on one side and the other, they make a right angle. A right angle is a peculiar geometrical shape. Accordingly, the equality of a right angle with its adjacent angle should not be intended as the equality in magnitude of these two angles, but

[49]The only exception being *Elements* XI, def. 5–7, which circularly state that an inclination between lines is an angle. These definitions in Book XI are however absent from some Greek manuscripts, added by another hand in some others, not to be found in the Arabic tradition, nor in indirect sources, and are never used or further mentioned in the *Elements*. They are, therefore, extremely likely to be late interpolations. An anonymous Greek scholiast (Heiberg and Menge 1883–1899, vol. 5, 240 [XI, 5]) seems to attribute the definition of an inclination through the notion of an angle to the Stoics. See also Vitrac 1990–2001, vol. 4, 77–79.

[50]Autolycus does not define angles anywhere in his extant works. In Prop. 8 of his *De sphaera* (28 Hultsch), however, he says that a certain angle is the inclination (κλίσις) between two circumferences, and the term is here recurrent and seems to be technical. A late scholium (162 Hultsch) explains Autolycus' term through the notion of an angle, referring to *Elements* XI, def. 6 (cf. the previous note). Note that Heath (1925, vol. 1, 176) remarked that the word κλίσις is not to be found in Aristotle, and therefore erroneously concluded that this definition of an angle was an original creation of Euclid's.

[51]Aristotle maintains that angles are qualities in several passages. In *De caelo* B 14, 296^{b}20 (and B 14, 297^{b}19; Δ 4, 311^{b}34) he talks about *similar* (ὅμοιαι) angles; and in *Cat.* 6, 6^{a}31-35, he adds that only quantities may be said to be equal (ἴσα), whereas non-quantities may be said to be similar (ὅμοια). The observation that Aristotle considered angles as qualities was made by Simplicius (*In de caelo* 538), and quoted by Heath (1925, vol. 1, 176). It is clear, in any case, that angles were not magnitudes according to Aristotle, who never mentions them in his exemplifications of μεγέθη; see for instance *Metaph.* Δ 13, 1020^{a}9–14; *Cat.* 6, 4^{b}23–25. Proclus adds that calling equal angles ὅμοιαι is an old-fashioned way of speaking, and attributes this usage to Thales (*In Euclidis* 251). The reference to Eudemus' theory and his book on angles is to be found in Proclus, *In Euclidis* 125, where an angle is called a κλάσις τῶν γραμμῶν. I assume here that this Eudemus is actually Eudemus of Rhodes, the disciple of Aristotle; the same conclusion is in Wehrli 1955, 21–22, fr. 30.

[52]Proclus, *In Euclidis* 125, explicitly states that for Euclid angles are *relations*, and frames this discussion in a very scholastic dispute on the Aristotelian categories to which an angle should belong (here, the πρὸς τί). While Euclid probably knew nothing about Aristotelian categories, the point that an inclination between lines is indeed a relation between these seems to be well taken.

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rather as their equality in shape, *viz.* their *congruence*. It is indeed very common that Euclid calls congruent figures "equal" (ἴσα).[53]

I would claim that with such a conception of angles, *the Fourth Postulate is neither needed nor meaningful*. There is no room for doubt, indeed, that the relation (the κλίσις, inclination) obtaining between two perpendicular straight lines is the same relation as the one that obtains between any other pair of perpendicular straight lines. They are both configurations of straight lines, and they are both symmetrical: all perpendiculars are perpendicular in the same way. Asking about the equality of distinct pairs of right angles would be tantamount to asking whether two straight lines are straight in the same way, or whether two circles are circular in the same way. Euclid has no principles (besides the definitions) concerning the straightness of straight lines, or their reciprocal congruence and fitting, or their shape in general. The same is valid for a right angle: it is not a magnitude that might be greater or lesser, but a particular shape and a unique configuration. It cannot be different in different instances, otherwise it would no longer be a right angle.

This is, I would argue, the view vehiculated by the definitions of angles in the *Elements*. It may also be regarded as a consequence of Euclid's geometry being a geometry of figures rather than space. In modern geometry we do accept, in fact, that two straight lines may be different from one another: geodesics may have different shapes and properties according to the curvature of space. But in a spaceless geometry, in which no underlying spatial background affects the properties of the figures—which are the only objects of investigation—there is no conceptual room for meaningfully asking if straight lines are identically "straight" (as if they might have different shapes *here* or *there*), or if all right angles are identically "right." The self-identity of the (non-spatially located) figures is the main assumption of classical geometry, which lacks sufficient reason (the space structure) for ever doubting it. This also explains why all ancient commentators stressed that Euclid's definition of a right angle spells out a unique configuration of lines, whereas Euclid's definitions of acute and obtuse angles encompass a plurality of different angles.[54] This

[53]It is well known that Euclid never defines equality, and employs the notion to mean either equality in measure or congruence. In *Elements* I, 16, 33, and 34, and *Elements* III, 24 and 25, for instance, "equal" triangles and circular arcs are to be understood as congruent figures, and the proof would not work if they were just equal in measure. In the theory of equivalence of *Elements* I, 35–45, on the other hand, "equal" clearly applies to non-congruent but equivalent figures. I claim that in the case of angles the meaning of "equal" essentially refers to congruence. In *Elements* I, 4, in which Euclid mentions angular equality for the first time, indeed, he tacitly assumes throughout that "equal" angles are congruent angles (and vice versa), and the whole demonstration is based on the technique of superposition. Even Proclus, in commenting on such a proposition, straightforwardly states that equal angles are those which may be superposed: "A rectilinear angle is said to be equal to a rectilinear angle when, if one of the sides containing it is placed upon one of the sides containing the other, the second side of the first coincides with the second side of the other. When the other sides fail to coincide, that angle is greater whose side falls outside, and that angle less whose side falls inside" (*In Euclidis* 237–38, transl. Morrow; cf. also a similar stance in 261). This is especially true concerning right angles. In *Elements* I, 11 and 12, Euclid teaches how to erect a perpendicular on a straight line, and in order to do so he proves that the erected line forms two equal angles with the given straight. The proof is carried out by constructing equal (right) triangles, and the equality of these triangles is grounded in *Elements* I, 8 (the side-side-side congruence criterion), which is proven through superposition. In the end, then, the equality of the angles forming a perpendicular is explained by the fact that two such angles may be superposed. (Note that the adjacent right angles cannot be directly superposed, since they are given in position—they are adjacent to one another. Euclid has to recur to triangles in order to indirectly show the superposability of the angles, since these triangles were not given in position in *Elements* I, 8: see above, note 40 and the paragraph to which it is appended.) We may conclude that a right angle is, in the definitional system of Euclid's *Elements*, an angle *congruent* to its adjacent angle. A very clear statement on the matter is to be found in al-Sijzī's discussion on the foundations of Euclid's geometry (tenth century): see Rashed and Crozet 2023, 86. A detailed reconstruction of the proofs of *Elements* I, 11–12 is offered in Panza 2012.

[54]Aristotle explicitly says that acute and obtuse angles are defined starting from the prior definition of a right angle (Arist. *Metaph.* Z 10, 1034^{b}28–32, 1035^{b}6–8, and M 8, 1084^{b}7–8; cf. *Elements* I, def. 11–12), and in the *Problemata*, a text that may be spurious but was nonetheless written in Aristotle's time, the author mentions a perpendicular making "similar opposite" (i.e. equal adjacent) angles (Arist. *Probl.* Iς 4, 913^{b}36). Even more interestingly, in *An. Post.* B 11, 94^{a}28–36, Aristotle states that a right angle should be defined as "a half of two right angles", and that this definition expresses the "essence" of a right angle. The priority of right angles is further stressed by Proclus and Simplicius: Proclus in *In Euclidis* 131–35 and 293–94; Simplicius

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apparently naïve observation is deeply rooted in the fact that a unique configuration cannot be different in different instances, so that the equality of all right angles is a straightforward consequence of their definition.

I find confirmations of this interpretation in the later Greek commentators. Heron of Alexandria adopted, in his *Definitiones*, a Euclidean perspective on the foundations of geometry and discussed at length the configurational properties of the perpendiculars forming two right angles, endorsing Euclid's understanding of angles as inclinations. Heron, indeed, did more than this, placing the statement that all right angles are equal *among the definitions*, since he clearly saw it as an immediate consequence of the definition of a right angle. Heron did not even mention the fact that the same statement had once been regarded as a postulate.[55] Elsewhere, Heron discussed the issue again, and just stated that all right angles are congruent to one another, without making reference to their equality.[56]

Further sources are Proclus and Simplicius. They seem to expand on Heron's remark when they claim that there is a second proof of the Fourth Postulate (besides the one we saw in the previous section), which is purely logical. This proof is just that all right angles are formed by lines that "are *not* inclined" to one another (i.e. by perpendicular lines) and therefore must be equal to one another. The truth of P4 flows, therefore, from the definition of a right angle. This "logical" proof is clearly grounded on a conception of a right angle as a unique and symmetrical configuration of lines rather than a magnitude, and also elsewhere Proclus states that (according to Euclid) "when the inclination is one, there is one angle."[57]

My hypothesis is that this was Euclid's original understanding of the equality of all right angles —a statement following the definition of a right angle, that he did not deem necessary to spell out as a postulate. Had he wanted to spell it out, arguably, he would have proceeded just as Heron did, and added this remark to the definitions. Since we do not know the exact sources of Heron's *Definitiones* and to what extent they rely on older Euclidean material, we are not able to further substantiate this conjecture and claim that this was indeed the case in an ancient version of the text.[58]

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in al-Nayrīzī's commentary, who also emphasized the uniqueness of the right angle as opposed to the plurality of acute and obtuse angles (Arnzen 2002; English translation in Lo Bello 2009, 8). Proclus, *In Euclidis* 132, quotes some Pythagoreans claiming that the right angle is constituted by the "Limit", and therefore "the idea which proceeds from the Limit should produce the one right angle, ruled by equality and similarity to every other right angle, always determinate and fixed in nature, not admitting of either growth or diminution." This seems to be a sort of metaphysical belaboring of P4, reminiscent of Heron's (much less metaphysical) statements on the equality of right angles and the inequality of acute and obtuse angles (*Definitiones* 20).

[55]See Heron's *Definitiones* 14 (definition of an angle as a relation), 17 (definition of a right angle), and 20 (equality of all right angles and general discussion on inclination and perpendicularity). The Fourth Postulate is mentioned in passing in *Def.* 20. It is not entirely clear whether Heron was expounding his own views or rather attempting to explain Euclid's stance. In definition 14 he also mentions Apollonius' definition of an angle as a contraction, and the same is restated in definition 117 as well. Moreover, we have a dubious reference to a *different* definition of angles attributed to Heron by Albert the Great (Tummers 1984, vol. 2, 10; English translation in Lo Bello 2003, 13).

[56]See Heron's *Metrica*, preface to Book I, where he says that every right angle is congruent ("fits", ἐφαρμόζει) to any other right angle (Acerbi Vitrac 2014, 146:24).

[57]Proclus, *In Euclidis* 122.

[58]The logical proof expounded in Proclus, *In Euclidis* 188, is rather obscure, but Simplicius' version is far clearer and more evidently related to Heron's *Definitiones* 20 (Besthorn and Heiberg 1893–1932, vol. 1, 21–22). Proclus' and Simplicius' "logical" proof was further developed by Thābit ibn Qurra, in his commentary on Euclid's principles appended to Isḥāq ibn Hunayn's Arabic translation of the *Elements*. In this commentary, in fact, Thābit accepted a modified definition of right angles, according to which a right angle is formed by a straight line having *no inclination whatsoever* with another straight line. Thābit's commentary was partially edited in Djebbar 2003 (see 306–307 and 310–11). On the Arabic translation of Heron's commentary, see Sezgin 1974, 152. Albert the Great highlighted the difference between a "demonstrative" proof of P4 and a "logical" proof of the same statement (Tummers 1984, vol. 2, 20; English translation in Lo Bello 2003, 25). A few years later, Thomas Aquinas inferred from Albert's discussion that P4 is an analytical consequence of the definition of a right angle (*In an. post.* Lib.1, lect. 5; in Aquinas 1989, 25), and this view continued to be accepted by philosophers and mathematicians for centuries (see, for example, Peletier 1557, 10).

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Finally, Proclus reports that Pappus had discussed at length the "converse" of P4, which states that if a right angle is equal to another angle, then the latter is a right angle as well. Pappus argued that this is not the case in general, since a curvilinear angle may be equal to a right angle (in the sense of *having the same magnitude*), without being itself a right angle. It is clear that Pappus' qualms may only arise within the framework of a distinction between the shape of an angle and its magnitude, and would make no sense if angles were considered to be just magnitudes. A curvilinear angle may have the same magnitude as a right angle, but it is not itself a right angle—because the latter is a configuration of straight lines and not a magnitude.[59]

Ancient sources therefore agree that Euclid had a non-quantitative conception of angles and, moreover, that the equality of all right angles was for him an immediate consequence of the definition of a right angle as a configuration.

4.2. An objection, a reply, and a further conjecture: Angles and angular magnitudes

Despite this evidence, some modern interpreters have claimed that for Euclid angles were magnitudes.[60] There is, indeed, a major objection against the interpretation of angles as relations: Euclid, sometimes, treats angles as if they were quantities and magnitudes (μεγέθη). He compares angles according to the lesser and the greater, he adds and subtracts angles, and takes multiples of given angles. In the single instance of *Elements* VI, 33, Euclid even puts angles in a ratio, and according to Book V of the *Elements* only magnitudes may have ratios.[61] Moreover, in the *Data* Euclid states that angles may be "given in magnitude" (τῷ μεγέθει δεδόσθαι).[62] These

[59]Proclus, *In Euclidis* 189–90; cf. also *In Euclidis* 238. There is a parallel passage in al-Nayrīzī (Besthorn and Heiberg 1893–1932, vol. 1, 21–24), which, however, does not mention Pappus as the author of this consideration. Pappus is referred to again in this connection in a scholium to a Greek manuscript of the *Elements* that may draw on Proclus (Heiberg and Menge 1883–1899, vol. 5, 112–13 [I, 13]). Euclid makes sometimes inferences of this kind: see for instance the proof of *Elements* XI, 6, in which an angle is said to be right since it is proven equal to a given right angle. In 1634, Pierre Herigone first used Pappus' converse principle in order to axiomatize elementary geometry in his *Cursus mathematicus*. More recently, Väisälä (1935) has shown that, in the system of Hilbert's *Grundlagen*, the statement that an angle congruent to a right angle is itself right may replace the axiom III, 3 on segment addition and transportation. For a modern treatment of P4 together with Pappus' statement, see Chapter 2, Theorem 67 in Borsuk and Szmielew 1960, 115.

[60]See for instance Heath's view that Euclid "certainly regarded angles as magnitudes" (Heath 1925, vol. 1, 178). By contrast, Kurt von Fritz has claimed that angles, according to Euclid, were essentially shapes. Von Fritz thought, however, that they were *also* taken as magnitudes, and that "bei den geradlinig begrenzten Winkeln Gestalt und Größe miteinander identisch sind" (von Fritz 1959, 40; cf. also 56). This view is still widespread in the literature.

[61]The proof of *Elements* VI, 33 makes explicit mention of the Eudoxian definition of ratios in *Elements* V, def. 5. Since the latter theory states that only magnitudes can be in ratios, the current formulation of *Elements* VI, 33 implies that angles are magnitudes. A corollary of *Elements* VI, 33 was proven by Theon of Alexandria, using the same Eudoxian technique as the main proposition—Theon, clearly, saw no problem in considering angles as magnitudes. Theon was probably reworking (to make it consistent with *Elements* VI, 33) a proof of this corollary that had been given by Pappus a few decades earlier. The latter proof was itself derived from Archimedes and has been speculated to be older than Archimedes himself and indeed pre-Euclidean. Such a proof does not make use of the theory of proportions set forth in Book V of the *Elements* and rather uses a different notion of proportionality. It is not impossible that *Elements* VI, 33 itself was originally proven in this way (i.e., without using the theory of proportions of Book V), and we can reconstruct how such an alternative demonstration may have been. Should this be the case, we might ask whether Euclid or a later author (such as Theon himself) reworked this hypothetical first version of *Elements* VI, 33 into the one we read in the *Elements* today. If it is so, no inference should be drawn from *Elements* VI, 33 about angles being magnitudes. Note that *Elements* VI, 33 is the last proposition in this book and has no obvious connection with the main topics of it. It is used only in two propositions in Book XIII, which is generally attributed to Theaetetus, who probably had a different conception of ratios compared to the Eudoxean one set forth in Book V. On the whole question, see the important essay by Knorr (1978), who first discussed the Archimedes-Pappus' proof of the corollary of *Elements* VI, 33 and attributed it to pre-Euclidean sources. See also Vitrac, 1990–2001, vol. 2, 243-44, for a hypothetical reworking of *Elements* VI, 33 along these lines, and 525-26 for an appraisal of Knorr's thesis (Vitrac thinks that the alternative proportion theory is in fact Archimedean and post-Euclidean). Further arguments are provided in Saito 2003 and Mendell 2007, which distinguish the theory of proportions of Book V from the techniques used in Book VI of the *Elements*.

[62]See *Data*, def. 1, and cf. (e.g.) prop. 40. It may be objected that angles are never "given in form" in the *Data*, which seems to speak against Euclid's fundamental notion of them as configurations. It should be noted, however, that in the *Data* only

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discrepancies on the use of angles in the *Elements* and elsewhere should not be surprising. There is no doubt that ancient mathematicians used, and probably mixed together, different conceptions of angles in their mathematical practice. Greek geometers and astronomers had to measure and calculate angular magnitudes and, as a consequence, they considered ratios and proportions between angles and, in some cases, measured angles by numbers. Euclid's *Elements* are a composite text and the different essays collected in the thirteen books display different mathematical practices concerning angles.[63]

However, I think that it is a mistake to take this variety of approaches to the notion of angles as an argument for claiming—against all ancient sources and textual evidence—that Euclid had a quantitative conception of them. This view stems from a confusion between the level of concrete mathematical practices and that of foundational analysis. The point is not how angles are used in the *Elements*, but how Euclid believed he could logically ground the theory of angles on a system of definitions—or a postulate. The question about P4 is not about mathematical practices or tacit knowledge (no Greek mathematician has ever doubted that all right angles are equal), but about the formulation of explicit foundations of geometry.

If we admit that Euclid had a design for the foundation of the theory of angles, we must show that it is at least possible to accommodate a definition of angles as relations with the mathematical practice of using angles as magnitudes. In the system of definitions of the *Elements*, there is no explicit trace of this sort of foundation, and so we can only speculate how Euclid, and perhaps the other mathematicians of the time, envisioned coordinating the two notions.[64]

The simplest conjecture to this effect, I suggest, is to assume that Euclid regarded angles as *relations with the property of having magnitudes*. Euclid distinguished between an angle and its magnitude, and the latter was conceived as a property and a consequence of the relation defining the angle. He considered angles to be inclinations and shapes at a foundational level, and nonetheless admitted that their magnitudes could be compared, summed, and put into ratios. The idea was not entirely unprecedented, because ratios, in the theory of Eudoxus developed in the *Elements*, are also relations at a foundational level, that nevertheless have certain quantitative properties (they are, for example, greater than, equal to, or less than other ratios). Similarly, an angle was for Euclid an inclination having a quantity rather than the quantity of an inclination.[65]

"figures" may be given in form, and an angle is not a figure in the proper sense (i.e. according to *Elements* I, def. 14). Moreover, *Data* 29 may be read as: if an angle is given in magnitude, then the configuration of straight lines is also given, in the sense that if a side and the vertex are given in position, then the other side is also given in position. Note that this fact is established by a theorem, whereas the opposite inference, that if two intersecting straight lines are given in position then the angle they make is given in magnitude, is never proven. I would say that Euclid considered the latter statement an immediate consequence of the definition of angles—just like P4.

[63]Before Apollonius, both Aristarchus (*On the Size and Distance*) and Archimedes (*Arenarius*) measured angles by means of fractions of the circumference or the right angle, and took ratios between angles. They never explicitly stated that angles are magnitudes, and it is quite possible that they were working outside the definitional tradition of *Elements* V, which requires ratios to be between μεγέθη (the same might be said, as we have seen in note 62, for alternative versions of *Elements* VI, 33). Archimedes, however, applied a form of *Elements* X, 1 to angles in Prop. 3 and 4 of *Sphere and Cylinder*, as well as in Prop. 22 and 23 of *Spirals*. It is not clear that Archimedes was referring here to the *Elements*, and he may instead have had Eudoxus' book in mind, from which this proposition in the *Elements* was taken. Yet, in the *Elements* this proposition explicitly refers to "magnitudes", and this may also have been the case for Eudoxus' original formulation. Should this be the case, Archimedes would here be taking angles as μεγέθη in the technical sense. Archimedes gave general theorems on magnitudes in *Sphere and Cylinder*, but the definitional system of this work seems to assume only lines, surfaces and solids as μεγέθη. The same seems to happen in Archimedes' other works, in which the named magnitudes are always extended figures rather than angles.

[64]Faced with the plurality of mathematical practices concerning angles, Proclus and other scholastic authors of Late Antiquity sought a concordist (but shallow) solution, arguing that angles are quantities, but also qualities, but also relations, and in short categorial mishmashes (*In Euclidis* 123-125). But this, even according to Proclus and the other eclectic commentators, was not the view of Euclid—who had rather defined angles as "relations."

[65]An extensive discussion on angles not being magnitudes but having magnitudes is to be found in ibn al-Haytham's first commentary on Euclid. Here, ibn al-Haytham employs some Aristotelian scholastic terminology to claim that angles, not being magnitudes, are not divisible *per se* but *per accidens*; and that an angle is a shape that determines a magnitude (i.e. the

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The distinction between angles and angular magnitudes is indeed required by the fact that Euclid considers only angles less than a straight angle. Since an angle is defined by him as an inclination between two lines, Euclid denies that a straight angle is an angle, and also excludes non-convex configurations of lines. This fact, which is everywhere confirmed by Euclid's mathematical practice, rules out the possibility that angles can simply be identified with their magnitudes. The interior angle sum of a triangle, which Euclid proves to be equal to "two right angles," for instance, is *an angular magnitude which is not an angle*. These angular magnitudes, distinguished from the angles from which they are derived, are summed, multiplied and put into ratios in the theorems of the *Elements*.[66]

This even occurs at the foundational level. The proposition of *Elements* I, 13, for instance, states that "if a straight line set up on a straight line make angles, it will make either two right angles or angles equal to two right angles." Should we take angles to be magnitudes, the theorem would be a triviality. It states, in fact, that angles forming together by hypothesis a straight angle are equal to two right angles, that is to say, to a straight angle. But, as we have seen, Euclid has no notion of a straight angle as a geometrical configuration, and therefore in this case he needs a propositional articulation (going beyond diagrammatic evidence) of the equality of two magnitudes, which is not an equality of two angles. The distinction between angles and angular magnitudes requires an explicit proof of this fact.[67]

The fact that angles *are not magnitudes* but *have magnitudes* is in sharp contrast with Euclid's treatment of lines, plane figures and solid bodies: all these objects are considered *to be* magnitudes in the proper sense. The identification of figures and magnitudes is a pervasive and fundamental feature of ancient geometry and epistemology of mathematics, and it is at the foundations of the theory of content (*viz.* measure) in Greek geometry. I will not belabor the point here, but will note at least that the definability of a standard of measure for angles (i.e. the right angle) as opposed to the indefinability of such a standard for lengths (say, the "standard meter") seems to be the core reason for the very different theories of angles and extensive magnitudes in classical geometry. An extensive magnitude may only be shown in intuition through a particular instance of it, and it is therefore natural to identify segments or squares with their own magnitudes. By contrast, an angular magnitude may be defined in a purely propositional way (e.g. "three right angles"), and it is therefore possible to refer to it without exhibiting it in intuition as a concrete figure. The distinction between angles and angular magnitudes is a natural byproduct of this more fundamental divide.[68]

convex part of the plane intercepted by its sides). I think that ibn al-Haytham had seen an important feature of Euclid's conception of angles and magnitudes. For an English translation of this commentary, see Hooper Sude 1974, 37–42. For a French translation and a detailed discussion of it, see Rashed 2015. In the latter book, three texts on ibn Sīnā's theory of angles are also offered. In them, Avicenna considers angles as convex surfaces delimitated by lines, and disregards the notion of inclination altogether.

[66]These issues were already being noticed in Antiquity, and represented important arguments to the effect that for Euclid angles were relations rather than quantities. Simplicius, *apud* al-Nayrīzī, states that a quantity, when doubled, remains a quantity. This does not happen with the right angle. Therefore, Simplicius concludes, angles are not quantities (Arnzen 2002; English translation in Lo Bello 2009, 6).

[67]It has been noted that Euclid never assumed or proved a proposition to the effect that: given the segment AB and a point X on it, then AX+XB = AB. This missing assumption is easily explained by the fact that, for Euclid, segments are themselves magnitudes, and therefore this proposition would appear to him as a triviality (as far as I know, such a proposition on segments was first discussed in the very different epistemological context of Leibniz' essays on the foundations of geometry; cf. his unpublished *Specimen analyseos figuratae*, in LH XXXV, I, 14, Bl. 21–22). It is clear that this proposition on segments is analogous to *Elements* I, 13 on angles, which Euclid saw the necessity to *prove*. I would surmise that Euclid would have not felt the need to prove that any two "complementary angles" (angles made by tracing a ray inside a right angle) together make one right angle. This was rather trivial, for him, like the case of segments above: a right angle (contrary to a straight angle) is a proper angle and a configuration of lines.

[68]There is no need to think that for Euclid an angular magnitude was an abstract magnitude. Although the sum of the angles of a triangle is not an angle, one can imagine that it was for Euclid "angles," that is, the magnitude of two right angles was represented as a pair of configurations of perpendicular lines. This was Dodgson's view (1879, 192-93), also accepted by

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The present conjecture about the distinction between angles and angular magnitudes explains the puzzle of the apparent incongruence of Euclid's definitions with the mathematical practices found in the *Elements*. But even if this conjecture were not accepted, and other hypotheses were advanced as to the compatibility, or lack thereof, of Euclid's definition of angles with the mathematical practices of the *Elements*, there would still be no need to infer from those practices that for Euclid angles were magnitudes. At the foundational level, as it is clear from the definitions and the ancient testimonies, an angle was for him a relation and a shape. This in turn, I maintain, makes the Fourth Postulate useless in Euclid's own theory.

4.3. Apollonius on angles and the Fourth Postulate

Euclid's conception may be usefully contrasted with Apollonius' idea of angles as magnitudes in the full sense. Proclus, indeed, ascribed to Apollonius the first *definition* of angles as magnitudes in Greek mathematics. By defining angles as magnitudes, then, Apollonius consciously opposed Euclid and all other previous geometers. He knew, like Euclid before him, that mathematical practices on angles varied widely and that angles were treated as forms, qualities, relations, or quantities. But he believed that the correct foundational path for the theory of angles should begin with the notion of a magnitude.

Apollonius' definition of angles is famously obscure: he called a plane angle "the contraction of a surface at a point under a broken line" (συναγογὴ ἐπιφανείας πρὸς ἑνὶ σημείῳ ὑπὸ κεκλασμένῃ γραμμῇ).[69] Apollonius' definition is given by Proclus out of context and without any real mathematical application. Moreover, the definition itself does not make any reference to magnitudes, despite the fact that Proclus stated that Apollonius and his successors took angles to be magnitudes.[70]

I would say that the most obvious reason for introducing Apollonius' definition may be found in the aim of comparing rectilinear and curvilinear angles with one another. Such a comparison may be needed in the treatment of tangents, and requires greater resources than Euclid's own configurational definition, his conception of angular equality as congruence, and his diagrammatic inferences grounded on the intuitive notion of mereological containment. A curvilinear angle with the same vertex and side as a rectilinear angle, indeed, may neither contain nor be contained by the latter (see figure 2).

This makes their standard diagrammatic comparison impossible, since neither of them is a part of the other. The consideration of a *small enough* part of the plane around the vertex of the angles, however, allows us to tell which angle is inside the other. This would make available once again a

Heath (1925, vol. 2, 275-76), and can perhaps be supported by an analysis of the notion of "equimultiple" in the theory of proportions (e.g., the equimultiple of a circle was probably not regarded by Euclid as a big circle, but as a plurality of circles). But even if angular magnitudes are considered as concrete objects, it would still remain true (1) that they are distinct from angles, because there is no biunivocal correspondence between the possible inclinations of two lines and the possible angular magnitudes; (2) that they would be concrete objects that can be defined in the abstract, because there is no need to show a right angle to know its magnitude.

[69]Proclus, *In Euclidis* 123.

[70]I have already mentioned (in note 20 above) that Proclus reports a discussion on the angles at the vertex of a cone (*In Euclidis* 123). Apparently, this discussion had the aim of providing an argument that angles cannot be inclinations. Indeed, an angle on the vertex of a cone comprehends two different extended magnitudes (μεγέθη): the surface of the triangle cutting the cone and forming the angle, and the surface of the cone itself included in the angle. But, the objection continues, the same inclination cannot determine two different angles, and therefore an angle is not an inclination. It should be noted here that: (1) Proclus agrees with the argument that the angle conceived as an inclination determines a magnitude (rather than being identical with it); (2) The magnitude of an angle is indeed uniquely determined by its inclination; (3) The magnitude of the angle is taken to be a small surface (either a plane surface, in the triangle, or a curved surface, on the cone), just as in Apollonius' definition. I would suggest that the argument reported by Proclus may indeed come from Apollonius himself, whose definition of angles is mentioned by Proclus immediately thereafter. I may add that what Apollonius normally calls the "angle of a cone" (γωνία κώνου) is in fact the vertex of a cone: see *Conics* I, 11 and Mugler 1958, 110.

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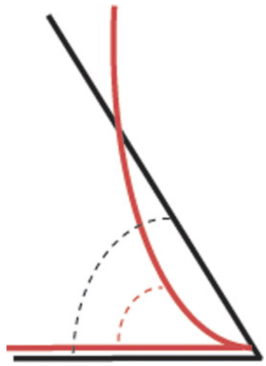

Figure 2. Diagrammatic comparison of curvilinear angles.

criterion based on diagrammatic inspection, which would allow us to say that (for instance) the curvilinear angle is less than the rectilinear angle. I surmise, then, that Apollonius' definition of an angle as "the contraction of a surface at a point under a broken line" had the aim of making these comparisons possible, and that the need for a consistent treatment of curvilinear angles in Greek geometry was the mathematical reason behind the switch from Euclid's configurational definition to Apollonius' quantitative definition.[71]

It is important, indeed, to understand that Apollonius' definition takes angles to be (finite, small) portions of the plane. We know, from Proclus' report, that this was exactly the topic of discussion of the many further refinements and adjustments of Apollonius' definition. Syrianus, Proclus' master, seems to have corrected Apollonius by saying that an angle is a *contracted surface*, rather than the act of contraction itself. Another Neoplatonist, Plutarch of Athens, took this contraction to result in the *first extension* (πρῶτον διάστημα) under the point, possibly meaning a minimal, atomic extension. Carpus of Antioch maintained that a plane angle is a quantity extended in one dimension (rather than two), without it being a line.[72] Others argued whether a plane angle may have a dimension in between those of a line and a surface.[73]

Whatever we may think of these elliptic fragments of debate provided by Proclus, it seems clear that all these conceptions regarded an angle as a piece of a surface (or a line, or something in between), and moreover a finite piece of it (a contraction, or contracted surface, or minimal surface, according to various definitions). An angle is therefore a finite extended quantity of the same kind of the standard magnitudes—segments, plane figures, solid bodies. For this reason, an angle, according to Apollonius and his followers, is indeed a μέγεθος in the full sense of Eudoxus,

[71] I can add that Bernard Vitrac (1998) conjectured that the propositions on mixtilinear angles (in *Elements* III, 16 and 31) were added to the work of Euclid in order to discuss the paradoxical nature of such angles in Apollonius' time or shortly thereafter. Indeed, a long part of the statement in *Elements* III, 16 is also to be found (with the same words) in the statement of *Conics* I, 32. This may be regarded as a reference to Euclid made by Apollonius, but may also signal Apollonius' authorship of both statements. Vitrac's conjecture would fit with my interpretation of the meaning and aims of Apollonius' new definition of angles. I am skeptical, on the other hand, of Heath's over-modernizing attribution to Apollonius of some sort of dynamic or proto-fluxional ideas concerning angles as vanishing quantities (Heath 1925, vol. 1, 177).

[72] Proclus, *In Euclidis* 124-26. See above, note 44.

[73] This opinion is attributed to Apollonius himself by Simplicius, who endorsed it (Arnzen 2002; English translation in Lo Bello 2009, 7). Simplicius adds that an angle comprehends a surface even if not from every side, just as "a harbor comprehends a ship." In Latin Europe, the earliest instance of the view that angles are "spaces" comprehended by lines is probably to be found in the tenth-century *Geometria* by Gerbert of Aurillac (1880, 100). I can also mention that, in the sixteenth century, Niccolò Tartaglia interpreted P4 as a statement asserting that all right angles are equal irrespective of the length of the segments forming them (Tartaglia 1543, xii–xiii). Such an erratic interpretation of the postulate becomes understandable in the framework of the Apollonian definition of angles, which are finite portions of the plane.

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Aristotle and Euclid. The Apollonian definitions of angles as magnitudes may therefore have been regarded, in Antiquity, as a solid foundation of the use of proportions in the theory of angles.

Proclus does not provide us with any direct application of Apollonius' new definition of angles in geometry. Nonetheless, in the course of his commentary on Book I of the *Elements*, he mentions a few other elementary theorems about angles attributed to Apollonius that may hint at Apollonius' foundational endeavors in the theory of angles. We know, indeed, that Apollonius offered alternative demonstrations of *Elements* I, 10–11, teaching how to construct perpendiculars and thereby right angles.[74] Another extant demonstration ascribed to him reformulates Euclid's theorem of angle transportation (*Elements* I, 23).[75] While the two former proofs on perpendiculars are quite similar to Euclid's, the last one widely diverges from the material in the *Elements*, suggesting that Apollonius' theory of angle transportation might have been itself very different from Euclid's. We have seen, indeed, that Euclid grounded angle transportation on the congruence of triangles obtained by superposition (with no references to P4). Apollonius seems to follow an approach closer to the modern one, in which the transportation of angles is established before proving the congruence theorems for triangles. Proclus objected that, by proceeding in this way, Apollonius was obliged to assume a few results on circular arcs coming from Book III of the *Elements*. But it is not evident that this was the case, and Apollonius may have employed definitions (or *postulates*) on angle equality that made his proof more direct than Euclid's. In particular, Apollonius (according to Proclus) seems to have assumed that two angles subtending equal arcs are equal, and this may well be a consequence (or just a reformulation) of his definition of an angle as a magnitude and the contraction of a surface.[76]

Apollonius' definition of angles clearly has important consequences on the meaning of the Fourth Postulate and its foundational significance. According to this definition, indeed, P4 would not mean that all right angles are congruent, but would rather state that *all right angles are equal magnitudes*. In this interpretation, P4 means that (say) a small portion of the plane around the point of intersection of a straight line and its perpendicular is equal to every other such portion formed by other perpendicular lines. This is a non-trivial statement that cannot be grounded on the definition of a right angle and requires, indeed, an explicit propositional assumption.

It is impossible to establish, from Proclus' overly terse account, whether Apollonius' new theory made explicit use of P4. We have seen that there are two possible references to P4 in the Arabic translation of Apollonius' *Conics*, and that it is difficult to ascertain whether they may have been interpolated in the text—and therefore whether Apollonius may have made use of P4 as a principle. The very fact that Apollonius reworked the entire foundations of the theory of angles shows that his new definition of angles as magnitudes was not a "local" amendment with the aim of fixing some philosophical difficulties, but required a reassessment of the entire axiomatic theory. It is possible, then, that P4 was first formulated as the outcome of Apollonius' new definition of an angle, either by Apollonius himself or by some other mathematician of his time, to cope with the demands of this new, thoroughly quantitative understanding of angles.

[74]Proclus, *In Euclidis* 279–83. It is remarkable that the only theorems employed in these proofs by Apollonius (as reported by Proclus) are *Elements* I, 4 and 8, which are those employing superposition. We cannot say whether Apollonius had discussed and possibly re-demonstrated them in a different way.

[75]Proclus, *In Euclidis* 335–36.

[76]Euclid proves that two angles subtending equal arcs are equal in *Elements* III, 27, itself employing in its proof *Elements* I, 23 on the transportation of angles and thus depending on *Elements* I, 8 (the second criterion of congruence for triangles). Apollonius may well have taken the opposite direction, assuming *Elements* III, 27 (or proving it from P4), and then proving *Elements* I, 23 with the demonstration offered by Proclus, and from this latter (with some further assumptions) the congruence theorems for triangles. Apollonius may have also held a view different from Euclid's concerning the transportation of segments, and Heiberg ascribes to him an alternative proof of *Elements* I, 2 mentioned by Proclus (*In Euclidis* 227). This is quite natural if we think of the equality of segments as the possibility of superposition ("occupying the same place", according to Apollonius; cf. Proclus, *In Euclidis* 194–95). Should this be the case, we see that Apollonius' way through the theorems of congruence may have been very different from Euclid's and much more similar to modern axiomatizations.

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It may still be asked why Apollonius or other mathematicians of his time did not prove P4 through superposition rather than assuming it as a postulate. Perhaps they did: while Euclid does not seem to assume any provable principle in the *Elements*, the philosophy of mathematics in the following centuries was sufficiently flexible to admit these as well.[77] Alternatively, Apollonius may have had his own reasons not to employ superposition in this connection, and have thought that this technique could prove nothing concerning angles as magnitudes. After all, he seems not to have proven angle transportation (*Elements* I, 23) through triangle transportation and superposition.[78] Had he done so, P4 may have been first formulated as an unprovable principle and later proven in a still different epistemological context (Posidonius' replies to Zeno's criticisms).

It is not implausible that, if P4 was first formulated in Apollonius' circle, it was later added to the text of the *Elements*. Apollonius, we are told, was "a student of the students of Euclid,"[79] and we know from Apollonius' preface to the *Conics* that he was corresponding with a group of mathematicians working in Alexandria, who were most likely these followers of Euclid. Apollonius himself wrote some *elements* (στοιχεία) of geometry, which may have been a reworking of Euclid's, discussed the axiomatization of geometry and proposed demonstrations of Euclid's common notions.[80] More importantly, a Greek scholium tells us that three definitions in Euclid's *Data* (two of which concern angles) had in fact been conceived by Apollonius. These definitions are never further employed in the *Data* (just as P4 is not explicitly employed in the *Elements*), and it is difficult to fathom their mathematical significance. The early interpolation of these Apollonian definitions into Euclid's work is important evidence that other principles coming from Apollonius may easily have been added to the *Elements*.[81] In sum, it is wholly possible that Apollonius' discussions on the foundations of mathematics, carried out with Euclid's students in Alexandria, contributed to the modification of the text of the *Elements*.

In conclusion, we may take a second look at Geminus' statement about P4 as we find it in Proclus, which is our earliest testimony of the existence of P4 in the *Elements*. Geminus' aim is to show that P4 is not a postulate. The argument seems to run as follows: if P4 is a principle, then it must be a common notion, since it is not constructive in the sense that the other postulates are; and if it is not a principle but rather a provable proposition (according to the proof that Geminus expounded), then it is not a postulate either, since it should be considered a theorem.[82] It is surely possible that Geminus' dichotomy (P4 is a principle or a provable proposition) was merely dialectic and constructed in order to better make his point. It is more likely, however, that

[77]For instance, some of the principles (called προλαμβανόμενα) in Archimedes' *Method* are proven in the *Equilibrium of Planes* and in *Conoids and Spheroids*. In the *Method*, Archimedes stated *explicitly* that they can be proven.

[78]I may also mention here that Apollonius himself does employ superposition in his surviving works, but he makes use of it in a different way from Euclid in *Elements* I, 8 or III, 24, and the proof of P4 reported by Proclus. In particular, Apollonius does not employ superposition as a construction that allows the drawing of diagrammatic inferences—as is done in all the above-mentioned proofs. Therefore, it is possible that Apollonius did not accept the ancient proof of the Fourth Postulate and rather took it as an unprovable assumption. I will not belabor further this difficult topic in the present essay. Paul Tannery, who believed in the spuriousness of P4 (see above, note 2), also thought that Apollonius had offered its proof (Tannery 1881).

[79]Pappus, *Collectiones* H 35.

[80]We are informed by Pappus (*Collectiones* H 21) that Apollonius wrote some στοιχεία of geometry and by Marinus (*In Euclidis data* 234) that he wrote a *general treatise* (καθόλου πραγματεία) on mathematics which may, or may not, be identified with the former. Apollonius had a strong interest in the foundations of mathematics and worked in Euclid's tradition of geometry, and the first four books of Apollonius' *Conics* were possibly a reworking of a lost work by Euclid (cf. Pappus, *Collectiones* H 30). Proclus tells us about Apollonius' attempt to prove Euclid's common notions in *In Euclidis* 194–95. On Apollonius' involvement with the foundations of geometry, see Tannery 1881 and Acerbi 2010.

[81]The interpolated definitions are *Data*, def. 13–15, and Apollonius' authorship of these may be read in scholium 13 to the *Data* (Heiberg and Menge 1883–1899, vol. 6, 264). See also the critical discussion in Heiberg's *Prolegomena* (Heiberg and Menge 1883–1899, vol. 6, xlix), stating that these definitions are to be found *in all manuscripts* we have of this work (just as in the case of P4 and the *Elements*), and therefore the Apollonian addition must have been in a "*quasi archetypum*" exemplar of the text.

[82]Proclus, *In Euclidis* 188.

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Geminus was simply reporting existing discussions. In the mathematical literature with which he was familiar, someone had posited P4 among the postulates, while someone else (possibly Posidonius) had proposed its demonstration and placed it among the theorems. It is even possible that in Geminus' time there was a debate on the need for P4 in the foundations of geometry, and Geminus was attempting (though failing) to have it excluded from the list of the postulates of the *Elements*. Should this be the case, the text of the *Elements* would show a remarkable fluidity even in the first century BCE. Given that the next historical source at our disposal is Heron (first century CE), who mentioned P4 among the definitions, this state of uncertainty may have persisted for some time.

This, then, is my conjecture on the addition of P4 to the *Elements*. It is grounded on an interpretation of Euclid's notion of angles as non-quantitative geometrical configurations, and an inference to the uselessness of P4 in such a context. The conjecture also offers an interpretation of Apollonius' theory of angles as magnitudes and an inference to the need for P4 in this different foundational context. Finally, it shows the historical plausibility that Apollonius' ideas may have entered the system of principles of the *Elements*. The conjecture, therefore, provides positive arguments for the missing references to P4 in the demonstrations in the *Elements*, without claiming that Euclid simply overlooked this principle: Euclid did not need P4 at all. A later interpolation of P4 may account, in turn, for the non-constructive features of P4, since the epistemology of mathematics was rapidly changing in the centuries after Euclid. The same story also accounts for the puzzlement about P4 which we find among ancient interpreters, who clearly did not know what to do with it: the principle, indeed, came from a different mathematical tradition, and attempted to solve problems that were not in the *Elements*. Finally, this conjecture offers non-trivial mathematical and logical reasons for the addition of P4 to the *Elements*, thus giving an interpretation of the meaning and significance of P4 in ancient geometry.

The conjecture complements with positive arguments the negative textual and mathematical results of the previous sections, which had established that we have no evidence that the Fourth Postulate was in the *Elements* before the time of Geminus, and that its foundational aims could not possibly be those attributed to it by modern mathematicians. These latter conclusions are more certain, and set the stage for conceiving several different stories about the role and the interpolation of the Fourth Postulate in the text of the *Elements*. These stories tell of Hellenistic professors inventing the postulate to justify the use of common notions about the addition of angles; of sophists and atomists discussing angles that are almost right angles, but not quite right angles; of Posidonius, who may have believed that the equality of right angles was a condition for the Parallel Postulate; and many more can be dreamed up. It seems to me that the conjecture about the transformation of the definition of an angle between the times of Euclid and those of Apollonius is the most consistent with the scanty evidence we have, and our best guess for explaining several puzzles in the foundations of Greek mathematics.

We do not know who, between the third and the first century BCE, conceived the idea that the equality of all right angles should be a principle of geometry. Since then, the significance of this statement has changed many times over the centuries. Yet the Fourth Postulate, for more than two thousand years, has remained at the core of the foundations of mathematics.

Acknowledgements. I am especially grateful to Orna Harari, Mattia Mantovani, and two anonymous referees for their attentive reading and innumerable suggestions: without them the present essay would have never appeared. I also thank Michael Beeson, Pierre Boutry, Micheline Decorps-Foulquier, Michael Fried, José Gil-Férez, Jeremy Gray, Victor Pambuccian, Marco Panza and Bernard Vitrac for their advice.

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Cite this article: De Risi, Vincenzo. 2022. "Euclid's Fourth Postulate: Its authenticity and significance for the foundations of Greek mathematics," *Science in Context* **35** (1): 49–80. doi:10.1017/S0269889723000145

https://doi.org/10.1017/S0269889723000145 Published online by Cambridge University Press