SUBGROUPS OF FINITELY PRESENTED METABELIAN GROUPS OF FINITE RANK

JAMES BOLER

Abstract

Let G be a finitely generated metabelian group whose derived group G' has finite rank. It is shown that G can be embedded in a finitely presented metabelian group H with H' of finite rank.

Introduction

Baumslag (1973) has shown that each finitely generated metabelian group can be embedded in a finitely presented metabelian group. Our objective here is to prove

THEOREM 4.1. *Let G be a finitely generated metabelian group whose derived group G' has finite rank. Then there exists a finitely presented metabelian group H with H' of finite rank, and an embedding of G into H.*

The proof of Theorem 4.1 makes use of the well-known connection between finitely generated metabelian groups and modules over finitely generated commutative rings. This relationship is mentioned in Section 1, along with some basic facts about the notion of rank. In Section 2 a residual property of certain modules is established. This property is used in Section 3 and Section 4 to prove Theorem 4.1.

1. Preliminaries

We denote the ring of integers by Z and the field of rational numbers by Q.

If x and y are elements of a group, we use the standard notation

$$x^y = y^{-1}xy \,; [x, y] = x^{-1}y^{-1}xy.$$

501

If G is a group and R is a commutative ring with unit, RG is the R-algebra consisting of finite sums $\Sigma\, r_i g_i$ with $r_i \in R$ and $g_i \in G$. Addition is linear, and multiplication is induced by the multiplication of G.

If A is an abelian normal subgroup of a group G, G acts on A by conjugation: $ag = g^{-1}ag$. Since A is abelian, this gives A a (G/A)-module structure. Equivalently, A becomes a module over the integral group ring $Z(G/A)$ of G/A.

If A is a G-module, the semidirect product $A\,]\,G$ of A by G is the group consisting of elements ag ($a \in A$, $g \in G$) under the multiplication

$$(a_1 g_1)(a_2 g_2) = (a_1 + a_2 g_1^{-1})(g_1 g_2).$$

A group G is *metabelian* if there is an abelian normal subgroup A of G with G/A abelian. When G is finitely generated, A satisfies the ascending chain condition for submodules and hence G satisfies the ascending chain condition for normal subgroups (Hall, 1954).

DEFINITION 1.1. *We say an abelian group A has finite rank $\leqq n$ if each of its finitely generated subgroups can be generated by n elements.*

This notion of rank can be defined for arbitrary groups and has been called *special rank* (Mal'cev, 1948) and *Prüfer rank* (Wehrfritz, 1973).

It should perhaps be mentioned that this definition of rank is more restrictive than the usual definition of rank of an abelian group (where the rank of A is the dimension of $A \otimes Q$). However, the two notions coincide for torsion-free abelian groups.

The following proposition is elementary.

PROPOSITION 1.1. *Let A and C be abelian groups of finite rank.*
(i)   *If $B \leqq A$, B has finite rank.*
(ii) *If B is a homomorphic image of A, B has finite rank.*
(iii) *If*

$$0 \to A \to B \to C \to 0$$

is a short exact sequence of abelian groups, B has finite rank.

PROPOSITION 1.2. *Let A be an abelian normal subgroup of the finitely generated group G, and suppose G/A is abelian. If A has finite rank, the torsion subgroup T of A is finite.*

PROOF. Let $T = \bigoplus_{i=1}^{\infty} T_i$ where $p_1, p_2, \cdots, p_n, \cdots$ are the prime numbers and T_i consists of those elements of T of order a power of p_i. Now, $T_i = 0$ for all but finitely many primes p_i. Otherwise, putting $A_n = \bigoplus_{i=1}^{n} T_i$,

$$A_1 \leqq \cdots \leqq A_n \leqq A_{n+1} \leqq \cdots$$

would be an infinite chain of normal subgroups of G which is proper at infinitely many places, contradicting (Hall, 1954). Thus $T = \bigoplus_{i=1}^{n} T_i$ for some n, and it is enough to show that the p_i-primary group T_i is finite ($1 \leqq i \leqq n$). This can be done by noticing that, again because of (Hall, 1954), there is an integer N with $p_i^N t = 0$ for all t in T_i-otherwise a properly ascending infinite chain

$$A_1 \leqq \cdots \leqq A_j \leqq A_{j+1} \leqq \cdots$$

could be constructed by letting

$$A_j = \{t \in T_i \mid P_i^j t = 0\}.$$

Thus T_i is a p_i-primary abelian group with a finite exponent hence is a direct sum of cyclic groups (see Kurosh (1955), p. 171). Now $T_i \leqq A$ has finite rank by Proposition 1.1, and since T_i has finite exponent, it must be finitely generated. This can be seen by noting that a direct sum of m cyclic groups of order p^n, p a prime and n fixed, cannot be generated by fewer than m elements. Thus T_i is finite and Proposition 1.2 is proved.

2. Separating subspaces of QT-modules

Throughout this section T is a finitely generated abelian group. In order to prove Theorem 4.1, we will make use of

THEOREM 2.1. *Let A be a finitely generated QT-module and $0 \neq S$ a subspace of A with $\dim_Q S < \infty$. Then there is a submodule K of A with $\dim_Q (A/K) < \infty$ and $K \cap S = 0$.*

Note that Theorem 2.1 implies that nonintersecting finite-dimensional subspaces of A can be separated by a submodule of finite codimension.

PROOF OF THEOREM 2.1. We shall prove Theorem 2.1 by a sequence of lemmas. Following Hall (1959), we say a QT-module A is *monolithic* with *monolith M* if the intersection M of the nonzero submodules of A is itself nonzero.

LEMMA 2.1. *Let R be a noetherian algebra over a field F and let I be an ideal of R with $I^m = 0$ for some positive integer m. If $\dim_F (R/I) < \infty$, then $\dim_F R < \infty$.*

PROOF. The lemma is clearly true when $m = 1$. Let $m \geqq 2$ and suppose it is true for $m - 1$. Then there is a short exact sequence

$$0 \to I^{m-1} \to R \to R/I^{m-1} \to 0$$

of R-modules. Since $(I/I^{m-1})^{m-1} = I^{m-1}$, $\dim_F (R/I^{m-1}) < \infty$ by induction. Now I^{m-1} is finitely generated because R is noetherian, and $I(I^{m-1}) = 0$. Thus I^{m-1} is a finitely generated R/I-module, so that the finite dimensionality of R/I implies that of I^{m-1}. Thus

$$\dim_F R = \dim_F I^{m-1} + \dim_F (R/I^{m-1}) < \infty$$

and Lemma 2.1 is proved.

LEMMA 2.2. *Let R be a homomorphic image of QT and suppose R is monolithic with monolith M. Then each element of $I = \{r \in R \mid rM = 0\}$ is nilpotent.*

PROOF. Let $u \in I$ and suppose u is not nilpotent. Then the elements

$$u, u^2, \cdots, u^n, \cdots$$

are all nonzero and hence each Ru^n ($n = 1, 2, \cdots$) is a nonzero ideal of R and so must contain M. If $0 \neq a \in M$ is fixed, we must have

$$y_1, y_2, \cdots, y_n, \cdots$$

in R with $y_n u^n = a$.

Now let $A_n = \{r \in R \mid ru^n = 0\}$ so that

(2.1) $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n \subseteq A_{n+1} \subseteq \cdots$

is an infinite ascending sequence of submodules of R. Now $y_n u^{n+1} = au = 0$ since $u \in I$. Thus $y_n \in A_{n+1} - A_n$ and hence (2.1) is a properly ascending infinite chain, a contradiction since each finitely generated QT-module satisfies the ascending chain condition for submodules (Hall, 1959, Theorem 4). Thus Lemma 2.2 is proved.

LEMMA 2.3. *Let R be a homomorphic image of QT. If R is monolithic, $\dim_Q R < \infty$.*

PROOF. Let M be the monolith of R and let $I = \{r \in R \mid rM = 0\}$. By Lemma 2.2 each element of I is nilpotent. Since I is finitely generated, there is an integer m with $I^m = 0$.

Now M, being the monolith of R, is irreducible, so that $M \approx R/I$ and I is a maximal ideal of R. By the Hilbert Nullstellensatz (see Atiyah–MacDonald (1969), p. 67), $\dim_Q (R/I) < \infty$. Invoking Lemma 2.1 completes the proof of Lemma 2.3.

LEMMA 2.4. *Let A be a finitely generated monolithic QT-module. Then $\dim_Q A < \infty$.*

PROOF. Let M be the monolith of A and let $a_1, \cdots, a_k$ be a set of nonzero generators for A. If $R = QT$, then each Ra_i $(1 \leqq i \leqq k)$ is a nonzero submodule of A and so must contain M. It follows that each Ra_i is monolithic with monolith M. By Lemma 2.3,

$$\dim_O Ra_i < \infty \qquad (1 \leqq i \leqq k).$$

But A is a homomorphic image of $\bigoplus_{i=1}^k Ra_i$, so that

$$\dim_O A \leqq \sum_{i=1}^k \dim_Q (Ra_i) < \infty.$$

LEMMA 2.5. Let A be a finitely generated QT-module and let $0 \neq a \in A$. Then there is a submodule K of A with $a \notin K$ and $\dim_Q (A/K) < \infty$.

PROOF. Let K be a submodule of A maximal with respect to $a \notin K$. Then each nonzero submodule of A/K contains $a + k$, so that A/K is monolithic. By Lemma 2.4, $\dim_O (A/K) < \infty$ and Lemma 2.5 is proved.

PROOF OF THEOREM 2.1. Let A be a finitely generated QT-module and S a nonzero subspace of A with basis $x_1, \cdots, x_n$. We argue by induction on n.

If $n = 1$, the result follows from Lemma 2.5.

Assume $n > 1$ and that the theorem holds for subspaces of dimension less than n. Let

$$S_0 = Qx_1 + \cdots + Qx_{n-1}.$$

Then there is a submodule K_0 with $K_0 \cap S_0 = 0$ and $\dim_Q (A/K_0) < \infty$. If $S \cap K_0 = 0$, we're done. If not, let

$$0 \neq y = a_1 x_1 + \cdots + a_n x_n \in S \cap K_0 \qquad (a_1, \cdots, a_n \in Q).$$

Then $a_n \neq 0$, so that $z = ya_n^{-1}$ lies in $S \cap K_0$ and

$$x_1, \cdots, x_{n-1}, z$$

is a basis for S. It follows that $S \cap K_0 = Qz$.

Invoking Lemma 2.5 again, let K_1 be a submodule with $z \notin K_1$ and $\dim_Q (A/K_1) < \infty$. Putting $K = K_0 \cap K_1$, we have

$$\dim_O (A/K) \leqq \dim_O (A/K_0) + \dim_O (A/K_1) < \infty$$

and

$$S \cap K = S \cap (K_0 \cap K_1) = Qz \cap K_1 = 0.$$

This completes the proof of Theorem 2.1.

3. Embedding torsion-free groups

In this section we begin the process of deducing Theorem 4.1 from Theorem 2.1. At this point it is convenient to introduce the theorem of Baumslag (1973) mentioned in the introduction. We state this result in the following form.

THEOREM 3.1 (Baumslag, 1973). *Let A be an abelian normal subgroup of a finitely generated group G with G/A abelian. Then there is a finitely presented semidirect product $H = B]U$ with B and U abelian, and an embedding ϕ of G into H. ϕ may be chosen so that $\phi(A) \subseteq B$.*

We now use Theorems 2.1 and 3.1 to deduce

THEOREM 3.2. *Let G be a finitely generated metabelian group. Suppose G has a normal subgroup A which is maximal with repect to A and G/A both being abelian, and which is torsion free and of finite rank. Then there is a finitely presented group $L = C]U$, with both C and U abelian and of finite rank, and an embedding ψ of G into L. We may choose L so that C is torsion-free and $\psi(A) \subseteq C$.*

PROOF. Let A and G be as in the statement of the theorem and let ϕ: $G \to B]U = UH$ be given by Theorem 3.1. In particular, $\phi|_A : A \to B$ is an embedding. Tensoring with the field of rational numbers, we obtain maps

$$A \xrightarrow{i} A \otimes Q \xrightarrow{\phi|_A \otimes 1} B \otimes Q$$

where $i(a) = a \otimes 1$. Because A is torsion-free and Q is a flat Z-module, the mapping $\beta = (\phi|_A \otimes 1) \circ i$ is an embedding of A into $B \otimes Q$.

Let $\gamma: B]U \to (B \otimes Q)]U$ be defined by

$$\gamma(bu) = (b \otimes 1)u \qquad (b \in B, u \in U)$$

Now $\beta(A) \subseteq A \otimes Q$ and since A has finite rank, $A \otimes Q$ is a finite dimensional subspace of $B \otimes Q$. By Theorem 2.1, there is a submodule K of $B \otimes Q$ with

$$\dim_O ((B \otimes Q)/K) < \infty \quad \text{and} \quad K \cap \beta(A) = 0.$$

Let

$$\delta: (B \otimes Q)]U \to \left(\frac{B \otimes Q}{K}\right)] U$$

be defined by

$$\delta((b \otimes q)u) = ((b \otimes q) + K)u.$$

Then $\delta\gamma\phi$ is a map of G into $\delta\gamma H$. By the constructions above, $\delta\gamma H = D]U$, where $D = \delta\gamma B$ and $\delta\gamma\phi A \subseteq D$. Now, D has finite rank since it is a subgroup of the finite rank group $\left(\dfrac{B \otimes Q}{K}\right)$. Since A is a maximal abelian subgroup of G, $\phi(G) \cap B = \phi(A)$ and $\delta\gamma\phi G \cap D = \delta\gamma\phi A$. It follows that $\delta\gamma\phi$ is an embedding of G into $D]U$, with D having finite rank.

Now since D has finite rank, it torsion part F is finite by Proposition 1.2. Since A is torsion-free, $\delta\gamma\phi A \cap F = 0$. Thus $\delta\gamma\phi G$ embeds in $\left(\dfrac{D}{F}\right)]U$. Let

$$\eta : D]U \to \frac{D}{F}]U \quad \text{be defined by} \quad \eta(du) = (d + F)u.$$

Now putting

$$C = D/F \qquad L = C]U \qquad \psi = \eta\delta\gamma\phi$$

we see that the conclusion of Theorem 3.2 is satisfied: the fact that L is finitely presented follows from its being a factor group of the finitely presented metabelian group H. Because of the maximum condition for normal subgroups, any factor group of a finitely presented metabelian group is finitely presented. This completes the proof of Theorem 3.2.

4. Conclusion of proof of Theorem 4.1

Let G be a finitely generated metabelian group whose derived group G' has finite rank. Let $G/G' = T$ and let F be the torsion subgroup of G', which is finite by Proposition 1.2. Let $G_0 = G/F$ and let A_1/F be a normal subgroup of G_0 containing G'/F and maximal with respect to being abelian. Let F_1/F be the torsion subgroup of A_1/F. Now let $G_1 = G/F_1$ and iterate this process to obtain sequences

$$F = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_i \subseteq F_{i+1} \subseteq \cdots$$

$$A' = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_i \subseteq A_{i+1} \subseteq \cdots$$

of normal subgroups of G with F_{i+1}/F_i finite and A_i/F_{i-1} maximal in $G_{i-1} = G/F_{i-1}$ with respect to being abelian $(i = 1, 2, \cdots)$. Since G satisfies the ascending chain condition for normal subgroups, there is an integer n with $F_n = F_{n+1}$ and $A_n = A_{n+1}$. Then G_n has a torsion-free normal subgroup A_n/F_n which is maximal with respect to being abelian. Since $G' \subseteq A_n$, A_n/F_n has finite rank by Proposition 1.1. By Theorem 3.2, there is an embedding ψ of G_n into a finitely presented metabelian group $L = C]U$ with C of finite rank. Since $L' \subseteq C$, L' also has finite rank.

that finitely generated metabelian groups are residually finite, there is a normal subgroup N of G with $N \cap F_n = 1$ and G/N finite. The map ζ: $G \to L \times G/N$ defined by $\zeta(g) = (\psi(gF_n), gN)$ has kernel $N \cap F_n = 1$ and so is an embedding into $L \times G/N$, which is clearly finitely presented and metabelian. Since L' has finite rank and G/N is finite, $(L \times G/N)' = L' \times (G/N)'$ clearly has finite rank, and the proof of Theorem 4.1 is complete.

Acknowledgements

References

M. F. Atiyah and I. G. MacDonald (1969), *Introduction to Commutative Algebra* (Addison-Wesley, Reading, Massachusetts; London; Don Mills, Ontario).

Gilbert Baumslag (1973), 'Subgroups of finitely generated metabelian groups', *J. Austral. Math. Soc.* **16**, 98–110.

James S. Boler (1974), 'Embedding and conjugacy in metabelian groups', (PhD thesis, Rice University, Texas).

P. Hall (1954), 'Finiteness conditions for soluble groups', *Proc. London Math. Soc.* (3) **4**, 419–436.

P. Hall (1959), 'On the finiteness of certain soluble groups', *Proc. London Math. Soc.* (3) **9**, 595–622.

P. Hall (1961), 'The Frattini subgroups of finitely generated groups', *Proc. London Math. Soc.* (3) **11**, 327–352.

A. G. Kurosh (1955), *Theory of Groups I* (translated by K. A. Hirsch) (Chelsea, New York, New York).

А. И. Мальцев [A. I. Mal'cev], 'О группах коречного ранга' [On groups of finite rank], *Math. Sb. N.S.* **22 (64)**, 351–352.

B. A. F. Wehrfritz (1973), *Infinite Linear Groups. An account of the Group-Theoretic Properties of Infinite Groups of Matrices* (Ergebnisse der Mathematik und ihrer Grenzgebiete, 76. Springer-Verlag, Berlin, Heidelberg, New York).

Department of Mathematics,
Oklahoma State University,
Stillwater, Oklahoma 74074, U.S.A.