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PRODUCTS OF RADON MEASURES: A COUNTER-EXAMPLE

BY
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ABSTRACT. I show that if X is the hyperstonian space of Lebesgue measure on $[0, 1]$, then there are open sets in $X\times X$ which are not measurable for the simple product outer measure. This answers a question of M. C. Godfrey and M. Sion.

1. **Products of measure spaces.** If (X, Σ, μ) and (Y, T, ν) are two measure spaces of finite magnitude (i.e. $\mu X<\infty, \nu Y<\infty$), we may define an outer measure φ on $X\times Y$ by

$$\varphi(C)=\inf\left\{\sum_{n\in\mathbf{N}} \mu E_n \cdot \nu F_n : E_n\in\Sigma \ \&\ F_n\in T\ \forall\ n\in\mathbf{N}, C\subseteq \bigcup_{n\in\mathbf{N}} E_n\times F_n\right\};$$

if λ_φ is the measure defined from φ by Carathéodory's construction, $\lambda_\varphi(E\times F)$ exists and is equal to $\mu E\cdot\nu F$ for every $E\in\Sigma$, $F\in\mathrm{T}$. ([8], §29. λ_φ is the completion of the product measure defined by [7] or [1] on the σ-algebra of sets generated by $\{E\times F: E\in\Sigma, F\in T\}$). On the other hand, if $(X, \mathfrak{T}, \Sigma, \mu)$ and $(Y, \mathfrak{S}, \mathrm{T}, \nu)$ are compact Radon measure spaces, we have a linear functional θ on $(C(X\times Y)$ defined by

$$\theta(w)=\int \nu(du)\int w(t, u)\mu(dt)\ \forall\ w\in C(X\times Y),$$

and θ defines a Radon measure λ_R on $X\times Y$. (See [2], chap. III, §4, no. 1.) It is well known that $\lambda_R(E\times F)$ exists and is equal to $\mu E\cdot\nu F$ for every $E\in\Sigma$ and $F\in\mathrm{T}$; consequently, if we follow the convention that Radon measures are taken to be complete (see [5], §73), λ_R is an extension of λ_φ. In many cases (e.g. if one of $(X, \mathfrak{T})$ and $(Y, \mathfrak{S})$ is compact and metrizable) λ_R and λ_φ coincide. In [6] the question is raised, whether they coincide in all cases; I show below that they do not.

2. **Hyperstonian spaces.** For my example, I need a compact Radon measure space which is extremally disconnected (i.e. the closure of an open set is open), diffuse (i.e. without atoms), of non-zero magnitude, and which has the property that $\bar{G}$ and G have the same measure for every open set G. These properties are possessed by any "hyperstonian" space derived from a diffuse probability space by the method of [3], Theorem 1 (p. 169).

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3. **A result of Erdös & Oxtoby.** I need the following fact, proved in the general case in [4]. Let (X, Σ, μ) and (Y, T, ν) be diffuse measure spaces of finite magnitude, and $\varepsilon > 0$. Then there exist sequences $\langle I_n\rangle_{n\in\mathbf{N}}$ in Σ and $\langle J_n\rangle_{n\in\mathbf{N}}$ in T such that (i) $\sum_{n\in\mathbf{N}} \mu I_n \cdot \nu J_n \le \varepsilon$ (ii) if $E\in\Sigma$, $F\in\mathrm{T}$ are such that $\mu E \cdot \nu F > 0$, then there is an $n\in\mathbf{N}$ such that $\mu(E\cap I_n)\cdot \nu(F\cap J_n) > 0$; i.e.

$$\lambda_\varphi(H_0\cap(E\times F))>0, \text{ where } H_0=\bigcup_{n\in\mathbf{N}} I_n\times J_n.$$

(R. O. Davies has given the following much easier proof of this result. We may suppose, without loss of generality, that $\mu X = \nu Y = 1$. Take $\langle \varepsilon_n\rangle_{n\in\mathbf{N}}$ to be any sequence of positive numbers, ≤ 1, such that $\sum_{n\in\mathbf{N}} \varepsilon_n = \infty$, $\sum_{n\in\mathbf{N}} \varepsilon_n^2 \le \varepsilon$. Take $\langle I_n\rangle_{n\in\mathbf{N}}$, $\langle J_n\rangle_{n\in\mathbf{N}}$ to be independent sequences of measurable sets such that $\mu I_n = \nu J_n = \varepsilon_n$ for every n. Of course $\sum_{n\in\mathbf{N}} \mu I_n \cdot \nu J_n \le \varepsilon$. if $E\in\Sigma$, $F\in\mathrm{T}$ are such that $\mu(E\cap I_n)\cdot\nu(F\cap J_n)=0$ for every $n\in\mathbf{N}$, set $P=\{n:\mu(E\cap I_n)=0\}$, $Q=\{n:\nu(F\cap J_n)=0\}$. Then $P\cup Q=\mathbf{N}$, so one of $\sum_{n\in P}\varepsilon_n$, $\sum_{n\in Q}\varepsilon_n$ is infinite; suppose the former. We have E essentially included in $X\setminus I_n$ for each $n\in P$, so that

$$\mu E \le \mu\Big(\bigcap_{n\in P} X\setminus I_n\Big) = \prod_{n\in P}(1-\varepsilon_n) = 0.$$

Similarly, if $\sum_{n\in Q}\varepsilon_n = \infty$, then $\nu F = 0$.)

4. **A lemma.** The following lemma is in some ways the core of my argument.

Let (X, Σ, μ) be a measure space of finite magnitude, and $\mathfrak{T}_0$ a second-countable topology on X (not necessarily Hausdorff) such that $\mathfrak{T}_0 \subseteq \Sigma$. Let $H_0 \subseteq X\times X$ be a $\mathfrak{T}_0\times\mathfrak{T}_0$-open set such that $\lambda_\varphi(H_0\cap(E\times F))>0$ whenever E and F are non-negligible measurable subsets of X. (Here λ_φ is the "classical" product measure defined in §1, and H_0 is λ_φ-measurable because $\mathfrak{T}_0$ is second-countable, so that H_0 must be a countable union of products of open sets.) Then, for λ_φ-almost all *$(t, u)\in X\times X$, there exist $E, F\in\mathfrak{T}_0$ such that $E\times F\subseteq H_0$ and $t\in\bar{E}$, $u\in\bar{F}$, the closures being taken for $\mathfrak{T}_0$.*

Proof. Let $\langle W_n\rangle_{n\in\mathbf{N}}$ be a sequence such that $\{W_n : n\in\mathbf{N}\}$ is a base for $\mathfrak{T}_0$. For each $n\in\mathbf{N}$, set

$$G_n=\{t:\exists u\in W_n, (t, u)\in H_0\}.$$

Because H_0 is $\mathfrak{T}_0\times\mathfrak{T}_0$-open, $G_n\in\mathfrak{T}_0\subseteq\Sigma$ and $D_n = X\setminus G_n\in\Sigma$. As $D_n\times W_n$ does not meet H_0, $\lambda_\varphi(D_n\times W_n)=\mu D_n\cdot\nu W_n = 0$, by the hypothesis on H_0. Set $C_1=(X\times X)\setminus\bigcup_{n\in\mathbf{N}}(D_n\times W_n)$; then $\lambda_\varphi((X\times X)\setminus C_1)=0$. But C_1 is precisely $\{(t, u)$:for every $\mathfrak{T}_0$-neighbourhood U of u, $\exists s\in U$, $(t, s)\in H_0\}$. Similarly, if $C_2=\{(t, u)$:for every $\mathfrak{T}_0$-neighbourhood U of t, $\exists s\in U$, $(s, u)\in H_0\}$, then $\lambda_\varphi((X\times X)\setminus C_2)=0$. Set $C=C_1\cap C_2$; then C is λ_φ-almost the whole of $X\times X$.

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Let (t, u) be any point of C. Let $\langle U_n\rangle_{n\in\mathbf{N}}$ and $\langle V_n\rangle_{n\in\mathbf{N}}$ be non-increasing sequences such that $\{U_n : n\in\mathbf{N}\}$ and $\{V_n : n\in\mathbf{N}\}$ are bases of neighbourhoods of t, u respectively. Choose strictly increasing sequences $\langle m_k\rangle_{k\in\mathbf{N}}$ and $\langle n_k\rangle_{k\in\mathbf{N}}$ in $\mathbf{N}$, and sequences $\langle E_k\rangle_{k\in\mathbf{N}}$, $\langle F_k\rangle_{k\in\mathbf{N}}$ in $\mathfrak{T}_0$ as follows. Set $m_0=0$. Given m_k, observe that, as $(t, u)\in C_2$, $U_{m_k}\times\{u\}$ meets H_0; choose $n_k>m_k$ and a non-empty open set $E_k\subseteq U_{m_k}$ such that $E_k\times V_{n_k}\subseteq H_0$. Now $(t, u)\in C_1$, so $\{t\}\times V_{n_k}$ meets H_0, and we can find a non-empty open set $F_k\subseteq V_{n_k}$ and an $m_{k+1}\geq n_k$ such that $U_{m_{k+1}}\times F_k\subseteq H_0$. Continue.

Now, for $j\leq k$,

$$E_j\times F_k\subseteq E_j\times V_{n_k}\subseteq E_j\times V_{n_j}\subseteq H_0,$$

while for $j>k$

$$E_j\times F_k\subseteq U_{m_j}\times F_k\subseteq U_{m_{k+1}}\times F_k\subseteq H_0.$$

So if $E=\bigcup_{j\in\mathbf{N}} E_j$ and $F=\bigcup_{k\in\mathbf{N}} F_k$, then $E\times F\subseteq H_0$. Since $E\cap U_{m_k}\supseteq E_k\neq\varnothing$ and $F\cap V_{n_k}\supseteq F_k\neq\varnothing$ for each $k\in\mathbf{N}$, $t\in\bar{E}$ and $u\in\bar{F}$. As (t, u) is an arbitrary point of C, this proves the lemma.

5. **The example.** Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a compact Radon measure space of the kind described in §2; e.g. the hyperstonian space of Lebesgue measure on $[0, 1]$. Let $\langle I_n\rangle_{n\in\mathbf{N}}$ and $\langle J_n\rangle_{n\in\mathbf{N}}$ be sequences in Σ such that $\sum_{n\in\mathbf{N}}\mu I_n\cdot\mu J_n<(\mu X)^2$ but, for every pair E, F of non-negligible members of Σ, there is an $n\in\mathbf{N}$ such that $\mu(E\cap I_n)\cdot\mu(F\cap J_n)>0$ (§3 above); because μ is outer regular, we may (enlarging I_n and J_n slightly if necessary) take them to be open. Set $H_0=\bigcup_{n\in\mathbf{N}} I_n\times J_n$. Then H_0 is open and is measurable for the classical product measure λ_φ.

Let

$$H=\bigcup\{\bar{E}\times\bar{F} : E, F\in\mathfrak{T}, E\times F\subseteq H_0\}.$$

Then H is open in $X\times X$ (because X is extremally disconnected), but H is **not** λ_φ-measurable.

For suppose, if possible, that $\lambda_\varphi(H)$ exists. In this case, $\lambda_\varphi(H)=\lambda_R(H)$, where λ_R is the product Radon measure on $X\times X$. Because λ_R is a Radon measure,

$$\lambda_R(H)=\sup\left\{\lambda_R\left(\bigcup_{i\leq n}\bar{E}_i\times\bar{F}_i : E_0,\ldots, E_n, F_0,\ldots, F_n\in\mathfrak{T}, E_i\times F_i\subseteq H_0\forall i\leq n\right\}.$$

But, given open sets $E_0,\ldots, E_n$, $F_0,\ldots, F_n$ such that $E_i\times F_i\subseteq H_0$ for each $i\leq n$, note that $\mu\bar{E}_i=\mu E_i$, $\mu\bar{F}_i=\mu F_i$ for each $i\leq n$. So

$$\lambda_R((\bar{E}_i\times\bar{F}_i)\setminus H_0)\leq\lambda_R((\bar{E}_i\times\bar{F}_i)\setminus(E_i\times F_i))=0\forall i\leq n,$$

and

$$\lambda_{\mathrm{R}}\Big(\bigcup_{i\leq n}\bar{E}_i\times\bar{F}_i\Big)\leq\lambda_{\mathrm{R}}H_0=\lambda_{\varphi}H_0.$$

So

$$\lambda_{\varphi}(H)=\lambda_{\mathbf{R}}(H)\leq\lambda_{\varphi}H_0<(\mu X)^2.$$

So we have $\varphi H<(\mu X)^2$, where φ is the product outer measure, and there exist sequences $\langle E_n\rangle_{n\in\mathbf{N}}$ and $\langle F_n\rangle_{n\in\mathbf{N}}$ in Σ such that $H\subseteq\bigcup_{n\in\mathbf{N}}E_n\times F_n$ and $\sum_{n\in\mathbf{N}}\mu E_n\cdot\mu F_n<(\mu X)^2$; in the same way as I_n, J_n above could be taken to be open, we may suppose that E_n and F_n are open for each $n\in\mathbf{N}$.

Let $\mathfrak{T}_0$ be the topology on X generated by the countable set

$$\{I_n:n\in\mathbf{N}\}\cup\{J_n:n\in\mathbf{N}\}\cup\{X\backslash\bar{E}_n:n\in\mathbf{N}\}\cup\{X\backslash\bar{F}_n:n\in\mathbf{N}\}.$$

Then $\mathfrak{T}_0$ is second-countable and $\mathfrak{T}_0\subseteq\mathfrak{T}\subseteq\Sigma$. As I_n, $J_n\in\mathfrak{T}_0$ for each $n\in\mathbf{N}$, H_0 is open for $\mathfrak{T}_0\times\mathfrak{T}_0$.

At the same time,

$$C=(X\times X)\backslash\bigcup_{n\in\mathbf{N}}(\bar{E}_n\times\bar{F}_n)$$

is not λ_{φ}-negligible, because $\sum_{n\in\mathbf{N}}\mu\bar{E}_n\cdot\mu\bar{F}_n=\sum_{n\in\mathbf{N}}\mu E_n\cdot\mu F_n<(\mu X)^2$, using the fact that X is hyperstonian. By the lemma in §4, there exists a point $(t,u)\in C$ and E, $F\in\mathfrak{T}_0$ such that $E\times F\subseteq H_0$ and $(t,u)\in\tilde{E}\times\tilde{F}$, where $\tilde{E}$ and $\tilde{F}$ are the $\mathfrak{T}_0$-closures of E, F respectively. Now if $\bar{E}$ and $\bar{F}$ are the $\mathfrak{T}$-closures of E and F, we have $\bar{E}\times\bar{F}\subseteq H\subseteq\bigcup_{n\in\mathbf{N}}E_n\times F_n$. As $\bar{E}$ and $\bar{F}$ are $\mathfrak{T}$-compact, while $\bar{E}_n$ and $\bar{F}_n$ are $\mathfrak{T}$-open for every $n\in\mathbf{N}$, there is some $n\in\mathbf{N}$ such that

$$E\times F\subseteq\bar{E}\times\bar{F}\subseteq\bigcup_{i\leq n}\bar{E}_i\times\bar{F}_i=B\quad\text{say.}$$

B is $\mathfrak{T}_0\times\mathfrak{T}_0$-closed, so $\tilde{E}\times\tilde{F}\subseteq B$ and $(t,u)\in B$. But (t,u) was chosen to lie in $C=(X\times X)\backslash\bigcup_{i\in\mathbf{N}}(\bar{E}_i\times\bar{F}_i)$. This is the required contradiction.

Remark. What I have really shown above is that $\varphi H=(\mu X)^2$, while $\varphi((X\times X)\backslash H)=(\mu X)^2-\lambda_{\varphi}H_0>0$.

6. **Completion regular measures.** R. A. Johnson has pointed out that the example above also settles a question raised in [1], §70, Exercise 8. Following [1] or [7], let us say that a Radon measure μ on a compact Hausdorff space X is *completion regular* if for every Borel set $E\subseteq X$ there exist Baire sets G, H such that $G\subseteq E\subseteq H$ and $\mu(H\backslash G)=0$. It is easy to see that this is equivalent to saying: for every open set $G\subseteq X$, there is a Baire set H such that $H\supseteq G$ and $\mu H=\mu G$. Compact hyperstonian spaces are always completion regular, since if G is open, then $\bar{G}$ is a Baire set, and $\mu\bar{G}=\mu G$. However, the product Radon measure of two hyperstonian spaces is not in general completion regular, for it

is easy to see that every Baire set of the product is in the σ-algebra generated by measurable rectangles, and is therefore measurable for the classical product measure; so that a completion regular product Radon measure must coincide with the classical product measure, which in the example above is not the case. Thus we have here two completion regular Radon measures with a Radon product measure which is not completion regular.

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