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SEMIGROUPS UNDER A SANDWICH OPERATION

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For any element a of a semigroup $(S,\cdot)$, we may define a "sandwich" operation $\circ$ on the set S by $x\circ y = xay$ $(x, y \in S)$. Under this operation the set S is again a semigroup; we denote this semigroup by (S, a) and call it a *variant* of S. Variants of semigroups of binary relations have been studied by Chase [**6, 7**]. In this paper we consider variants of arbitrary semigroups.

A related idea is that of a *mididentity* (i.e. an element u in a semigroup S such that $xuy = xy$ for all $x, y \in S$). Such elements were called "middle units" in the work of Yamada [**13**] and Blyth [**3, 4, 5, 10**] and in [**8**, §3.2, Exercise 12]. They have also been studied by Ault [**1, 2**] who called them "midunits".

In Section 1 we review some known results [**13, 1, 2**] on mididentities. In Section 2 we prove some introductory results on variants; Green's relations on a variant are considered and it is shown that, if an element a in a semigroup S satisfies a weak cancellation condition, then (S, a) is isomorphic to a subdirect product of the subsemigroups Sa and aS of $(S,\cdot)$. In Section 3 we consider semigroups with regular elements. This produces a method of manufacturing semigroups with mididentities; in particular, every variant of a regular semigroup is a semigroup with mididentity. Further, every variant of a completely simple semigroup is shown to be isomorphic to a rectangular group.

In Section 4 we introduce the idea of a regularity-preserving element in a semigroup S containing a regular element. The set $RP(S)$ of these elements in a semigroup S with mididentity is seen to be a subsemigroup of S; it provides a generalisation to S of the unit group in a semigroup with identity and contains the subsemigroup M_S [**13, 1, 2**] of S. Another generalisation of the unit group is furnished by the set $RRP(S)$ of regular regularity-preserving elements in an arbitrary semigroup S: if $RRP(S)$ is non-empty it forms a completely simple subsemigroup of S; if S has a mididentity then $RRP(S)$ coincides with an obvious subsemigroup of M_S (that consisting of regular elements of S) and $RP(S)$ is an ideal extension of it by a null semigroup. The final result of the section is that, for S a regular semigroup, every variant of S is isomorphic to S if and only if S is a rectangular group.

We show in Section 5 that consideration of variants of a regular semigroup leads in a natural way to a partial order on the semigroup. It is shown that this partial order coincides with Nambooripad's partial order [**11**] on a regular semigroup.

1. Preliminaries

We use the notation of [**8**] and [**9**] throughout.

The definition of an inflation of a semigroup is given in [**8**, §3.2, Exercise 10], but the following description, due to Petrich [**12**], will serve as an alternative (see also [**2**, Lemma 1.7]). Let T be a semigroup, let Q be a (possibly empty) set disjoint from T and let $\theta: Q \to T$ be a mapping. Extend the multiplication in T to a multiplication in $V = T \cup Q$ by taking

$$xy = \begin{cases} x(y\theta) & \text{if} \quad x \in T, y \in Q, \\ (x\theta)y & \text{if} \quad x \in Q, y \in T, \\ (x\theta)(y\theta) & \text{if} \quad x, y \in Q. \end{cases}$$

Then V is a semigroup, called an *inflation of* T (*by* Q, *with associated mapping* θ).

An element u of a semigroup S will be called a *mididentity* if $aub = ab$ for all $a, b \in S$. We now review some results on such elements [**13, 1, 2**]. Note first that if a and b are mididentities in a semigroup S then ab is an idempotent mididentity in S; also, a regular mididentity in S is always idempotent. We need to distinguish between idempotent and non-idempotent mididentities and will use the following notation: if S is a semigroup with a mididentity, $K(S)$ [$I(S)$] will denote the set of mididentities [idempotent mididentities] in S. Clearly $I(S)$ and $K(S)$ are both subsemigroups of S.

Lemma 1.1. [**2**]. *Let S be a semigroup with a mididentity. Then $I(S)$ is a rectangular band and $K(S)$ is an inflation of $I(S)$ by $K(S) \backslash I(S)$. The associated mapping $\theta: K(S) \backslash I(S) \to I(S)$ is given by $x\theta = x^2$.*

We note that if S is regular then $K(S) = I(S)$ is a rectangular band. Further, if S is a rectangular band then $K(S) = S$. An example of a non-regular semigroup S with the property that $K(S) = S$ is afforded by any null semigroup S with $|S| > 1$.

We now establish some additional terminology. Let a be an element of a semigroup S. By a *pre-inverse* [*post-inverse*] of a we shall mean an element $b \in S$ such that $aba = a$ [$bab = b$]. We denote the set of pre-inverses [post-inverses] of a by $\mathrm{Pre}(a)$ [$\mathrm{Post}(a)$]. Clearly $V(a) = \mathrm{Pre}(a) \cap \mathrm{Post}(a)$, where, as usual, $V(a)$ denotes the set of inverses of a in S.

2. Variants of a semigroup

Let $(S, \cdot)$ be a semigroup and let $a \in S$. Define a binary operation $\circ$ on S by

$$x \circ y = xay \qquad (x, y \in S).$$

Then S becomes a semigroup with respect to this operation. We denote it by $(S, \cdot\,; a, \circ)$, or, more briefly, by (S, a), and we refer to (S, a) (for any $a \in S$) as a *variant* of $(S, \cdot)$. Since we now may have more than one multiplication defined on the same set S we make the following convention: if it is stated or implied that S (or a subset of it) is a semigroup, then the multiplication in question will be that in (or inherited from) $(S, \cdot)$. Variants of semigroups of binary relations have been studied by Chase [**6, 7**].

If $(S, +, \cdot)$ is a ring and $a \in S$, then $(S, +, \circ)$ is a ring, where $\circ$ is the multiplication in (S, a). Thus, if a semigroup S admits ring structure, so does (S, a) for each $a \in S$.

We now consider Green's relations on a variant of a semigroup S. So let a be a fixed element of S and, in an obvious notation, let Green's relations on (S, a) be denoted by $\mathscr{L}'$, $\mathscr{R}'$, $\mathscr{H}'$, $\mathscr{J}'$ and $\mathscr{D}'$. Then [c.f. **6**, Lemma 1.2]

$$\mathscr{L}' \subseteq \mathscr{L}, \quad \mathscr{R}' \subseteq \mathscr{R}, \quad \mathscr{H}' \subseteq \mathscr{H}, \quad \mathscr{J}' \subseteq \mathscr{J} \quad \text{and} \quad \mathscr{D}' \subseteq \mathscr{D}.$$

Further, it is easily seen that S is left simple [right simple, simple] if and only if (S, a) is left simple [right simple, simple]. If S has a zero element 0 then clearly (S, a) has 0 as a zero element. We may show then that, if S has zero element 0,

$$(S, a) \text{ is 0-simple [0-bisimple]} \Rightarrow S \text{ is 0-simple [0-bisimple]}.$$

The converse of each of these statements is false as can be seen by taking $a = 0$ in an appropriate 0-simple [0-bisimple] semigroup.

We may note here that if a is a mididentity in a semigroup S, then (S, a) coincides with S. Thus if a is any element of a rectangular band S then $(S, a) = S$. In the case when S is a semilattice we again have a straightforward and easily verified result on variants: for any $a \in S$, the semigroup (S, a) is an inflation of Sa by $S \setminus Sa$, with associated mapping $\theta: S \setminus Sa \to Sa$ given by $x\theta = xa$.

If a is an element of a semigroup S we define mappings $\lambda_a^*: S \to aS$, $\rho_a^*: S \to Sa$ by

$$\lambda_a^*(x) = ax, \quad \rho_a^*(x) = xa \qquad (x \in S).$$

So λ_a^*, ρ_a^* are restrictions of the usual inner left and right translations λ_a, ρ_a, respectively, of S.

Lemma 2.1. *Let S be a semigroup and let $a \in S$. Then λ_a^*, ρ_a^* are surjective homomorphisms from (S, a) onto aS, Sa, respectively.*

Proof. In (S, a), $x \circ y = xay$ for $x, y \in S$. Then

$$\lambda_a^*(x \circ y) = a(xay) = \lambda_a^*(x)\lambda_a^*(y),$$

so λ_a^* is a homomorphism; similarly for ρ_a^*. The mappings are clearly surjective.

This shows that, if $a \in S$ is such that either λ_a or ρ_a is a permutation of S, then $(S, a) \cong S$. In particular, if S has an identity element and a is an element of the unit group then $(S, a) \cong S$. We note here that if S is a rectangular band and $a \in S$, then $(S, a) = S$, but neither λ_a nor ρ_a is a permutation in general.

Now let $(S, \cdot)$ be a semigroup, let $a \in S$ and let H be a subset of S. If H is closed under the operation in (S, a), we will denote the resulting semigroup by (H, a). Conversely, the statement "(H, a) is a subsemigroup of (S, a)" will mean that H is closed under the above operation. In this case Ha and aH are both subsemigroups of $(S, \cdot)$ and it is readily verified that the subset $T_a(H;\cdot)$ of the cartesian product $Ha \times aH$ defined by

$$T_a(H;\cdot)=\{(xa, ax): x\in H\}$$

is in fact a subdirect product of Ha and aH.

For an arbitrary subset H of S, we will say that $a\in S$ is *weakly cancellable on* H (*in* $(S,\cdot)$) if, whenever $x, y\in H$,

$$xa=ya \text{ and } ax=ay \Rightarrow x=y.$$

Lemma 2.2 *Let a be an element and H a subset of a semigroup S such that (H,a) is a subsemigroup of (S,a). Then the mapping $\theta:(H,a)\to T_a(H;\cdot)$ defined by*

$$\theta(x)=(xa, ax) \qquad (x\in H)$$

is a surjective homomorphism. If a is weakly cancellable on H in $(S,\cdot)$, θ is an isomorphism.

The proof is straightforward and will be omitted.

It follows that if a is weakly cancellable on S itself in $(S,\cdot)$ then the variant (S,a) is isomorphic to a subdirect product of the subsemigroups Sa and aS of $(S,\cdot)$.

Suppose now that S is completely simple and that $a\in S$. It is readily verified, using the Rees theorem, that a is weakly cancellable on S itself, so (S,a) is isomorphic to a subdirect product of Sa and aS. In fact we show later that (S,a) is a rectangular group (i.e. the direct product of a rectangular band and a group). As a first step towards this result we note here that every variant of S is completely simple. For, taking S to be the Rees matrix semigroup $\mathcal{M}(G,I,\Lambda;P)$ over the group G, and letting $a=(\alpha,g,\beta)$ $(\alpha\in I, \beta\in\Lambda, g\in G)$ be an arbitrary element of S, we get that $(S,a)=\mathcal{M}(G,I,\Lambda;P')$, where $P'=(p'_{\lambda i})$ with

$$p'_{\lambda i}=p_{\lambda\alpha}g p_{\beta i}.$$

Thus (S,a) is a Rees matrix semigroup with the same $\mathcal{L}$- and $\mathcal{R}$-classes as S but with a (possibly) modified sandwich matrix. It follows that (S,a) is completely simple.

Lemma 2.2 gives some information on bands with mididentity elements. For let S be a band and let $a\in S$ be a mididentity. Then a is weakly cancellable on S, as is easily seen, so, by Lemma 2.2, (S,a) is isomorphic to $T_a(S;\cdot)$, i.e. S is isomorphic to $T_a(S;\cdot)$. In the case when S is a rectangular band and a is any element of S, this reduces to saying that S is isomorphic to the direct product of the left zero semigroup Sa and the right zero semigroup aS.

The above result on bands with mididentity is a special case of a theorem of Blyth [**5**]. He shows that, if S is an orthodox semigroup with a mididentity u, then S is isomorphic to a spined product (being a pull-back for a related diagram) of uS and Su. En route to this result he shows that u is weakly cancellable on S [**5**, Lemma 6].

Lemma 2.3. *Let S be a semigroup and let $a \in S$. If the semigroup (S, a) has identity element 1 then*

(i) *S has identity element,*
(ii) *the elements $a, 1$ lie in the unit group of S and are inverse to each other,*
(iii) $(S, a) \cong S$.

Proof. Let (S, a) have identity element 1. Then, for all $x \in S$, $x \circ 1 = 1 \circ x$ in (S, a), i.e. $xa1 = 1ax = x$ in S. Thus $a^2 1 = 1a^2 = a$, so $a\mathscr{H}a^2$ in S, i.e. H_a is a subgroup of S by [**8**, Theorem 2.16]. Let e be the identity element of H_a. Then, since $a^2 1 = 1a^2 = a$, we have $a1 = 1a = e$. Further, for any $x \in S$,

$$xa1 = 1ax = x \Rightarrow xe = ex = x.$$

So S has identity element e, and the elements $a, 1$ lie in the unit group of S and are inverse to each other. Finally we have $(S, a) \cong S$ by an earlier remark.

We note here that if S is a semigroup and $a \in S$, then an element that is regular in (S, a) is also regular in S. Thus, if (S, a) is regular then so is S. Furthermore, if a is a mididentity in S and $x \in S$, then x is regular in $(S, a) \Leftrightarrow x$ is regular in S.

3. Variants and regularity

We quickly find that there is a natural connection between pre-inverses [post-inverses] of an element $a \in S$ and mididentities [idempotents] in (S, a).

Lemma 3.1. *Let S be a semigroup and let $a \in S$.*

(i) *If b is a pre-inverse of a in S, then b is a mididentity in (S, a).*
(ii) *If a is regular in S then b is a pre-inverse of a in S if and only if b is a mididentity in (S, a).*
(iii) *b is a post-inverse of a in S if and only if b is idempotent in (S, a).*

Proof. Let $b \in \mathrm{Pre}(a)$ and let $x, y \in S$. Then, in (S, a), $x \circ b \circ y = xabay = xay = x \circ y$, proving (i). Suppose now that a is regular in S and that $b \in K((S, a))$. Then, for any $x, y \in S$, we have $xabay = xay$. Taking $x = y = z$, where $z \in \mathrm{Pre}(a)$, we get $azabaza = azaza$, i.e. $aba = a$, proving (ii). The result (iii) is obvious.

Lemma 3.1 (i) provides a way of constructing semigroups with mididentities: (S, a) is such a semigroup whenever a is regular in the semigroup S. In particular, if S is regular then every variant of S has a mididentity. That the converse of this is not true is evident from consideration of a null semigroup S with $|S| > 1$. The same example shows that part (ii) of the lemma does not hold in general without the stipulation that a be regular.

If we take a to be a mididentity in S, we obtain the following corollary to Lemma 3.1. As usual, $E(S)$ denotes the set of idempotents in S.

Corollary 3.2. *Let a be a mididentity in a semigroup S. Then*

(i) $\mathrm{Pre}(a) \subseteq K(S)$,
(ii) *if a is regular in S*, $\mathrm{Pre}(a) = K(S)$,
(iii) $\mathrm{Post}(a) = E(S)$.

We recall that a mididentity in a semigroup is idempotent if and only if it is regular. This enables us to prove the following corollary to Lemma 3.1.

Corollary 3.3. *Let a be a regular element in a semigroup S. Then a pre-inverse of a in S is an inverse of a in S if and only if it is regular in* (S, a).

Proof. Let $b \in \mathrm{Pre}(a)$. Then b is a mididentity in (S, a), by Lemma 3.1 (i). So

$$\begin{aligned} b \in V(a) &\Leftrightarrow b \in \mathrm{Post}(a) \\ &\Leftrightarrow b \text{ is idempotent in } (S, a) \quad \text{(by Lemma 3.1 (iii))} \\ &\Leftrightarrow b \text{ is an idempotent mididentity in } (S, a) \\ &\Leftrightarrow b \text{ is regular in } (S, a), \end{aligned}$$

giving the result.

Suppose now that S is regular and that a is a mididentity in S. Then $(S, a)(= S)$ is regular and, by Corollary 3.3, every pre-inverse of a in S is an inverse of a in S. By virtue of Corollary 3.2 (ii), we have here

$$\mathrm{Pre}(a) = V(a) = K(S) \quad (= I(S))$$

(c.f. [**2**, Lemma 1.2 (ii)]).

We now turn our attention again to completely simple semigroups. Let S be such a semigroup and let $a, b \in S$. By [**9**, Exercise 11, p. 88],

$$bab = b \Leftrightarrow aba = a.$$

It follows that $\mathrm{Post}(a) = \mathrm{Pre}(a)(= V(a))$. The regularity of S and Lemma 3.1 (ii), (iii) now give

$$E((S, a)) = K((S, a)) = I((S, a)).$$

Lemma 3.4. *Every variant of a completely simple semigroup is isomorphic to a rectangular group.*

Proof. Let S be completely simple and let $a \in S$. We recall that (S, a) is completely simple. Also, by the remarks above, $E((S, a)) = I((S, a))$. But $I((S, a))$ forms a band in

(S,a), so (S,a) is orthodox. Hence (S,a) is isomorphic to a rectangular group [**8**, Exercise 2(b) for §3.2].

Corollary 3.5. *Let S be a completely simple semigroup with a mididentity. Then S is isomorphic to a rectangular group and $(S,a)\cong S$ for every $a\in S$.*

Proof. If $u\in S$ is a mididentity then $S=(S,u)$ is isomorphic to a rectangular group by Lemma 3.4. We may take S to be the direct product $B\times G$, where B is the rectangular band on the cartesian product $I\times\Lambda$, and G is a group. Then if $a=((i,\lambda),g)\in S$ ($i\in I$, $\lambda\in\Lambda$, $g\in G$), the mapping $\theta:(S,a)\to(S,\cdot)$ defined by

$$((j,\mu),h)\theta=((j,\mu),gh) \qquad (j\in I,\ \mu\in\Lambda,\ h\in G)$$

is easily seen to be an isomorphism.

We can remark further on this following the next lemma.

Lemma 3.6. *Let S be a semigroup. Then S has a regular element a such that $(S,a)\cong S$ if and only if S has a mididentity.*

Proof. Suppose $(S,a)\cong S$ where a is regular in S. Then a has a pre-inverse in S, so (S,a) has a mididentity. It follows that S must have a mididentity.

Conversely, if $u\in S$ is a mididentity, we put $a=u^2$ and clearly $(S,a)\cong S$.

We can now say, using Corollary 3.5, that if a completely simple semigroup S has one element x for which $(S,x)\cong S$ then $(S,a)\cong S$ for every $a\in S$.

To complete this section we broaden the discussion and consider a semigroup S containing a regular element a but otherwise arbitrary. We have $\mathrm{Pre}(a)=K((S,a))$ and $V(a)=I(S,a))$ by Lemma 3.1. So $(\mathrm{Pre}(a),a)$ and $(V(a),a)$ are both subsemigroups of (S,a) and, by Lemma 1.1, $(\mathrm{Pre}(a),a)$ is an inflation of the rectangular band $(V(a),a)$. The latter semigroup can, of course, be exhibited as the direct product of a left zero and a right zero semigroup, and we now do this. The element a is weakly cancellable on $V(a)$ in $(S,\cdot)$, as is easily checked. So, by Lemma 2.2, $(V(a),a)$ is isomorphic to

$$T_a(V(a);\cdot)=\{(xa,ax):x\in V(a)\},$$

which is the direct product of the left zero semigroup $V(a)a=E(S)\cap L_a$ and the right zero semigroup $aV(a)=E(S)\cap R_a$.

4. Generalisations of the unit group

For a semigroup S with mididentity, the following two subsets of S are important [**13, 1, 2**]:

$$M_S=\{x\in S:x\in K(S) \text{ or } x \text{ has inverse } x' \text{ with } xx',\ x'x\in I(S)\},$$

$$RM_S=\{x\in S:x \text{ has inverse } x' \text{ with } xx',\ x'x\in I(S)\}.$$

Then we have [**2**, Theorem 1.8]

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Lemma 4.1. *Let S be a semigroup with mididentity. Then M_S and RM_S are both subsemigroups of S. The semigroup RM_S is a rectangular group and M_S is an inflation of RM_S. The set of elements of M_S that are regular in S is precisely RM_S.*

Clearly M_S provides, for a semigroup S with mididentity, a generalisation of the unit group in a semigroup with identity. We will discuss now other generalisations of the unit group.

Let S be a semigroup with a regular element and let $a \in S$. If x is a regular element of S, we will say that *a preserves the regularity of x* if x is also regular in (S, a). If a preserves the regularity of every regular element in S then we will say that a is a *regularity-preserving element* in S. The set of such elements in S will be denoted by $RP(S)$. The set of regular regularity-preserving elements in S will be denoted by $RRP(S)$. If u is a mididentity in S then clearly $u \in RP(S)$ and $u^2 \in RRP(S)$.

Lemma 4.2. *Let S be a semigroup, let $a \in RP(S)$ and let b be a regular element of S. Then $b\mathscr{L}ab$, $b\mathscr{R}ba$ in S.*

Proof. Since a preserves the regularity of b we have $b \circ x \circ b = b$ in (S, a) for some $x \in S$, i.e. $baxab = b$. The result follows.

Lemma 4.3. *Let S be a semigroup.*

(i) *If $a \in RRP(S)$ then H_a is a subgroup of S.*
(ii) *If $a \in RP(S)$ and $b \in S$ is such that $a \in bS \cap Sb$ then $b \in RP(S)$.*

Proof. (i) For $a \in RRP(S)$ we have $a\mathscr{H}a^2$ in S by Lemma 4.2, so H_a is a subgroup of S by [**8**, Theorem 2.16].

(ii) Let $a \in RP(S)$, let $b \in S$ and suppose that $a = bc = db$, where $c, d \in S$. Let x be an arbitrary regular element of S. Then, for some $w \in S$, we have

$$x = xawax = x(bc)w(db)x$$

from which it follows that x is regular in (S, b). Thus $b \in RP(S)$ as required.

Theorem 4.4. *Let S be a semigroup with $RRP(S) \neq \emptyset$. Then $RRP(S)$ is a completely simple subsemigroup of S.*

Proof. Let $R = RRP(S)$. First suppose that $a, b \in R$. Then $a = abcba$, $b = badab$ for some $c, d \in S$. Then $ab = ab(cb)ab$, so ab is regular in S. Now let x be any regular element in S. Then x is regular in both (S, a) and (S, b), so there exist $y, z \in S$ such that

$$x = xayax = xbzbx.$$

Thus

$$\begin{aligned} x &= xaya(xbzbx) \\ &= x(abcba)yaxbz(badab)x \\ &= xabvabx \end{aligned}$$

where $v \in S$. Thus x is regular in (S, ab). It follows that R is a subsemigroup of S.

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Again, let $a \in R$ and let H_a denote the $\mathscr{H}$-class of a in S. By part (i) of Lemma 4.3, H_a is a subgroup of S, and, by part (ii) of the same lemma, H_a is contained in R. The group inverse of a in H_a is thus an inverse of a in R, so R is regular.

Finally, let $a, b \in R$. Then, by Lemma 4.2, $a\mathscr{L}ba$, $a\mathscr{R}ab$, $b\mathscr{L}ab$, $b\mathscr{R}ba$ in S. Suppose that $aba = a$. Then by Green's lemma [**8**, Lemma 2.2] applied in S, $\rho_b|L_a, \rho_a|L_b$ are mutually inverse $\mathscr{R}$-class-preserving bijections from L_a onto L_b and from L_b onto L_a, respectively. Thus $bab = b$. It follows from [**9**, Exercise 11, p. 88] that R is completely simple. This completes the proof.

We note here that there can be no further restriction on $RRP(S)$ in general, since if S is an arbitrary completely simple semigroup then $RRP(S) = S$.

We now consider $RP(S)$ for a semigroup S with a mididentity, u say. Since u^2 is regular in S, both $RP(S)$ and $RRP(S)$ are non-empty here.

Lemma 4.5. *Let S be a semigroup with a mididentity and let $a \in S$. Then the following statements are equivalent:*

(i) $a \in RP(S)$;
(ii) *aS and Sa both contain a mididentity for S;*
(iii) *a preserves the regularity of an idempotent mididentity.*

Proof. Let u be an idempotent mididentity in S.

Suppose (i) holds. Then u is regular in (S, a) so there exists $z \in S$ such that $u = uazau$. Thus $aza \in \mathrm{Pre}(u)$ in S, i.e. aza is a mididentity in S, by Corollary 3.2 (i). This gives (ii).

Suppose (ii) holds and that aw, ya $(w, y \in S)$ are both mididentities in S. Now let x be any regular element of S, say $x = xvx$ $(v \in S)$. Then

$$x = x(aw)v(ya)x = xa(wvy)ax,$$

so that a preserves the regularity of x. In particular, we have (iii).

Suppose, finally, that (iii) holds and that a preserves the regularity of an idempotent mididentity u. Then $u = uazau$ for some $z \in S$. Now let $x = xyx$ $(y \in S)$ be any regular element of S. Then

$$\begin{aligned} x = xuyux &= x(uazau)y(uazau)x \\ &= xa(zayaz)ax, \end{aligned}$$

so that x is regular in (S, a). This gives (i) and the lemma is proved.

We recall [**12**, Chapter III] that if S is an ideal of a semigroup T then T is said to be an ideal extension of S by the Rees quotient semigroup T/S.

Theorem 4.6. *Let S be a semigroup with a mididentity. Then $RP(S)$ is a subsemigroup of S, being an ideal extension of $RRP(S)$ by a (possibly empty) null semigroup. It contains the subsemigroup M_S of S.*

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Proof. We recall that $RP(S) \neq \varnothing$ here. Suppose then that $a, b \in RP(S)$. By Lemma 4.5 there exist mididentities ax, by $(x, y \in S)$ in S. Let $u \in S$ be an idempotent mididentity. Then

$$u = u^2 = u(ax)u = ua(by)xu = u(abyx)u,$$

so, by Corollary 3.2 (i), $abyx$ is a mididentity. Thus abS (and, similarly, Sab) contains a mididentity, so that $ab \in RP(S)$ by Lemma 4.5. Thus $RP(S)$ is a subsemigroup of S and we easily see, again using Lemma 4.5, that it contains M_S.

To complete the proof it suffices to show that the product of two elements of $RP(S)$ is a regular element in S. So again, let $a, b \in RP(S)$ and let by, wa $(y, w \in S)$ be mididentities in S. Then $ab(yw)ab = ab$, so ab is regular in S. This proves the theorem.

An example shows that $M_S \neq RP(S)$ in general. Let S be the finite cyclic semigroup $\langle a \rangle$, where a has index 2 and arbitrary period $m \in \mathbb{N}$. Let K denote the subgroup $\{a^2, \ldots, a^{m+1}\}$ of S. Then S has a unique mididentity, namely the identity element of K. Since $xS = Sx = K$ for every $x \in S$, Lemma 4.5 gives that $RP(S) = S$. But clearly $M_S = K$.

It is noted in [2] that if a semigroup S has an identity element then that element is a unique mididentity for S. This fact, together with Lemma 4.5, tells us that, in a semigroup S with an identity element, $RP(S)$ (and hence $RRP(S)$, from its definition) coincides with the unit group of S.

We return now to $RRP(S)$ for the case when S has a mididentity.

Theorem 4.7. *Let S be a semigroup with a mididentity and let $a \in S$. Then the following statements are equivalent:*

(i) $a \in RRP(S)$;
(i) *a is $\mathscr{H}$-equivalent in S to an idempotent mididentity;*
(iii) $a \in RM_S$.

Proof. Suppose that (i) holds. Then, by Lemma 4.3 (i), H_a is a subgroup of S, with identity element e, say. Let u be an idempotent mididentity in S. Then $(ue)^2 = ue$. Also, there exists $y \in S$ such that

$$u = uayau = (ue)ayau.$$

It follows that $ueu = u$, so e is an idempotent mididentity by Corollary 3.2 (i), giving (ii).

It is obvious that (ii) implies (iii). Suppose then that (iii) holds, i.e. that $a \in RM_S$. Then $a \in M_S$ implies that $a \in RP(S)$ by Theorem 4.6. Further a is regular, so $a \in RRP(S)$, giving (i). This completes the proof.

We recall that, in a semigroup S with mididentity, RM_S is a rectangular group (Lemma 4.1).

Corollary 4.8. *Let S be a semigroup with a mididentity. Then $RRP(S) = RM_S$ is a rectangular group; if S is regular, $RP(S) = M_S$ is a rectangular group.*

The following result is obviously true.

Lemma 4.9. *Let S be a semigroup and let $a \in S$. Then (S, a) is regular $\Leftrightarrow S$ is regular and $a \in RP(S)$.*

Theorem 4.10. *Let S be a regular semigroup. Then the following statements are equivalent:*

(i) *$(S, a) \cong S$ for all $a \in S$;*
(ii) *S has a mididentity and (S, a) is regular for all $a \in S$;*
(iii) *S is a rectangular group.*

Proof. Suppose (i) holds. Then clearly (S, a) is regular for all $a \in S$. Also, by Lemma 3.6, S has a mididentity, giving (ii).

Suppose (ii) holds. Then $S = RP(S)$ by Lemma 4.9, and, since S has a mididentity, (iii) holds by Corollary 4.8.

Finally, (iii) $\Rightarrow$ (i) by Corollary 3.5.

We may note here that a null semigroup S with $|S| > 1$ is an example of a non-regular semigroup with the property that $(S, a) \cong S$ for all $a \in S$.

5. A partial order for regular semigroups

Any element of a regular semigroup appears as an idempotent in a suitable variant of the semigroup. This observation leads in a natural way to a partial order on a regular semigroup. For let S be a regular semigroup; if $x, y \in S$, we will say that x is *ρ-related to* y if there exists a variant of S in which x and y are both idempotents and satisfy $x \leqq y$.

We may readily check that ρ is a partial order on S. It transpires, however, that ρ coincides with Nambooripad's partial order [11] on S and we will simply verify that this is the case.

Recall that, for a regular semigroup S, Nambooripad's partial order $\leqq$ on S may be defined by

$$x \leqq y \Leftrightarrow x \in ySy \text{ and } x = xy'x \text{ for some } y' \in V(y).$$

Theorem 5.1. *Let S be a regular semigroup and let a relation ρ be defined on S by*

$$x \rho y \Leftrightarrow \exists a \in S \text{ such that } x, y \in E((S, a)) \text{ and satisfy } x \circ y = y \circ x = x \text{ in } (S, a).$$

Then ρ coincides with Nambooripad's partial order on S and thus is itself a partial order on S.

Proof. Suppose $x, y \in S$ satisfy $x \leqq y$. Then there exist elements $s \in S$, $y' \in V(y)$ such that

$$x = ysy = xy'x.$$

Then clearly $x, y \in E(S, y'))$ and

$$xy'y = (ysy)y'y = ysy = x,$$

$$yy'x = yy'(ysy) = ysy = x,$$

so $x \circ y = y \circ x = x$ in $E((S, y'))$. Thus we have $x\rho y$.

Conversely, let $a \in S$ be such that

$$x = xax, \quad y = yay, \quad xay = yax = x.$$

Then

$$x = ya(xay) \in ySy,$$

and, if $y' \in V(y)$,

$$xy'x = (xay)y'(yax) = xayax = x.$$

This shows that $x \leqq y$, proving the result.

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