A NEW PROOF OF A THEOREM OF DIRAC

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In [1] Dirac has determined the structure of all 3-connected graphs which do not contain two independent (i.e., disjoint) circuits. We shall here provide a short proof of this theorem by applying Tutte's theory of 3-connected graphs [2].

For definitions of the terms used the reader is referred to [1]. We shall restrict our attention to simple graphs, i.e. graphs without multiple edges. $\langle k \rangle$ will denote the complete k-graph. W_k will denote the k-wheel: a graph whose vertices are labelled 0, 1, ..., k, with edges (0,1), ..., (0,k), (1,2), (2,3), ..., (k-1,k), (k,1). ${}_pK$ will denote the graph with vertices $x_1, x_2, x_3, y_1, \ldots, y_p$ and edges (x_i, y_j) $(i = 1, 2, 3; j = 1, 2, \ldots, p)$ $(p > 1)$. We define ${}_pK_1 = {}_pK + (x_1, x_2)$, ${}_pK_2 = {}_pK_1 + (x_2, x_3)$, ${}_pK_3 = {}_pK_2 + (x_3, x_1)$; the subscripts p may be suppressed. The class of graphs having two independent circuits will be denoted by Ω.

The theorem of Tutte we require states that every simple 3-connected graph having more than 3 vertices is either a wheel or can be obtained from a wheel by a sequence of operations of the following two types:

I. inscribing a new edge;

II. replacing a vertex x by two vertices x', x'' connected by an edge, such that every vertex formerly connected to x is connected to exactly one of x', x'', and such that each of x', x'' is connected to at least two of these vertices.

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We observe that any graph obtained from a graph in Ω by Operations I or II also lies in Ω.

Dirac's result [1, pp. 186-193] is

THEOREM. The only 3-connected (simple) graphs with at least 4 vertices and not in Ω are of the following types:

1) W_k $(k > 3)$;

2) ${}_pK$, ${}_pK_1$, ${}_pK_2$, ${}_pK_3$ $(p > 2)$;

3) ${}_2K_3$;

4) <5> .

We first observe that none of the graphs listed lies in Ω.

LEMMA 1. An application of I or II to W_k $(k > 4)$ yields a graph in Ω.

Proof:

a) Label the vertices so that an added edge is $(1, n)$ where $3 \leq n \leq [(k+1)/2]$. Then two independent circuits are $((1, 2, 3, \ldots, n))$ and $((0, n+1, n+2, \ldots, k))$.

b) The only vertex at which II can be applied is 0. Suppose one of $0'$, $0''$, say $0'$, is connected to two consecutive vertices, which can be taken to be 1 and 2. Then there must exist two vertices p and q connected to $0''$ and such that $2 < p < q \leq k$. Then two circuits are $((1, 0', 2))$ and $((p, p+1, \ldots, q, 0''))$. Otherwise the vertices can be so labelled that $0''$ is connected to all even vertices, and $0'$ to all odd vertices. Two circuits are $((1, 0', 3, 2))$ and $((4, 0'', 6, 5))$ (here $k \geq 6$).

It follows from Lemma 1 and the theorem of Tutte that the only graphs other than the wheels that we need consider are those obtainable from W_4 by a sequence of operations I and II. We note that W_4 is a ${}_2K_2$.

LEMMA 2. Any graph obtained by operation II from a K, K_1, K_2, or K_3 and which does not lie in Ω, is a K or a K_1.

Proof: The only vertices where II can be applied are the x's, as the y's have valency less than 4.

a) Let a K be given. Label the vertices so that x_1 is replaced by x_1' connected to $y_1, y_2, \ldots$, and x_1'' connected to $y_3, y_4, \ldots$; (p must be greater than 3). Then two circuits are $((x_1', y_1, x_2, y_2))$ and $((x_1'', y_3, x_3, y_4))$.

b) Let a K_1 be given. Applying II at x_3 yields a graph in Ω by the same argument as in a). Label the vertices so that x_1 is replaced by x_1' connected to $y_1, y_2, \ldots$ and x_1'' connected to $x_2, y_3, \ldots$; (here $p > 2$). Then two circuits are $((x_1', y_1, x_3, y_2))$ and $((x_1'', x_2, y_3))$.

c) Let a K_2 or K_3 be given. When $p > 2$, the only type of application of II not considered above is where x_2 is replaced by x_2' connected to x_1 and x_3 only, and x_2'' connected to $y_1, \ldots, y_p$. It can be seen by renaming x_3, x_2'' and x_2' respectively x_2, x_3, and y_{p+1} that the resulting graph is a K or K_1.

d) One further case remains when $p = 2$. (I am indebted to the referee for drawing this to my attention.) Given a ${}_2K_2$ or ${}_2K_3$, let x_2 be replaced by x_2' connected to x_1 and y_2, and x_2'' connected to x_3 and y_2. Two circuits are $((x_1, x_2', y_2))$ and $((x_2'', y_1, x_3))$.

LEMMA 3. Any graph not in Ω obtained by operation I from a 3-connected ${}_pK$, ${}_pK_1$, ${}_pK_2$, or ${}_pK_3$ is

a) a ${}_2K_3$ or <5> if $p = 2$;

b) a ${}_pK_1$, ${}_pK_2$, or ${}_pK_3$ if $p > 2$.

Proof: (We observe that ${}_2K$, ${}_2K_1$ are not 3-connected.) The lemma is obvious for a new edge connecting two x's. We consider the case where the new edge connects two y's and assume the vertices labelled so that the new edge is (y_1, y_2).

a) Let $p = 2$. K_2 yields a K_3; this can be seen by relabelling x_1, x_3, y_1, y_2 as y_1, y_2, x_1, x_3 respectively. K_3 yields a <5>.

b) Let $p = 3$. Clearly K yields a K_1, (interchange x's and y's). But K_1 yields a graph in Ω having circuits $((x_1, x_2, y_3))$ and $((y_1, y_2, x_3))$. Thus K_2 and K_3 also yield graphs in Ω.

c) Let $p > 3$. Then in all cases the circuits $((x_1, y_1, y_2))$ and $((x_2, y_3, x_3, y_4))$ are present.

To complete the proof of the theorem we need only remark that an operation II on <5> (and all such operations are equivalent) yields a graph in Ω. Thus, for $p > 2$, operations I on ${}_pK$, ${}_pK_1$, ${}_pK_2$, ${}_pK_3$ yield only ${}_pK_1$, ${}_pK_2$, or ${}_pK_3$ not in Ω; and operations II on ${}_pK$, ${}_pK_1$, ${}_pK_2$, ${}_pK_3$ yield only ${}_{p+1}K$ and ${}_{p+1}K_1$ not in Ω. The proof follows by induction on p.

REFERENCES

1. G. A. Dirac, Some results concerning the structure of graphs, Canad. Math. Bull. 6 (1963) 183-210.

2. W. T. Tutte, A theory of 3-connected graphs, Proc. Koninkl. Akad. Wetenschappen A 64 (1961) 441-455.

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