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A COMPARISON OF METHODS FOR CONSTRUCTING PROBABILITY MEASURES ON INFINITE PRODUCT SPACES

BY

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ABSTRACT. The construction, from a consistent family of finite dimensional probability measures, of a probability measure on a product space when the marginal measures are perfect is shown to follow from a classical theorem due to Ionescu Tulcea and known results on the existence of regular conditional probability functions.

1. **Introduction**. The famous theorem of Kolmogorov [5] gives the construction, from a consistent family of finite dimensional probability measures, of a probability measure on an arbitrary product of copies of the real line. Marczewski [6] extended the crucial compactness part of Kolmogorov's argument by defining, in a nontopological context, a compact class of sets. He called a probability measure compact if it is inner regular relative to some compact class and proved that Kolmogorov's theorem holds in an arbitrary product space if the marginal measures are compact. Ryll-Nardzewski [11] further extended the result to the case where the marginal measures are quasi-compact and proved that quasi-compactness is equivalent to perfection, a concept originally introduced by Gnedenko and Kolmogorov [3]. Without any compactness assumptions, von Neumann [7] obtained the general product probability theorem. Assuming only the existence of certain transition functions, Ionescu Tulcea [4] showed how to construct probability measures on infinite product spaces, and in the process obtained both von Neumann's result and Kolmogorov's original theorem as special cases. Dinculeanu [1] further extended Ionescu Tulcea's results to the case where there exist quasi-regular conditional probabilities.

Our purpose here is to show that the general result of Ryll-Nardzewski follows directly from Ionescu Tulcea's theorem and known results concerning the existence of regular conditional probabilities.

2. **Preliminaries**. Let $\{\Omega_i, \mathscr{F}_i)\}_{i\in I}$ be an arbitrary family of measurable spaces. A measurable rectangle is a subset of $\Pi_{i\in I}\Omega_i$ of the form $\Pi_{i\in I}S_i$, where $S_i \in \mathscr{F}_i$ for $i \in I$ and $S_i = \Omega_i$ for $i \in J^c$ with J finite. The algebra of subsets of $\Pi_{i\in I}\Omega_i$ consisting of finite disjoint unions of measurable rectangles will be denoted by $\mathscr{A}$, and $\mathscr{F} =$

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$\otimes_{i \in I} \mathcal{F}_i$ is the σ-algebra generated by $\mathcal{A}$. For $J \subset I$, f_J is the canonical projection of $\Pi_{i \in I} \Omega_i$ onto $\Pi_{i \in J} \Omega_i$, and if $J \subset I$ is finite, then a set of the form $f_J^{-1}(B_J)$, where $B_J \in \otimes_{i \in J} \mathcal{F}_i$, is called a cyclinder set based on J. The collection of all such cyclinder sets forms an algebra $\mathcal{B}$ and $\mathcal{A} \subset \mathcal{B} \subset \mathcal{F}$.

If $(\Omega_1, \mathcal{F}_1)$ and $(\Omega_2, \mathcal{F}_2)$ are measurable spaces, then μ is a transition function from $(\Omega_1, \mathcal{F}_1)$ to $(\Omega_2, \mathcal{F}_2)$ if

(i) for every $\omega_1 \in \Omega_1$, $\mu(\omega_1, .)$ is a probability measure on $(\Omega_2, \mathcal{F}_2)$;

(ii) for every $A_2 \in \mathcal{F}_2$, $\mu(., A_2)$ is a measurable function on $(\Omega_1, \mathcal{F}_1)$.

Given a probability measure P_1 on $(\Omega_1, \mathcal{F}_1)$ and a transition function μ from $(\Omega_1, \mathcal{F}_1)$ to $(\Omega_2, \mathcal{F}_2)$, we may define a probability measure $P_1\mu$ on $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2)$ by the formula

$$P_1\mu(A) = \int_{\Omega_1} P_1(d\omega_1) \int_{\Omega_2} I_A(\omega_1, \omega_2)\mu(\omega_1, d\omega_2).$$

Compact and perfect measures are discussed in detail by Ramachandran ([9], [10]) and Sazanov [12]. We only mention here that a probability measure P on $(\Omega, \mathcal{F})$ is perfect, or that $(\Omega, \mathcal{F}, P)$ is a perfect probability space, if for any real valued random variable X there is a Borel subset E of the real line such that $E \subset X(\Omega)$ and $P(X^{-1}(E)) = 1$. The restriction of a perfect measure to a sub-σ-algebra of $\mathcal{F}$ is obviously perfect. Other than this simple fact we need only the following result: if $(\Omega_1, \mathcal{F}_1, P_1)$ and $(\Omega_2, \mathcal{F}_2, P_2)$ are probability spaces with P_2 perfect and $\mathcal{F}_2$ countably generated, and if P is a probability measure on $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2)$ with marginal measures P_1 and P_2, then there is a transition function μ from $(\Omega_1, \mathcal{F}_1)$ to $(\Omega_2, \mathcal{F}_2)$ such that $P = P_1\mu$. A particularly simple proof of this result was given by Faden [2].

3. **Results.** In what follows the index set I is assumed to be infinite and $\mathcal{J}$ will denote the finite subsets of I. We first restate, in our terminology, Theorem (B) of [4].

THEOREM 1. *Let $\{(\Omega_i, \mathcal{F}_i)\}_{i \in I}$ be a family of measurable spaces. Assume that for each $J \in \mathcal{J}$ there is a probability measure P_J on $(\Pi_{i \in J} \Omega_i, \otimes_{i \in J} \mathcal{F}_i)$ such that*

(i) *$\{P_J\}_{J \in \mathcal{J}}$ is a consistent family;*

(ii) *if $J \in \mathcal{J}$ and $j \in J^c$, then there is a transition function $\mu_{J,j}$ from*

$$(\Pi_{i \in J} \Omega_i, \otimes_{i \in I} \mathcal{F}_i) \text{ to } (\Omega_j, \mathcal{F}_j) \text{ such that } P_{J \cup \{j\}} = P_J \mu_{j,j}.$$

Then there is a unique probability measure P on

$$(\Omega, \mathcal{F}) = (\Pi_{i \in I} \Omega_i, \otimes_{i \in I} \mathcal{F}_i)$$

such that $P_J = P f_J^{-1}$ for all $J \in \mathcal{J}$.

We now prove Ryll-Nardzewski's generalization of Kolmogorov's theorem.

THEOREM 2. *Let $\{(\Omega_i, \mathcal{F}_i)\}_{i \in I}$ be a family of measurable spaces. Assume that for each $J \in \mathcal{J}$ there is a probability measure P_J on $(\Pi_{i \in J} \Omega_i, \otimes_{i \in J} \mathcal{F}_i)$ such that*

(i) *$\{P_J\}_{J \in \mathcal{J}}$ is a consistent family;*

(ii) *$P_{\{i\}}$ is a perfect probability measure on $(\Omega_i, \mathcal{F}_i)$ for all $i \in I$.*

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Then there is a unique probability measure P on

$$(\Omega, \mathcal{F}) = (\Pi_{i\in I}\Omega_i, \otimes_{i\in I}\mathcal{F}_i)$$

such that $P_J = Pf_J^{-1}$ for all $J \in \mathcal{J}$.

PROOF. Let Q denote the finitely additive probability measure on $\mathcal{B}$ defined by the formula

$$Q(f_J^{-1}(B_J)) = P_J(B_J),$$

where $J \in \mathcal{J}$ and $B_J \in \otimes_{i\in J}\mathcal{F}_i$. Since $\mathcal{A} \subset \mathcal{B}$, Q is also a finitely additive probability measure when restricted to $\mathcal{A}$. $\mathcal{F}$ is generated by $\mathcal{A}$ and therefore it suffices to show that Q is countably additive on $\mathcal{A}$. That is, if $\{A_n\}_{n\geq 1}$ is a decreasing sequence of sets in $\mathcal{A}$ with empty intersection, then

$$\lim_{n\to\infty} Q(A_n) = 0.$$

For the purpose of proving that $\lim_{n\to\infty} Q(A_n) = 0$ for the fixed sequence $\{A_n\}_{n\geq 1}$, note that any particular representation of A_n as a finite disjoint union of measurable rectangles involves finitely many measurable subsets of any Ω_i. Hence $\{A_n\}_{n\geq 1}$ is a sequence of sets in the algebra generated by the rectangles which are measurable with respect to certain countably generated sub-σ-algebras $\mathcal{F}_i^*$ of $\mathcal{F}_i$. If P_J^* denotes the restriction of P_J to $\otimes_{i\in J}\mathcal{F}_i^*$, then (i) holds with P_J replaced by P_J^* and (ii) continues also to hold, since the restriction of a perfect measure is also perfect. If $\mathcal{A}^*$, $\mathcal{B}^*$, $\mathcal{F}^*$ and Q^* are defined analogously, then Q^* is the restriction of Q to $\mathcal{A}^*$.

If $J \in \mathcal{J}$ and $j \in J^c$, then the results stated in Section 2 imply that there is a transition function $\mu_{J,j}$ from $(\Pi_{i\in J}\Omega_i, \otimes_{i\in J}\mathcal{F}_i^*)$ to $(\Omega_j, \mathcal{F}_j^*)$ such that $P_{J\cup\{j\}}^* = P_J^*\mu_{J,j}$. Hence Theorem 1 implies that there is a unique probability measure on P^* on $(\Omega, \mathcal{F}^*)$ such that $P_J^* = P^*f_j^{-1}$ for all $J \in \mathcal{J}$, and therefore

$$\lim_{n\to\infty} Q(A_n) = \lim_{n\to\infty} Q^*(A_n) = \lim_{n\to\infty} P^*(A_n) = 0,$$

where the last equality follows from the countable additivity of P^*. The proof is thus complete.

4. **Alternative approaches**. The point to be emphasized in the proof of Theorem 2 is that, in general, it is impossible to obtain transition functions $\mu_{J,j}$ with $P_{J\cup\{j\}} = P_J\mu_{J,j}$ as in Theorem 1, even if the marginal measures $P_{\{i\}}$ are compact. However, it suffices to assume that the σ-algebras $\mathcal{F}_i$ are countably generated and, in the presence of perfection, suitable transition functions then exist.

Recent work of Pachl [8], as described in Ramachandran [10], on disintegration of measures implies that if the marginal measures $P_{\{i\}}$ are compact, then $P_{J\cup\{j\}} = P_J\mu_{J,j}$, where $\mu_{J,j}$ is a quasi-transition function. In general, if $(\Omega_1, \mathcal{F}_1, P_1)$ is a probability space and $(\Omega_2, \mathcal{F}_2)$ is a measurable space, then μ is a quasi-transition function from $(\Omega_1, \mathcal{F}_1, P_1)$ to $(\Omega_2, \mathcal{F}_2)$ if

(i) for every $\omega_1 \in \Omega_1$ there is a sub-σ-algebra $\mathcal{F}_2^{\omega_1} \subset \mathcal{F}_2$ such that $\mu(\omega_1, .)$ is a probability measure on $(\Omega_2, \mathcal{F}_2^{\omega_1})$;

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(ii) for every $A_2 \in \mathcal{F}_2$ there is a set $N(A_2) \in \mathcal{F}_1$ with $P_1(N(A_2)) = 0$ such that $A_2 \in \mathcal{F}_2^{\omega_1}$ if $\omega_1 \in N(A_2)^c$ *and* $\mu(.,A_2)$ is $\mathcal{F}_1$-measurable on $N(A_2)^c$.

Note that the definition of a quasi-transition function makes sense only with respect to a given probability measure P_1 on $(\Omega_1, \mathcal{F}_1)$.

It is possible to obtain Marczewski's theorem (Theorem 2 with perfect replaced by compact) from a generalization of Theorem 1, where the transition functions in that theorem are replaced by quasi-transition functions. To follow this approach it is necessary to formulate and prove Ionescu Tulcea's classical theorem in the context of quasi-transition functions. This is a straightforward (but tedious) extension of the original result with extra care exercised in handling certain exceptional sets on probability zero.

Theorem 2 can also be proved by showing that the assumption of perfect marginal probability measures implies the existence of quasi-regular conditional probabilities (not to be confused with the quasi-transition functions defined above) and applying the results of Dinculeanu [1].

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