FORMULAE FOR ABSOLUTE MOMENTS

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The purpose of this note is to derive an alternative expression to that given in Lemma 1 below (due to Hsu, and to von Bahr) for the absolute moments of a random variable, in terms of the characteristic function.

Let X be a random variable ($r.v.$) with distribution function (d.f.) $F(x)$ and characteristic function (ch.f.)

$$\phi(t) = \int_{-\infty}^{\infty} e^{ixt} dF(x).$$

The r^{th} moment of X (or of F) is

$$EX^r = \mu_r = \int_{-\infty}^{\infty} x^r dF(x),$$

and the r^{th} absolute moment of X (or of F) is

$$E|X|^r = \beta_r = \int_{-\infty}^{\infty} |x|^r dF(x).$$

When $\beta_r < \infty$, $\phi(t)$ is r times differentiable with

$$\phi^{(r)}(0) = i^r \mu_r, \quad \text{and}$$

$$\phi^{(r)}(t) = i^r \int_{-\infty}^{\infty} x^r e^{itx} dF(x); \qquad r = 1, 2, \cdots$$

(e.g. Lukacs [4], p. 29).

It is well known that the moments μ_r, $r = 1, 2, \cdots$ can be identified as the coefficients of $(it)^r r!$ in a power series expansion of $\phi(t)$ (see Pitman [5], Loeve [3], p. 199, or equations (1), (2) of [1]), thus including absolute moments of even integer order. When $v > 0$ is *not* an even integer, absolute moments B_v of order v can be found from the following formula, due to Hsu [2], and von Bahr [6] (see also lemma 1 of [1]).

LEMMA 1. *If $v > 0$ is not an even integer, and $\beta_v < \infty$, then*

$$A_v \beta_v = \int_0^{\infty} \mathscr{R}l \left(\phi(t) - \sum_{j=0}^{m} \frac{(it)^j \mu_j}{j!}\right) t^{-(v+1)} dt,$$

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where $\nu = m+\delta$ *with* m *an integer*, $0 < \delta \leqq 1$, *and*

$$A_\nu = -\pi/2\Gamma(\nu+1)\cdot\sin(\nu\pi/2).$$

Now assume in addition that $X \geqq 0$ a.e., and let

$$G_r(x) = \int_0^x u^r \, dF(u),$$

with $r = 0, 1, \cdots, m < \nu \leqq m+1$ and $\beta_\nu < \infty$. Then $\beta_\nu = \mu_\nu$ is the $(\nu-r)^{\text{th}}$ moment of $G_r(x)$ and $i^{-r}\phi^{(r)}(t)$ is the ch.f. of $G_r(x)$. Let

$$\alpha_m(t) = \phi(t) - \sum_{j=0}^{m} (it)^j \mu_j / j!$$

According to Theorem 2 of [1], $\mu_\nu = \beta_\nu < \infty$ implies that

$$\alpha_m(t) = o(|t|^\nu) \text{ for non-integral } \nu,$$
$$\mathscr{R}l\alpha_m(t) = o(|t|^\nu) \text{ for odd integers } \nu, \text{ and}$$
$$\mathscr{I}m\alpha_m(t) = o(|t|^\nu) \text{ for even integers } \nu, \text{ as } t \to 0.$$

Applying this result to $G_r(x)$ and its ch.f. $i^{-r}\phi^{(r)}(t)$ (noting that ν is either non-integral or that $\nu, \nu-1, \nu-2, \cdots$ are odd, even, odd, $\cdots$ integers respectively) gives

LEMMA 2. *If* $\nu > 0$ *is not an even integer*, $\nu = m+\delta$ *with* m *an integer and* $0 < \delta \leqq 1$, $X \geqq 0$ *a.e. and* $\beta_\nu = \mu_\nu < \infty$, *then*

$$\mathscr{R}l\Big(\phi^{(r)}(t) - i^r \sum_{j=0}^{m-r} (it)^j \mu_{j+r}/j!\Big) = o(|t|^{\nu-r}) \qquad \text{as } t \to 0;$$

for $r = 0, 1, 2, \cdots m$.

COROLLARY 1.

$$\mathscr{R}l\Big(\phi^{(r)}(t) - i^r \sum_{j=0}^{m-r} (it)^j \mu_{j+r}/j!\Big) = o(|t|^{\nu-r}) \qquad \text{as } |t| \to \infty,$$

for $r = 0, 1, 2, \cdots m$.

PROOF. Observe that $\nu > m$ and that $|\phi^{(r)}(t)| \leqq \mu_r < \infty$ for $r \leqq m$. From lemma 1,

$$A_\nu \mu_\nu = \lim_{\substack{\varepsilon\to 0\\ R\to\infty}} \int_\varepsilon^R \mathscr{R}l\, \alpha_m(t) t^{-(\nu+1)}\, dt,$$

which, after integrating by parts m times and invoking Lemma 2 and Corollary 1 for $r = 0, 1, 2, \cdots m-1$ gives

COROLLARY 2.

$$(1) \qquad A_\nu \beta_\nu = \lim_{\substack{\varepsilon\to 0\\ R\to\infty}} \frac{\Gamma(1+\delta)}{\Gamma(\nu+1)} \int_\varepsilon^R \mathscr{R}l\,(\phi^{(m)}(t) - i^m \mu_m) t^{-(1+\delta)}\, dt$$

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If $\phi(t)$ is $(m+1)$ times differentiable, then a further integration by parts, and application of Lemma 2 and Corollary 1 for $r = m$, gives

COROLLARY 3.

$$A_\nu \beta_\nu = \lim_{\substack{\varepsilon\to 0\\ R\to\infty}} \frac{\Gamma(\delta)}{\Gamma(\nu+1)} \int_\varepsilon^R \mathscr{R}l\, \phi^{(m+1)}(t)\cdot t^{-\delta}\, dt \tag{2}$$

Now drop the assumption that $X \geqq 0$ a.e. Let $X_+ = \max(0, X)$, $X_- = \max(0, -X)$, with $X_+, X_- \geqq 0$ and $X = X_+ - X_-$. If $\phi_+(t)$, $\phi_-(t)$ are the ch.fs of X_+ and X_-, respectively, then

$$\phi(t) = \phi_+(t)+\phi_-(-t)-1,$$
$$\mathscr{R}l\phi(t) = \mathscr{R}l(\phi_+(t)+\phi_-(t)-1),$$
$$\mathscr{R}l\, \phi^{(r)}(t) = \mathscr{R}l\,(\phi_+^{(r)}(t)+\phi_-^{(r)}(t)), \quad \text{and}$$
$$\beta_\nu = E|X|^\nu = EX_+^\nu + EX_\nu^- .$$

Therefore, applying Corollaries 2 and 3 to X_+ and X_-, and adding, gives

THEOREM 1. *If $\nu > 0$ is not an even integer, $\nu = m+\delta$ with m an integer and $0 < \delta \leqq 1$, and $\beta_\nu < \infty$, then* (1) *holds.*

If $\phi(t)$ is $(m+1)$ times differentiable, then (2) *holds.*

We note that (i) $\mu_0 = 1$, (ii) $\phi^{(m+1)}(t)$ might exist for $t \neq 0$ even if $\phi^{(m+1)}(0)$ does not exist, (iii) the integrands in (1), (2) might not be absolutely integrable and thus (iv) contour integration might be needed to evaluate (1) and/or (2).

The possibility of obtaining Theorem 1 arose out of a discussion with Dr. G. K. Eagleson.

References

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