3

The Wess–Zumino model

The simplest four-dimensional quantum field theory with supersymmetry realized linearly, i.e. where the transformed field is a linear function of the original fields, was written down in 1974 by Julius Wess and Bruno Zumino.[1] The Wess–Zumino (WZ) model is interesting not only because it illustrates many of the characteristics of more complicated supersymmetric models within a toy framework (this forms the subject of this chapter), but also because Yukawa interactions of the supersymmetric SM can be written as a straightforward extension of this model.

3.1 The Wess–Zumino Lagrangian

3.1.1 The field content

Let us consider a field theory with the Lagrangian given by

$$\mathcal{L} = \mathcal{L}_{\text{kin}} + \mathcal{L}_{\text{mass}}. \tag{3.1a}$$

[1] J. Wess and B. Zumino, *Nucl. Phys.* **B70**, 39 (1974). This is not, however, the first paper on (relativistic) spacetime supersymmetry. This distinction belongs to Y. Golfand and E. Likhtman, *JETP Lett.* **13**, 323 (1971) who introduced the supersymmetric extension of the Poincaré algebra. Motivated by the possibility that the neutrino could be the Goldstone fermion (see Chapter 7) associated with the spontaneous breakdown of a fermionic symmetry, D. Volkov and V. Akulov, *JETP Lett.* **16**, 621 (1972) and *Phys. Lett.* **B46**, 109 (1973) *independently* constructed a model with non-linearly realized supersymmetry. Local supersymmetry was first considered by D. Volkov and V. Soroka, *JETP*, **18**, 312 (1973). In this remarkable paper, they noticed the need for dynamical spin 2 and spin $\frac{3}{2}$ fields, noted the connection with gravity, and also what we now refer to as the super-Higgs mechanism; see Chapter 10. Wess and Zumino wrote their seminal paper quite unaware of any of these developments in what was formerly the Soviet Union. Two-dimensional world sheet supersymmetry (which is conceptually distinct from the spacetime supersymmetry that is the subject of this book) was discovered in 1971 in string models by A. Neveu and J. Schwarz, *Nucl. Phys.* **B31**, 86, (1971), and by P. Ramond, *Phys. Rev.* **D3**, 2415 (1971), and recognized as such by J. Gervais and B. Sakita, *Nucl. Phys.* **B34**, 632 (1971). We refer the interested reader to SUSY 30, *Proc. of the International Symposium Celebrating 30 Years of Supersymmetry*, K. Olive, S. Rudaz and M. Shifman, Editors, *Nucl. Phys.* **B 101** (Proc. Suppl.) (2001), and to *The Supersymmetric World*, G. Kane and M. Shifman, Editors (World Scientific, 2000) for a view of these developments through the eyes of the pioneers of supersymmetry.

$$\mathcal{L}_{\text{kin}} = \frac{1}{2}(\partial_\mu A)^2 + \frac{1}{2}(\partial_\mu B)^2 + \frac{\mathrm{i}}{2}\overline{\psi}\not{\partial}\psi + \frac{1}{2}(F^2 + G^2). \tag{3.1b}$$

$$\mathcal{L}_{\text{mass}} = -m[\frac{1}{2}\bar{\psi}\psi - GA - FB]. \tag{3.1c}$$

Here, A and B are real scalar fields with mass dimension $[A] = [B] = 1$, while ψ is a 4-component *Majorana* spinor field with mass dimension $[\psi] = 3/2$. A Majorana spinor is its own charge conjugate, so that

$$\psi = \psi^c = C\bar{\psi}^T, \tag{3.2a}$$

where the charge conjugation matrix C satisfies

$$C\gamma_\mu^T C^{-1} = -\gamma_\mu \tag{3.2b}$$

$$C^T = C^{-1} = -C \tag{3.2c}$$

and

$$[C, \gamma_5] = 0. \tag{3.2d}$$

Notice that (3.2a) is a constraint equation that says only two of the four components of ψ are independent. For instance, projecting out the right chiral component of (3.2a) yields

$$\psi_{\mathrm{R}} \equiv \frac{1+\gamma_5}{2}\psi = C\gamma^0\frac{1-\gamma_5}{2}\psi^* = C\gamma^0\psi_{\mathrm{L}}^* \tag{3.3}$$

which shows that ψ_{R} is completely determined by ψ_{L}.[2]

The fields F and G in (3.1b) and (3.1c) are also real scalar fields with mass dimension $[F] = [G] = 2$. Since they have no kinetic energy term, these fields do not propagate, and their Euler–Lagrange equations are purely algebraic. It is, therefore, simple to write F and G in terms of the propagating fields, and eliminate them from the Lagrangian altogether. For this reason, these fields are customarily referred to as *auxiliary* fields. Explicitly, the Euler–Lagrange equations,

$$\frac{\partial\mathcal{L}}{\partial\phi_i} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)} = 0, \tag{3.4}$$

for the Lagrangian (3.1a), with $\phi_i = F$ and G, give

$$F = -mB, \qquad G = -mA. \tag{3.5}$$

We thus see that F and G are not dynamically independent. The reason for introducing the auxiliary fields F and G, as we will soon see, is that it allows us to

[2] Throughout this book we use the convention that γ_5 is a real, symmetric matrix with $\gamma_5^2 = \mathbf{1}$.

https://doi.org/10.1017/9781009289801.004 Published online by Cambridge University Press

write supersymmetric variations as *linear* transformations on the fields, even in an interacting theory. It is interesting to see that the number of bosonic and fermionic degrees of freedom in the Lagrangian (3.1a) exactly balance, regardless of whether the Euler–Lagrange equations are satisfied: without equations of motion, there are four real components for the Majorana spinor field which are balanced by the four real scalars, A, B, F, and G. We can, however, eliminate the auxiliary fields using (3.5) to obtain the Lagrangian for the dynamically independent fields which then takes the form,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu A)^2 + \frac{1}{2}(\partial_\mu B)^2 + \frac{\mathrm{i}}{2}\bar{\psi}\partial\!\!\!/\psi - \frac{1}{2}m^2(A^2 + B^2) - \frac{1}{2}m\bar{\psi}\psi. \tag{3.6}$$

This is the Lagrangian for free fields A, B, and ψ. When these fields obey their respective equations of motion, their quanta correspond to two spin zero particles A and B, and a self-conjugate, spin $\frac{1}{2}$ particle, all with the same mass. Once again, we see that there is an exact match between the bosonic and fermionic degrees of freedom.

3.1.2 SUSY transformations and invariance of the action

In quantum field theory, a symmetry transformation is a transformation which leaves the equations of motion for the fields of the theory invariant. This is guaranteed if the action $S = \int \mathrm{d}^4x\,\mathcal{L}$ is left invariant under the transformation. In particular, if the Lagrangian $\mathcal{L}$ is invariant, or if it changes by a total derivative $\mathcal{L} \to \mathcal{L}' = \mathcal{L} + \partial_\mu\Lambda^\mu$, the action remains invariant. This can be seen by applying the four-dimensional version of Gauss' theorem, $\int_V \mathrm{d}^4x\,\partial_\mu\Lambda^\mu = \int_{\partial V} \mathrm{d}\sigma\,\Lambda^\mu n_\mu$, to the transformed Lagrangian. The quantity Λ^μ vanishes on the boundary ∂V as long as it is assumed that the fields vanish at spatial infinity, and field variations are equal to zero at the end points of the time integration.

Wess and Zumino noted that under the following set of *infinitesimal* field transformations, where $A \to A + \delta A$, etc., with

$$\delta A = \mathrm{i}\bar{\alpha}\gamma_5\psi, \tag{3.7a}$$

$$\delta B = -\bar{\alpha}\psi, \tag{3.7b}$$

$$\delta\psi = -F\alpha + \mathrm{i}G\gamma_5\alpha + \partial\!\!\!/\gamma_5 A\alpha + \mathrm{i}\partial\!\!\!/ B\alpha, \tag{3.7c}$$

$$\delta F = \mathrm{i}\bar{\alpha}\partial\!\!\!/\psi, \tag{3.7d}$$

$$\delta G = \bar{\alpha}\gamma_5\partial\!\!\!/\psi, \tag{3.7e}$$

the Lagrangian (3.1a) changes by a *total derivative*. Here, α is a spacetime-independent anticommuting Majorana spinor parameter with dimension $[\alpha] = -1/2$. The linear transformations (3.7a–3.7e), which clearly mix boson fields with fermion fields, are known as *supersymmetry* transformations.

To verify the invariance of the action under the above transformations, we first note that bilinears of Majorana spinors have special symmetry properties. For example, for Majorana spinors ψ and χ,

$$\bar{\psi}\chi = \psi^T C\chi = \psi_a C_{ab}\chi_b = -\chi_b(-C_{ba})\psi_a = \chi^T C\psi = \overline{\chi}\psi, \tag{3.8a}$$

where the first minus sign in step three is due to the anticommutativity of spinor fields and the second is due to the antisymmetry of C.[3] In a similar fashion, using the properties $\gamma_5^T = \gamma_5$ and $C^{-1}\gamma_\mu^T C = -\gamma_\mu$, it is straightforward to show that

$$\bar{\psi}\gamma_5\chi = \bar{\chi}\gamma_5\psi, \tag{3.8b}$$

$$\bar{\psi}\gamma_\mu\chi = -\bar{\chi}\gamma_\mu\psi, \tag{3.8c}$$

$$\bar{\psi}\gamma_\mu\gamma_5\chi = \bar{\chi}\gamma_\mu\gamma_5\psi, \tag{3.8d}$$

$$\bar{\psi}\sigma_{\mu\nu}\chi = -\bar{\chi}\sigma_{\mu\nu}\psi. \tag{3.8e}$$

Exercise *As discussed in the previous footnote, when $\chi = \psi$, we have to worry that χ and $\bar{\psi}$ do not perfectly anticommute. Except for the case $\mu = 0$, in (3.8c), the unwanted delta function term disappears because $Tr(\gamma^0\Gamma) = 0$, for $\Gamma = \gamma_5, \gamma_k, \gamma_5\gamma_\mu$ and $\sigma_{\mu\nu}$. This trace does not, however, vanish for $\Gamma = \gamma_0$. Show that (3.8c) still holds if we understand the field product to be normal ordered.*

Now we apply the supersymmetry transformations to each term of $\mathcal{L}_{\text{kin}}$, and make use of the product rule $\partial_\mu(f \cdot g) = \partial_\mu f \cdot g + f \cdot \partial_\mu g$ and the relations (3.8a–3.8e):

$$\begin{aligned}
\frac{1}{2}\delta[(\partial_\mu A)^2] &= (\partial^\mu A)\partial_\mu\delta A = \mathrm{i}\partial^\mu A\bar{\alpha}\gamma_5\partial_\mu\psi, \\
&= \partial^\mu(\mathrm{i}\partial_\mu A\bar{\alpha}\gamma_5\psi) - \mathrm{i}\Box A\bar{\alpha}\gamma_5\psi,
\end{aligned} \tag{3.9a}$$

$$\begin{aligned}
\frac{1}{2}\delta[(\partial_\mu B)^2] &= -\partial^\mu B\bar{\alpha}\partial_\mu\psi, \\
&= \partial^\mu(-\partial_\mu B\bar{\alpha}\psi) + \Box B\bar{\alpha}\psi,
\end{aligned} \tag{3.9b}$$

$$\begin{aligned}
\frac{\mathrm{i}}{2}\delta[\bar{\psi}\not{\partial}\psi] &= \frac{\mathrm{i}}{2}[\delta\bar{\psi}\not{\partial}\psi + \bar{\psi}\not{\partial}\delta\psi] \\
&= \partial^\mu(-\frac{\mathrm{i}}{2}F\bar{\alpha}\gamma_\mu\psi) + \mathrm{i}\bar{\alpha}\not{\partial}F\psi + \partial^\mu(-\frac{1}{2}G\bar{\alpha}\gamma_5\gamma_\mu\psi) \\
&\quad -\bar{\alpha}\not{\partial}G\gamma_5\psi + \partial^\mu(\frac{-\mathrm{i}}{2}\bar{\alpha}\gamma_5\not{\partial}A\gamma_\mu\psi) + \mathrm{i}\bar{\alpha}\gamma_5\Box A\psi \\
&\quad + \partial^\mu(\frac{1}{2}\bar{\alpha}\not{\partial}B\gamma_\mu\psi) - \bar{\alpha}\Box B\psi,
\end{aligned} \tag{3.9c}$$

$$\frac{1}{2}\delta(F^2) = \mathrm{i}F\bar{\alpha}\not{\partial}\psi$$

[3] If $\chi = \psi$, then the fields (at equal times) do not anticommute to zero but to a multiple of γ_0 times a delta function. Since γ_0 is traceless, the result in (3.8a) still holds.

https://doi.org/10.1017/9781009289801.004 Published online by Cambridge University Press

$$= \partial^\mu(\mathrm{i}F\bar{\alpha}\gamma_\mu\psi) - \mathrm{i}\bar{\alpha}\partial\!\!\!/ F\psi, \tag{3.9d}$$

$$\frac{1}{2}\delta(G^2) = G\bar{\alpha}\gamma_5\partial\!\!\!/\psi$$
$$= \partial^\mu(G\bar{\alpha}\gamma_5\gamma_\mu\psi) + \bar{\alpha}\partial\!\!\!/ G\gamma_5\psi, \tag{3.9e}$$

where $\Box = \partial_\mu\partial^\mu = \partial^2/\partial t^2 - \partial^2/\partial x^2 - \partial^2/\partial y^2 - \partial^2/\partial z^2 = \partial\!\!\!/\partial\!\!\!/$. By combining the terms contributing to $\mathcal{L}_{\text{kin}}$ in Eq. (3.9a–3.9e), we see that

$$\delta\mathcal{L}_{\text{kin}} = \partial^\mu(-\frac{1}{2}\bar{\alpha}\gamma_\mu\partial\!\!\!/ B\psi + \frac{\mathrm{i}}{2}\bar{\alpha}\gamma_5\gamma_\mu\partial\!\!\!/ A\psi + \frac{\mathrm{i}}{2}F\bar{\alpha}\gamma_\mu\psi + \frac{1}{2}G\bar{\alpha}\gamma_5\gamma_\mu\psi), \tag{3.10a}$$

so that $\mathcal{L}_{\text{kin}}$ changes by a total derivative under a SUSY transformation. The reader can similarly check that $\delta\mathcal{L}_{\text{mass}}$ is a total derivative.

Exercise *Show that*

$$\delta\mathcal{L}_{\text{mass}} = \partial^\mu(mA\bar{\alpha}\gamma_5\gamma_\mu\psi + \mathrm{i}mB\bar{\alpha}\gamma_\mu\psi) \tag{3.10b}$$

under the supersymmetry transformations (3.7a–3.7e).

We now recall Noether's theorem which states that for every continuous symmetry transformation in a field theory, there is a corresponding current which is conserved, as long as the field equations are satisfied. For the case at hand, where $\delta\mathcal{L} = \partial^\mu\Lambda_\mu$, the current is given by

$$\bar{\alpha}j^\mu(x) = \sum_{\text{fields }\phi_i}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\delta\phi_i - \Lambda^\mu, \tag{3.11}$$

with $\phi_i = A$, B, and ψ. The variations $\delta\phi_i$ as well as the quantity Λ^μ depend linearly on the transformation parameter $\bar{\alpha}$. The contributions to j^μ from the A, B, and ψ fields are

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu A)}\delta A = \partial^\mu A\mathrm{i}\bar{\alpha}\gamma_5\psi, \tag{3.12a}$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu B)}\delta B = \partial^\mu B(-\bar{\alpha}\psi), \quad \text{and} \tag{3.12b}$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\delta\psi = \bar{\alpha}[\frac{1}{2}(\mathrm{i}F + G\gamma_5)\gamma^\mu + \frac{1}{2}\partial\!\!\!/(-\mathrm{i}A\gamma_5 - B)\gamma^\mu]\psi. \tag{3.12c}$$

Combining the above with (3.10a) and (3.10b), we can explicitly construct the current (known in this case as the *supercurrent*). Notice that the supercurrent itself carries a spinorial index since its time component has to integrate to the spinor generator of supersymmetry transformations. We find

$$j^\mu = \partial\!\!\!/(-\mathrm{i}A\gamma_5 - B)\gamma^\mu\psi + \mathrm{i}m(\mathrm{i}A\gamma_5 - B)\gamma^\mu\psi. \tag{3.13a}$$

https://doi.org/10.1017/9781009289801.004 Published online by Cambridge University Press

For later use, notice that the supercurrent may also be written as,

$$j^{\mu} = \partial\!\!\!/(-\mathrm{i}A\gamma_5 - B)\gamma^{\mu}\psi + (G\gamma_5 + \mathrm{i}F)\gamma^{\mu}\psi. \tag{3.13b}$$

Exercise *Show that $\partial_\mu j_a^\mu = 0$ if the fields A and B satisfy the Klein–Gordon equation, and ψ satisfies the Dirac equation.*

The conserved charges associated with the current $j_a^\mu(x)$ are then given by

$$Q_a = \int j_a^0(x)\mathrm{d}^3x. \tag{3.14}$$

In the next section, we will explicitly compute the super-charge Q_a for the WZ Model, and show that it indeed generates the SUSY transformations (3.7a–3.7e) as long as the field equations hold.

Exercise *Verify that if we substitute the solutions (3.5) to the Euler–Lagrange equations for F and G into the SUSY transformation laws (3.7d–3.7e), the resulting "on-shell" transformations are consistent with (3.7a) and (3.7b) as long as A, B, and ψ satisfy their equations of motion.*

3.1.3 The chiral multiplet

For the purposes of the development of superfield calculus, we remark that the fields of the WZ model can be conveniently written in terms of complex fields,

$$\begin{aligned} \mathcal{S} &= \frac{1}{\sqrt{2}}(A + \mathrm{i}B) \\ \psi_{\mathrm{L}} &= \frac{1-\gamma_5}{2}\psi \\ \mathcal{F} &= \frac{1}{\sqrt{2}}(F + \mathrm{i}G) \end{aligned} \tag{3.15}$$

where $\mathcal{S}$, ψ_{L}, and $\mathcal{F}$ transform into one another under the SUSY transformations (3.7a–3.7e). It is straightforward to check that these transformations can be written as,

$$\delta\mathcal{S} = -\mathrm{i}\sqrt{2}\bar{\alpha}\psi_{\mathrm{L}}, \tag{3.16a}$$

$$\delta\psi_{\mathrm{L}} = -\sqrt{2}\mathcal{F}\alpha_{\mathrm{L}} + \sqrt{2}\partial\!\!\!/\mathcal{S}\alpha_{\mathrm{R}}, \tag{3.16b}$$

$$\delta\mathcal{F} = \mathrm{i}\sqrt{2}\bar{\alpha}\partial\!\!\!/\psi_{\mathrm{L}}\,. \tag{3.16c}$$

https://doi.org/10.1017/9781009289801.004 Published online by Cambridge University Press

Since ψ_R is not independent of ψ_L, we only have to specify how ψ_L transforms. Thus, $\mathcal{S}$, ψ_L, and $\mathcal{F}$ together constitute an irreducible supermultiplet in much the same way that the proton and neutron form a doublet of isospin. Further, analogous to the isospin formalism that treats the nucleon doublet as a single entity, there is a formalism known as the superfield formalism that combines all three components of the supermultiplet into a superfield $\hat{S}$. Since only one chiral component of the Majorana spinor ψ enters the transformations, such superfields are referred to as (left) chiral superfields. Because the lowest spin component of the multiplet has spin zero, this superfield is known as a left-chiral scalar superfield. We will defer detailed discussion of the superfield formalism until Chapter 5.

3.1.4 Algebra of the SUSY charges

We have already seen in Chapter 1 that a continuous symmetry transformation can be written in terms of the corresponding generator. This is also true of supersymmetry transformations, the difference being that the parameter of the transformation α is a Majorana spinor whose components anticommute with themselves and also with fermionic operators. Just as in (1.4), we may write the change of the field $\mathcal{S}$ under an infinitesimal SUSY transformation as,

$$\mathcal{S} \to \mathcal{S}' = \mathrm{e}^{\mathrm{i}\bar{\alpha}Q}\mathcal{S}\mathrm{e}^{-\mathrm{i}\bar{\alpha}Q} \approx \mathcal{S} + [\mathrm{i}\bar{\alpha}Q, \mathcal{S}] = \mathcal{S} + \delta\mathcal{S} \equiv (1 - \mathrm{i}\bar{\alpha}Q)\mathcal{S} \,. \tag{3.17}$$

Here, Q is the (Majorana) spinor generator of the SUSY transformation except in the last equality, where we have abused notation in that Q there denotes the representation of the super-charge generator (explicitly worked out in Chapter 5), in the same way that the translation generator P_μ is represented by $\mathrm{i}\partial_\mu$ when we write $[P_\mu, \mathcal{S}] = -\mathrm{i}\partial_\mu\mathcal{S}$. We thus write the change of the field $\mathcal{S}$ as $\delta\mathcal{S} = -\mathrm{i}\bar{\alpha}Q\mathcal{S}$. We can now work out the algebra for the Q's and their conjugates $\bar{Q}$ by considering the commutator of two successive SUSY transformations – the first by parameter α_1, and the second by parameter α_2. For the case of the scalar field $\mathcal{S}$, since $\delta_1\mathcal{S} = -\sqrt{2}\mathrm{i}\bar{\alpha}_1\psi_L$, then

$$\begin{aligned}
(\delta_2\delta_1 - \delta_1\delta_2)\mathcal{S} = -2\mathrm{i}&\left\{-\mathcal{F}\bar{\alpha}_1\frac{1-\gamma_5}{2}\alpha_2 + \bar{\alpha}_1\gamma^\mu\frac{1+\gamma_5}{2}\alpha_2\partial_\mu\mathcal{S}\right\}\\
+2\mathrm{i}&\left\{-\mathcal{F}\bar{\alpha}_2\frac{1-\gamma_5}{2}\alpha_1 + \bar{\alpha}_2\gamma^\mu\frac{1+\gamma_5}{2}\alpha_1\partial_\mu\mathcal{S}\right\}\\
=\ & 2\mathrm{i}\bar{\alpha}_2\gamma^\mu\alpha_1\partial_\mu\mathcal{S}\\
=\ & -2\bar{\alpha}_2\gamma^\mu\alpha_1[P_\mu, \mathcal{S}].
\end{aligned} \tag{3.18}$$

https://doi.org/10.1017/9781009289801.004 Published online by Cambridge University Press

We can work out the same commutator in terms of the SUSY generator Q using $\delta\mathcal{S} = [\mathrm{i}\bar{\alpha}Q, \mathcal{S}]$ to obtain,

$$\begin{aligned}\delta_2\delta_1\mathcal{S} &= [\mathrm{i}\bar{\alpha}_2 Q, \delta_1\mathcal{S}] = \left[\mathrm{i}\bar{\alpha}_2 Q, [\mathrm{i}\bar{Q}\alpha_1, \mathcal{S}]\right] \\ &= -\left[\mathrm{i}\bar{Q}\alpha_1, [\mathcal{S}, \mathrm{i}\bar{\alpha}_2 Q]\right] - \left[\mathcal{S}, [\mathrm{i}\bar{\alpha}_2 Q, \mathrm{i}\bar{Q}\alpha_1]\right],\end{aligned} \tag{3.19}$$

where in the last step we have used the Jacobi identity,

$$[[A, B], C] + [[B, C], A] + [[C, A], B] = 0,$$

that holds for any three bosonic operators, A, B, and C, as the reader may easily verify. Applying (3.19) twice, we readily obtain

$$(\delta_2\delta_1 - \delta_1\delta_2)\mathcal{S} = -\bar{\alpha}_{2a}\alpha_{1b}[\{Q_a, \bar{Q}_b\}, \mathcal{S}]. \tag{3.20}$$

Finally, by equating the right-hand sides of (3.18) and (3.20) we can write the algebra obeyed by the SUSY generators as,

$$\left\{Q_a, \bar{Q}_b\right\} = 2(\gamma^\mu)_{ab} P_\mu \tag{3.21}$$

where P_μ is the Poincaré group generator of spacetime translations.

A similar calculation can be performed for the commutator of SUSY transformations on the field ψ_L. It is straightforward to show

$$\begin{aligned}(\delta_2\delta_1 - \delta_1\delta_2)\psi_\mathrm{L} = &-2\mathrm{i}[(\bar{\alpha}_2\partial\!\!\!/\psi_\mathrm{L})\alpha_{1L} - (\bar{\alpha}_1\partial\!\!\!/\psi_\mathrm{L})\alpha_{2L}] \\ &- 2\mathrm{i}[(\bar{\alpha}_2\partial_\mu\psi_\mathrm{L})\gamma^\mu\alpha_{1R} - (\bar{\alpha}_1\partial_\mu\psi_\mathrm{L})\gamma^\mu\alpha_{2\mathrm{R}}].\end{aligned} \tag{3.22}$$

To proceed further, we need to apply a Fierz re-arrangement to the spinors, and combine the two α's into a bilinear.

Exercise *The set of 16 matrices* $\Gamma_i = \left\{\mathbf{1}, \gamma_5, \gamma_\mu, \mathrm{i}\gamma_\mu\gamma_5, \sigma_{\mu\nu}\right\}$ *(with* $\sigma_{\mu\nu} = \frac{\mathrm{i}}{2}[\gamma_\mu, \gamma_\nu]$ *for* $\mu > \nu$*), with* Γ^i *defined the same way except with all the indices upstairs, have the properties* $\mathrm{Tr}\Gamma^i = 0$ *(for* $\Gamma^i \neq 1$*) and* $\mathrm{Tr}\Gamma^i\Gamma_j = 4\delta^i_j$*. These matrices can be used as a basis of expansion for any other* 4×4 *matrix. In particular, for the combination of spinors*

$$\bar{\psi}(1)\psi(2)\psi_b(3) \equiv \bar{\psi}_a(1)\psi_a(2)\psi_b(3) = \psi_b(3)\bar{\psi}_a(1)\psi_a(2),$$

the quantity can be written as $\psi_b(3)\bar{\psi}_a(1) = \sum_i c_i\Gamma^i_{ba}$*. Multiplying both sides of this expansion by* Γ_{jab} *and summing over a and b, show that* $c_j = -\frac{1}{4}\bar{\psi}(1)\Gamma_j\psi(3)$*, so that*

$$\bar{\psi}(1)\psi(2)\psi_b(3) = -\frac{1}{4}\sum_j \bar{\psi}(1)\Gamma^j\psi(3)(\Gamma_j\psi(2))_b. \tag{3.23}$$

---

https://doi.org/10.1017/9781009289801.004 Published online by Cambridge University Press

Applying the Fierz re-arrangement to (3.22) yields

$$(\delta_2\delta_1 - \delta_1\delta_2)\psi_{\mathrm{L}} = \frac{2\mathrm{i}}{4}\sum_i (\bar{\alpha}_2\Gamma^i\alpha_1 - \bar{\alpha}_1\Gamma^i\alpha_2)P_{\mathrm{L}}\Gamma_i\gamma^\mu\partial_\mu\psi_{\mathrm{L}} + \frac{2\mathrm{i}}{4}\sum_i (\bar{\alpha}_2\Gamma^i\alpha_1 - \bar{\alpha}_1\Gamma^i\alpha_2)P_{\mathrm{L}}\gamma^\mu\Gamma_i\partial_\mu\psi_{\mathrm{L}}, \tag{3.24}$$

where the chiral projection operators P_{L} allow only the vector and axial-vector forms of Γ_i to contribute. Using relations (3.8c) and (3.8d) on the Γ_A and Γ_V terms, we find

$$\begin{aligned}(\delta_2\delta_1 - \delta_1\delta_2)\psi_{\mathrm{L}} &= \mathrm{i}\bar{\alpha}_2\gamma_\mu\alpha_1[\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu]\partial_\nu\psi_{\mathrm{L}} \\ &= 2\mathrm{i}\bar{\alpha}_2\gamma^\mu\alpha_1\partial_\mu\psi_{\mathrm{L}}.\end{aligned} \tag{3.25}$$

Comparison of this expression with the corresponding expression involving the Q and $\bar{Q}$ operators again verifies the relation (3.21).

Exercise *Show that the commutator of two SUSY transformations applied to the auxiliary field $\mathcal{F}$ again leads to the anticommutator (3.21).*

We thus see that (3.21) is valid acting on each component of an arbitrary field, so that it may be regarded as an operator relation.

The appearance of the translation generator in (3.21) shows that supersymmetry is a spacetime symmetry. Conservation of supersymmetry implies

$$[Q_a, P^0] = 0, \tag{3.26a}$$

or, from Lorentz covariance,

$$[Q_a, P^\mu] = 0. \tag{3.26b}$$

The commutators of Q with the Lorentz group generators $J_{\mu\nu}$ are fixed because we have already declared Q to be a spin $\frac{1}{2}$ Majorana spinor.

The supersymmetry algebra described above is not a Lie algebra since it includes anticommutators. Such algebras are referred to as graded Lie algebras. Haag, Lopuszanski, and Sohnius have shown that (except for the possibility of neutral elements and of more than one spinorial charge Q) the algebra that we have obtained above is the most general graded Lie algebra consistent with rather reasonable physical assumptions. Models with more than one SUSY charge in the low energy theory do not lead to chiral fermions and so are excluded for phenomenological reasons. We will henceforth assume that there is just a single super-charge.

https://doi.org/10.1017/9781009289801.004 Published online by Cambridge University Press

3.2 Quantization of the WZ model

The main purpose of this section is to review the implementation of the WZ model as a quantum field theory. This provides us with an opportunity to set up our conventions for the field expansions as well as for the (anti)commutators of the creation and annihilation operators. In the process we will also see how the quantization of the Majorana field differs from the more familiar quantization of the Dirac field. We always use the four-component spinor notation that many particle physicists are most familiar with.

We adopt the canonical quantization procedure, wherein the fields are regarded as quantum operators acting upon a Fock space of particle states. For the scalar fields A and B, the conjugate field momenta are $\Pi_A = \partial\mathcal{L}/\partial(\frac{\partial A}{\partial t}) = \partial A/\partial t \equiv \dot{A}$ and $\Pi_B = \dot{B}$. The equal time commutators for the A and B fields are stipulated to be

$$[A(\mathbf{x}), \dot{A}(\mathbf{y})] = \mathrm{i}\delta^3(\mathbf{x}-\mathbf{y}), \quad [A(\mathbf{x}), A(\mathbf{y})] = [\dot{A}(\mathbf{x}), \dot{A}(\mathbf{y})] = 0, \tag{3.27a}$$

$$[B(\mathbf{x}), \dot{B}(\mathbf{y})] = \mathrm{i}\delta^3(\mathbf{x}-\mathbf{y}), \quad [B(\mathbf{x}), B(\mathbf{y})] = [\dot{B}(\mathbf{x}), \dot{B}(\mathbf{y})] = 0. \tag{3.27b}$$

The Hermitian field operators A and B can be Fourier expanded such that

$$A(x) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{2E_{\mathbf{k}}}\left(a_{\mathbf{k}}\mathrm{e}^{-\mathrm{i}kx} + a^\dagger_{\mathbf{k}}\mathrm{e}^{\mathrm{i}kx}\right), \tag{3.28a}$$

$$B(x) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{2E_{\mathbf{k}}}\left(b_{\mathbf{k}}\mathrm{e}^{-\mathrm{i}kx} + b^\dagger_{\mathbf{k}}\mathrm{e}^{\mathrm{i}kx}\right), \tag{3.28b}$$

where the a ($a^\dagger$) and b ($b^\dagger$) operators are annihilation (creation) operators satisfying

$$[a_{\mathbf{k}}, a^\dagger_{\mathbf{l}}] = (2\pi)^3 2E_{\mathbf{k}}\delta^3(\mathbf{k}-\mathbf{l}), \qquad [a_{\mathbf{k}}, a_{\mathbf{l}}] = [a^\dagger_{\mathbf{k}}, a^\dagger_{\mathbf{l}}] = 0, \tag{3.29a}$$

$$[b_{\mathbf{k}}, b^\dagger_{\mathbf{l}}] = (2\pi)^3 2E_{\mathbf{k}}\delta^3(\mathbf{k}-\mathbf{l}), \qquad [b_{\mathbf{k}}, b_{\mathbf{l}}] = [b^\dagger_{\mathbf{k}}, b^\dagger_{\mathbf{l}}] = 0. \tag{3.29b}$$

The usual four-component Dirac spinor field ψ_D is quantized by stipulating the equal-time anticommutators,

$$\begin{aligned}\{\psi_{Da}(\mathbf{x}), \psi^\dagger_{Db}(\mathbf{y})\} &= \delta_{ab}\delta^3(\mathbf{x}-\mathbf{y}),\\ \{\psi_{Da}(\mathbf{x}), \psi_{Db}(\mathbf{y})\} &= \{\psi^\dagger_{Da}(\mathbf{x}), \psi^\dagger_{Db}(\mathbf{y})\} = 0.\end{aligned} \tag{3.30}$$

The field is expanded using distinct creation and annihilation operators for particles and antiparticles as,

$$\psi_D(x) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{2E_{\mathbf{k}}}\sum_s [c_{\mathbf{k},s}u_{\mathbf{k},s}\mathrm{e}^{-\mathrm{i}kx} + d^\dagger_{\mathbf{k},s}v_{\mathbf{k},s}\mathrm{e}^{\mathrm{i}kx}]. \tag{3.31}$$

These creation and annihilation operators satisfy the well-known *anticommutation* relations which we will not write out here. Note that for a Dirac spinor,

$$\langle 0|T\psi_{Da}(x)\bar{\psi}_{Db}(y)|0\rangle = S_{Fab}(x-y), \quad \text{and} \tag{3.32a}$$

$$\langle 0|T\psi_{Da}(x)\psi_{Db}(y)|0\rangle = \langle 0|T\bar{\psi}_{Da}(x)\bar{\psi}_{Db}(y)|0\rangle = 0. \tag{3.32b}$$

A similar procedure for quantizing a four-component Majorana field ψ *cannot* be followed because the Majorana spinor is constrained by the Majorana condition $\psi = \psi^c = C\bar{\psi}^T$, i.e. only two of the four components of the Majorana spinor are independent. To proceed further, we evaluate the field expansion for the conjugate Dirac field ψ_D^c. Using the spinor relations $u^c \equiv C\bar{u}^T = v$ and $v^c \equiv C\bar{v}^T = u$, we find

$$\psi_D^c(x) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{2E_{\mathbf{k}}}\sum_s [c^\dagger_{\mathbf{k},s}v_{\mathbf{k},s}e^{ikx} + d_{\mathbf{k},s}u_{\mathbf{k},s}e^{-ikx}]. \tag{3.33}$$

Next, impose the constraint $\psi = \psi^c$. The constraint is respected if we require $c = d$ and $c^\dagger = d^\dagger$, so that the Majorana spinor field expansion is just

$$\psi(x) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{2E_{\mathbf{k}}}\sum_s [c_{\mathbf{k},s}u_{\mathbf{k},s}e^{-ikx} + c^\dagger_{\mathbf{k},s}v_{\mathbf{k},s}e^{ikx}], \tag{3.34}$$

with

$$\{c_{\mathbf{k},r}, c^\dagger_{\mathbf{l},s}\} = (2\pi)^3 2E_k\delta_{rs}\delta^3(\mathbf{k}-\mathbf{l}), \quad \{c_{\mathbf{k},r}, c_{\mathbf{l},s}\} = \{c^\dagger_{\mathbf{k},r}, c^\dagger_{\mathbf{l},s}\} = 0. \tag{3.35}$$

The condition $\psi = \psi^c$ is the analogue of the reality condition for the scalar fields A and B; the condition $c_{\mathbf{k}} = d_{\mathbf{k}}$ implies the identity of the particle and antiparticle quanta of this field. For a Majorana spinor field, we still have

$$\langle 0|T\psi_a(x)\bar{\psi}_b(y)|0\rangle = S_{Fab}(x-y), \tag{3.36}$$

but now, because $\psi = C\bar{\psi}^T$ and $\bar{\psi} = \psi^T C$, $\langle 0|T\psi_a(x)\psi_b(y)|0\rangle$ and $\langle 0|T\bar{\psi}_a(x)\bar{\psi}_b(y)|0\rangle$ do not vanish as in the case of a Dirac field. It is easy to show that

$$\langle 0|T\psi_a(x)\psi_b(y)|0\rangle = S_{Fac}(x-y)C^T_{cb} \quad \text{and} \tag{3.37a}$$

$$\langle 0|T\bar{\psi}_a(x)\bar{\psi}_b(y)|0\rangle = C^T_{ac}S_{Fcb}(x-y). \tag{3.37b}$$

We must not forget to include these contractions when computing matrix elements of operators involving products of Majorana spinor fields.

The four-momentum operator P^μ for the WZ model can now be explicitly constructed from the energy–momentum tensor $T^{\mu\nu}$. Recall

$$T^{\mu\nu} = \sum_{\text{fields } \phi_i}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\partial^\nu\phi_i - g^{\mu\nu}\mathcal{L}, \tag{3.38}$$

https://doi.org/10.1017/9781009289801.004 Published online by Cambridge University Press

where

$$P^{\mu} = \int T^{0\mu} d^3x. \tag{3.39}$$

Substituting the field expansions (3.28a), (3.28b), and (3.34) into (3.39) and performing a rather lengthy calculation leads to

$$\begin{aligned} P^{\mu} &= \int \frac{d^3k}{(2\pi)^3 2E_{\mathbf{k}}} k^{\mu} [a^{\dagger}_{\mathbf{k}} a_{\mathbf{k}} + \frac{1}{2}\delta^3(\mathbf{0}) + b^{\dagger}_{\mathbf{k}} b_{\mathbf{k}} + \frac{1}{2}\delta^3(\mathbf{0}) \qquad (3.40) \\ &\quad + \sum_{s} (c^{\dagger}_{\mathbf{k},s} c_{\mathbf{k},s} - \frac{1}{2}\delta^3(\mathbf{0}))] \\ &= \int \frac{d^3k}{(2\pi)^3 2E_{\mathbf{k}}} k^{\mu} [a^{\dagger}_{\mathbf{k}} a_{\mathbf{k}} + b^{\dagger}_{\mathbf{k}} b_{\mathbf{k}} + \sum_{s} c^{\dagger}_{\mathbf{k},s} c_{\mathbf{k},s}]. \qquad (3.40) \end{aligned}$$

Thus, in the WZ model, we see that for the field four-momentum operator, the zero-point energy terms exactly cancel due to equal and opposite bosonic and fermionic contributions. This is the first of several examples of the cancellation of infinities in supersymmetric models. Expressions for the rotation and boost generators of the Poincaré group can be similarly constructed, but we will not do so here.

It is, however, instructive to explicitly construct the super-charge from the supercurrent (3.13a) for the WZ model. We find,

$$\begin{aligned} Q &= \int j^0 d^3x \qquad (3.41) \\ &= \sum_{s} \int \frac{d^3k}{(2\pi)^3 2E_{\mathbf{k}}} \left\{ (a_{\mathbf{k}}\gamma_5 + \mathrm{i} b_{\mathbf{k}}) c^{\dagger}_{\mathbf{k},s} v_{\mathbf{k},s} - (a^{\dagger}_{\mathbf{k}}\gamma_5 + \mathrm{i} b^{\dagger}_{\mathbf{k}}) c_{\mathbf{k},s} u_{\mathbf{k},s} \right\}. \end{aligned}$$

It should be apparent from this expression that the action of Q on a bosonic (fermionic) state results in an admixture with a fermionic (bosonic) state.

Exercise *Verify Eq. (3.41).*

It is now possible to explicitly show that the generators P^{μ} and Q obtained above commute with each other as indeed they should.

We can now use the expression (3.41) for the super-charge to work out the effect on the dynamically independent field operators of the WZ model.

Exercise *Using the expression (3.41) for the super-charge in the WZ model, verify that for an infinitesimal SUSY transformation,*

$$\delta A = [\mathrm{i}\bar{\alpha} Q, A] = \mathrm{i}\bar{\alpha}\gamma_5\psi,$$
$$\delta B = [\mathrm{i}\bar{\alpha} Q, B] = -\bar{\alpha}\psi,$$
$$\delta\psi = [\mathrm{i}\bar{\alpha} Q, \psi] = \partial\!\!\!/\gamma_5 A\alpha + \mathrm{i}\partial\!\!\!/ B\alpha - \mathrm{i}mA\gamma_5\alpha + mB\alpha.$$

The first two of these expressions are just the transformations of the A and B fields in (3.7a) and (3.7b), whereas the last of these corresponds to the transformation (3.7c) for $\delta\psi$ where the auxiliary fields are eliminated via their Euler–Lagrange equations. The fact that F and G do not appear on the right-hand side could have been anticipated since these do not appear in the form of the supercurrent.

3.3 Interactions in the WZ model

Up to this point we have been discussing free field theory which, despite being supersymmetric, would not be of interest if interactions could not be incorporated. Following Wess and Zumino, we add interaction terms given by

$$\mathcal{L}_{\text{int}} = -\frac{g}{\sqrt{2}}A\bar{\psi}\psi + \frac{\mathrm{i}g}{\sqrt{2}}B\bar{\psi}\gamma_5\psi + \frac{g}{\sqrt{2}}(A^2 - B^2)G + g\sqrt{2}ABF, \tag{3.43}$$

to the Lagrangian (3.1a). It can be verified by brute force that $\mathcal{L}_{\text{int}}$ is separately supersymmetric. The calculation is rather messy. We will demonstrate the supersymmetry of this Lagrangian more elegantly in Chapter 5 using the superfield formalism.

Once again we can eliminate the auxiliary fields F and G via their Euler–Lagrange equations which get modified to,

$$F = -mB - g\sqrt{2}AB \tag{3.44a}$$
$$G = -mA - \frac{g}{\sqrt{2}}(A^2 - B^2), \tag{3.44b}$$

and obtain the total Lagrangian in terms of the dynamical fields as,

$$\begin{aligned}\mathcal{L} = {} & \frac{1}{2}(\partial_\mu A)^2 + \frac{1}{2}(\partial_\mu B)^2 + \frac{\mathrm{i}}{2}\bar{\psi}\partial\!\!\!/\psi - \frac{1}{2}m^2(A^2 + B^2) - \frac{1}{2}m\bar{\psi}\psi \\ & - \frac{g}{\sqrt{2}}A\bar{\psi}\psi + \frac{\mathrm{i}g}{\sqrt{2}}B\bar{\psi}\gamma_5\psi - gm\sqrt{2}AB^2 - \frac{gm}{\sqrt{2}}A(A^2 - B^2) \\ & - g^2A^2B^2 - \frac{1}{4}g^2(A^2 - B^2)^2. \end{aligned} \tag{3.45}$$

Several features of the Lagrangian in (3.45) are worth stressing.

https://doi.org/10.1017/9781009289801.004 Published online by Cambridge University Press

1. It describes the interaction of two real spin zero fields and a Majorana field with spin half. As before, the number of bosonic and fermionic degrees of freedom match.
2. There is a single mass parameter m common to all the fields.
3. Although the interaction structure of the model is very rich and includes parity-conserving scalar and pseudoscalar interactions of the scalar A and pseudoscalar B with the fermion, as well as all possible (renormalizable) parity conserving trilinear and quartic scalar interactions, there is just *one* single coupling constant g. We thus see that supersymmetry is like other familiar symmetries in that it relates the various interactions as well as masses. The mass and coupling constant relationships inherent in (3.45) are completely analogous to the familiar (approximate) equality of neutron and proton masses or the relationships between their interactions with the various pions implied by (approximate) isospin invariance.

Before closing we remark that Iliopoulos and Zumino observed that unlike (3.13a), the expression (3.13b) for the supercurrent holds also in the presence of interactions, provided of course that the auxiliary fields satisfy (3.44a) and (3.44b).[4] We will use this observation in Chapter 7 when we discuss the interactions of the massless Goldstone fermion that appears as a result of spontaneous supersymmetry breaking.

3.4 Cancellation of quadratic divergences

We have already mentioned that the existence of supersymmetric partners serves to remove the quadratic divergences that destabilize the scalar sector of a generic field theory. We will illustrate this cancellation of quadratic divergences for the simple case of the WZ model. Consider the corrections to the "one-point function"

$$\langle\Omega|A(x)|\Omega\rangle = \begin{array}{l}\text{sum of all connected diagrams}\\ \text{with one external point}\end{array}$$

of the field A to first order in the coupling constant g in (3.45). Here $|\Omega\rangle$ is the ground state of the interacting theory. The relevant interaction Hamiltonian from (3.45) is

$$\mathcal{H}_{\text{int}} = -\mathcal{L}_{\text{int}} \ni \frac{g}{\sqrt{2}}A\bar{\psi}\psi + \frac{g}{\sqrt{2}}mAB^2 + \frac{g}{\sqrt{2}}mA^3. \tag{3.46}$$

The loop corrections to the one-point function are represented by the tadpole diagrams shown in Fig. 3.1. Expanding the matrix element $\langle\Omega|TA(x)|\Omega\rangle$

[4] J. Iliopoulos and B. Zumino, *Nucl. Phys.* **B76**, 310 (1974).

https://doi.org/10.1017/9781009289801.004 Published online by Cambridge University Press

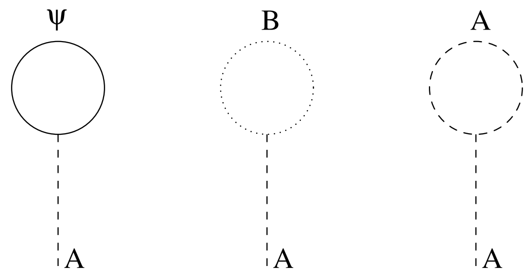

Figure 3.1 Lowest order diagrams contributing to quadratic divergences in the one-point function of A.

perturbatively to order g gives,[5]

$$-\mathrm{i}\frac{g}{\sqrt{2}}\int \mathrm{d}^4 y D_F^A(x-y)\left[(-1)\mathrm{Tr}S_F(y-y)+mD_F^B(y-y)+3mD_F^A(y-y)\right],$$

where the factor 3 in the last term arises from three possible contractions involving the A^3 interaction term. The factor in the square brackets is proportional to

$$\mathrm{Tr}\int\frac{\mathrm{d}^4 p}{\not p - m_\psi} - m\int\frac{\mathrm{d}^4 p}{p^2-m_B^2} - 3m\int\frac{\mathrm{d}^4 p}{p^2-m_A^2}$$

$$=\int\frac{\mathrm{d}^4 p}{p^2-m_\psi^2}4m_\psi - m\int\frac{\mathrm{d}^4 p}{p^2-m_B^2} - 3m\int\frac{\mathrm{d}^4 p}{p^2-m_A^2}. \tag{3.47}$$

Here, we have deliberately denoted the masses that enter via the propagators by m_A, m_B, and m_ψ, although these are exactly the same as the mass parameter m that enters via the trilinear scalar couplings in (3.45). We first see that because all these masses are exactly equal in a supersymmetric theory, the three contributions in (3.47) add to zero. Thus although each diagram is separately quadratically divergent, the divergence from the fermion loop exactly cancels the sum of divergences from the boson loops. Two remarks are in order.

1. In order for this cancellation to occur, it is crucial that the A^3, AB^2, and $A\bar{\psi}\psi$ couplings be exactly as given in (3.45).
2. The *quadratic* divergence in the expression (3.47) is independent of the scalar masses, m_A and m_B. It is, however, crucial that the fermion mass m_ψ is exactly equal to the mass m that enters via the trilinear scalar interactions in order for the cancellation of the quadratic divergence to be maintained. If the boson masses differ from the fermion mass m_ψ, the expression in (3.47) is at most

[5] For a review, see *Introduction to Quantum Field Theory*, M. Peskin and D. Schroeder, Perseus Press (1995), Chapter 4, where $D_F(x-y)$ is defined.

https://doi.org/10.1017/9781009289801.004 Published online by Cambridge University Press

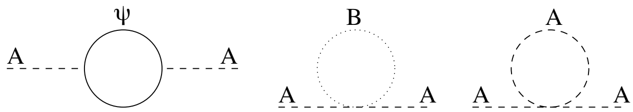

Figure 3.2 Lowest order diagrams contributing to quadratic divergences in the two-point function of A.

logarithmically divergent. As we have discussed, logarithmic divergences do not severely destabilize scalar masses.

It is also instructive to inspect the lowest order quadratic divergences in the two-point function of A defined as $\langle\Omega|TA(x)A(y)|\Omega\rangle$. The one-loop contributions to the quadratic divergences are shown in Fig. 3.2. [6] The first diagram of Fig. 3.2 gives a contribution

$$-\frac{g^2}{2}\int \mathrm{d}^4z\mathrm{d}^4z' D_F^A(x-z)D_F^A(z'-y)$$
$$\times\left[(-1)\mathrm{Tr}S_F(z-z')S_F(z'-z)+\mathrm{Tr}C^TC^TS_F(z-z')S_F(z-z')\right],$$

where the second term in the square parenthesis arises because contractions of the Majorana ψ (and $\bar{\psi}$) field with itself do not vanish as noted in (3.37a) and (3.37b). The integration over the intermediate points z and z' can be performed by writing the Fourier expansions of each of the propagators. One then finds that the correction from the fermion loop in Fig. 3.2 is given by,

$$g^2\int\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{1}{p^2-m_A^2}\mathrm{e}^{-\mathrm{i}p(x-y)}\int\frac{\mathrm{d}^4q}{(2\pi)^4}\mathrm{Tr}\left[\frac{1}{(\not{q}-m_\psi)}\frac{1}{(-\not{p}+\not{q}-m_\psi)}\right]\frac{1}{p^2-m_A^2}.$$

From the second diagram where the fields $A(x)$ and $A(y)$ can be contracted in two ways we get,

$$-\mathrm{i}\frac{g^2}{2}2\int \mathrm{d}^4zD_F^A(x-z)D_F^A(z-y)D_F^B(z-z),$$

while the third diagram for which we have twelve possible contractions yields,

$$-\mathrm{i}\frac{g^2}{4}12\int \mathrm{d}^4zD_F^A(x-z)D_F^A(z-y)D_F^A(z-z).$$

Once again, we can do the integration over z using the momentum expansion of the propagators. By combining the contributions from the diagrams in Fig. 3.2, we see that including the lowest order correction to the two-point function of A changes

[6] There are additional quadratic divergences in the two-point function from the tadpoles of Fig. 3.1 which, as we have just seen, separately cancel.

https://doi.org/10.1017/9781009289801.004 Published online by Cambridge University Press

the momentum space propagator as

$$\frac{\mathrm{i}}{p^2 - m_A^2} \to \frac{\mathrm{i}}{p^2 - m_A^2} + \frac{\mathrm{i}}{p^2 - m_A^2}(-\mathrm{i}\Pi(p))\frac{\mathrm{i}}{p^2 - m_A^2},$$

with the divergences all being contained in the function $\Pi(p)$ given by

$$\begin{aligned} \mathrm{i}\Pi(p) = g^2 \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \Bigg[\mathrm{Tr}\left(\frac{(\not{q} + m_\psi)}{q^2 - m_\psi^2} \cdot \frac{(-\not{p} + \not{q} + m_\psi)}{(q-p)^2 - m_\psi^2} \right) \\ - \frac{1}{q^2 - m_B^2} - 3\frac{1}{q^2 - m_A^2} \Bigg]. \end{aligned} \tag{3.48}$$

It is now straightforward to see that once again the quadratic divergences cancel between fermionic and bosonic loops. Moreover, this cancellation occurs for *all* values of particle masses. This is because trilinear scalar interactions do not contribute to the quadratic divergence that we have just computed. It is, however, crucial that the fermion Yukawa coupling ($g/\sqrt{2}$) is related to the quartic scalar couplings on the last line of (3.45).

Exercise *Verify Eq. (3.48) and check that the quadratic divergence cancels.*

Exercise *Verify that the quadratic divergence cancels in the one-loop tadpole and mass corrections to the B field.*

3.5 Soft supersymmetry breaking

The fact that the quadratic divergences continue to cancel even if the scalar boson masses are not exactly equal to fermion masses (as implied by SUSY) is absolutely critical for the construction of phenomenologically viable models. We know from observation that SUSY cannot be an exact symmetry of nature. Otherwise, there would have to exist a spin zero or spin one particle *with exactly the mass and charge of an electron.* Such a particle could not have evaded experimental detection. The only way out of this conundrum is to admit that supersymmetric partners cannot be degenerate with the usual particles. Thus, supersymmetry must be a broken symmetry.

Would the breaking of supersymmetry destroy the delicate cancellation of quadratic divergences in field theoretic models? Fortunately, it does not. We have just seen (by the two examples above) that if SUSY is explicitly broken because scalar masses differ from their fermion counterparts, no new quadratic divergences occur. We state here (without proof) that this is true for all processes, and to all orders in perturbation theory. It is, therefore, possible to introduce new terms such as independent additional masses for the scalars which break SUSY without the reappearance of quadratic divergences. Such terms are said to break SUSY *softly*. Not all SUSY breaking terms are soft. We have already seen that if $m_\psi \neq m$, the expression in (3.47) is quadratically divergent. Thus additional contributions to the fermion mass in the Wess–Zumino model results in a *hard* breaking of supersymmetry. Similarly, any additional contribution to just the quartic scalar interactions will result in the reappearance of a quadratic divergence in the correction to m_A^2 since these contributions only affect the last two diagrams in Fig. 3.2.

Are there other soft SUSY breaking terms possible for the WZ model? Recall the combinatorial factor 3 in the last term in (3.47). This tells us that the contribution of the A loop from the trilinear A^3 interaction is exactly three times bigger than the contribution from the B loop from the AB^2 interaction (the coupling constants for these interactions are exactly equal). Thus, there will be no net quadratic divergence in the expression (3.47) even if we add a term of the form,

$$\mathcal{L}_{\text{soft}} = k(A^3 - 3AB^2) \tag{3.49}$$

to our model, where k is a dimensional coupling constant. Obviously, this interaction does not give a quadratically divergent correction to the one-loop, contribution to m_A^2. It is an example of a soft supersymmetry breaking interaction term. We remark that this term can be written in terms of $\mathcal{S} = \frac{A+iB}{\sqrt{2}}$ as

$$\mathcal{L}_{\text{soft}} = \sqrt{2}k(\mathcal{S}^3 + \quad \text{h.c.}) \tag{3.50a}$$

while an arbitrary splitting in the masses of A and B can be incorporated by including a term,

$$\mathcal{L}_{\text{soft}} = m'^2(\mathcal{S}^2 + \quad \text{h.c.}) \tag{3.50b}$$

into the Lagrangian. It will turn out that super-renormalizable terms that are analytic in $\mathcal{S}$ are soft, while terms that involve products of $\mathcal{S}$ and $\mathcal{S}^*$ (except supersymmetric terms such as $\mathcal{S}^*\mathcal{S}$ already present in (3.45)) result in a hard breaking of SUSY.

Exercise *Check that an interaction proportional to* $(\mathcal{S}^2\mathcal{S}^* + \text{h.c.}) \sim (A^2 + B^2)A$ *leads to a quadratically divergent contribution to the expression in (3.47).*

Although we have illustrated the cancellation of quadratic divergences with just a few examples, it is important to stress that this is a general feature of supersymmetric theories. As we will elaborate upon in Section 6.7, this cancellation of quadratic divergences occurs to all orders in perturbation theory.