PERFECT SEMIGROUPS

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Introduction

A semigroup S with identity is (left) *perfect* if every unitary left S-system has a projective cover. This is the semigroup analogue of the definition of left perfect rings introduced in (1). The investigation of perfect semigroups was initiated by Isbell (4), who proved that a semigroup is perfect if and only if it satisfies two conditions referred to as conditions A and D.

Left perfect rings have several equivalent characterisations, one of which is that the rings satisfy the descending chain condition for principal right ideals. This is not enough for perfection of a semigroup as is shown by an example in (4). However, Isbell conjectures that in the presence of condition A, a semigroup satisfies condition D if and only if it satisfies M_R, the minimum condition for principal right ideals. One of the objects of this note is to show that this conjecture is correct. In fact, more is true. Björk (2) has shown that if a module satisfies the descending chain condition for cyclic submodules, then it also satisfies the descending chain condition for finitely generated submodules. The corresponding semigroup result is true with a similar but easier proof. Thus perfect semigroups satisfy the minimum condition for finitely generated right ideals.

Another characterisation of left perfect rings is that every direct limit of projective modules is projective. Isbell has shown that for semigroups the corresponding condition is sufficient but has left open the question of necessity. Following Strenström (7) we shall call an S-system which is a direct limit of projective S-systems, a weakly flat S-system. Such S-systems have also been considered by Kil'p in (5), where they are called strongly flat. Strenstöm gives several characterisations of weakly flat S-systems, one of which provides us with the main tool for our investigations. Using this, we show that for a perfect semigroup S, every weakly flat S-system is projective, thus answering the question of Isbell.

1. Preliminaries and statement of theorem

All semigroups considered have an identity. For a semigroup S, a *left S-system* is a set A together with a function $S \times A \rightarrow A$: $(s, x) \rightarrow sx$ satisfying $s(tx) = (st)x$ and $1x = x (x \in A, s, t \in S)$. Right S-systems are defined in a similar way. We shall refer to left S-systems simply as S-systems. A *subsystem* of an S-system A is a subset B of A which satisfies $SB \subseteq B$. Let A and B be

two S-systems. A function $f: A \to B$ is called an S-homomorphism if $f(sx) = sf(x)$ for all $s \in S$, $x \in A$.

A *free* S-system is simply a disjoint union of copies of S and a *projective* S-system is a retract of a free S-system. A *projective cover* of an S-system A is a surjective S-homomorphism $f: P \to A$, where P is a projective S-system and the restriction of f to any proper subsystem of P is not surjective.

A subset X of an S-system A is a set of *generators* for A if $A = SX$ and is a *free set of generators* if also $Sx \cap Sx' = \varnothing$ for x, $x' \in X$, $x \neq x'$ and $Sx \approx S$ for $x \in X$.

An S-system A is *decomposable* if there are non-empty subsystems B, C of A such that $A = B \cup C$ and $B \cap C = \varnothing$. Otherwise, A is called indecomposable. Clearly a cyclic S-system is indecomposable. From (6), we have the following two results.

Result 1. *Every S-system is uniquely decomposable into a disjoint union of indecomposable subsystems.*

Result 2. *A disjoint union of S-systems P_i is projective if and only if each P_i is projective. An S-system P is an indecomposable projective if and only if $P \approx Se$ for some idempotent e in S.*

The *tensor product* $A \otimes B$ of a right S-system A and an S-system B is the set $(A \times B)/\sim$, where $\sim$ is the equivalence relation generated by $(as, b) \sim (a, sb)$ for $a \in A$, $b \in B$, $s \in S$. The S-system B is *weakly flat* if the functor $\otimes B$ preserves equalisers and pullbacks. As noted in (5) and (7), projective S-systems are weakly flat. From (7) we also have the following two results.

Result 3. *A direct limit of weakly flat S-systems is weakly flat.*

Result 4. *The following properties of an S-system A are equivalent:*

(i) *A is weakly flat.*

(ii) *If $sx = ty$ where x, $y \in A$ and s, $t \in S$, then there is an element z in A and elements s', t' in S such that $x = s'z$, $y = t'z$ and $ss' = tt'$; moreover, if $x = y$, we may take $s' = t'$.*

(iii) *A is a direct limit of finitely generated free systems.*

An easy consequence of Results 1 and 4 is that every weakly flat S-system is a disjoint union of indecomposable weakly flat S-systems.

From (3, chapter 11) we have that a cyclic S-system is isomorphic to S/ρ where ρ is a left congruence on S. We call ρ weakly flat if S/ρ is a weakly flat S-system. As a corollary of Result 4, we have

Corollary. *For a left congruence ρ on S, the following are equivalent:*

(i) *ρ is weakly flat.*

(ii) *If $s \rho t$ where s, $t \in S$, then there are elements u, $v \in S$ such that $su = tu$ and $1 \rho uv$.*

Proof. The implication (i)$\Rightarrow$(ii) is immediate from (ii) of Result 4.

For (ii)$\Rightarrow$(i), let $a\rho$, $b\rho \in S/\rho$ and suppose that $s(a\rho) = t(b\rho)$. Then $sa\rho tb$ so that $sau = tbu$ and $1\rho uv$ for u, $v \in S$, that is, $ss' = tt'$ and

$$a\rho = s'(v\rho), \quad b\rho = t'(v\rho),$$

where $s' = au$, $t' = bu$. Thus (ii) of Result 4 is satisfied.

From **(4)** we have

Result 5. *For a subset B of S, we have $B = 1\rho$ for some left congruence ρ on S if and only if B is a left unitary subsemigroup of S.*

Condition A in **(4)** states that every locally cyclic S-system is cyclic or equivalently every S-system satisfies the ascending chain condition for cyclic subsystems.

Condition D in **(4)** states that every left unitary subsemigroup of S has a minimal left ideal generated by an idempotent.

The main theorem can now be stated as follows.

Theorem. *For a semigroup S, the following are equivalent:*

(i) *S is perfect.*

(ii) *S satisfies conditions A and D.*

(iii) *S satisfies conditions A and M_R.*

(iv) *S satisfies condition A and the descending chain condition for finitely generated right ideals.*

(v) *Every weakly flat S-system is projective.*

2. Proof of the main theorem

The equivalence (i)$\Leftrightarrow$(ii) is due to Isbell and in view of Result 4 so is the implication (v)$\Rightarrow$(ii). The implication (iv)$\Rightarrow$(iii) is obvious. We shall prove the implications (v)$\Rightarrow$(iii), (ii)$\Rightarrow$(v), (iii)$\Rightarrow$(v) and (iii)$\Rightarrow$(iv).

Lemma 1. *If every weakly flat S-system is projective, then S satisfies M_R.*

Proof. If $a_1 S \supseteq b_2 S \supseteq b_3 S \supseteq \ldots$ is a decreasing sequence of principal right ideals of S, then $b_2 = a_1 a_2$, $b_3 = a_1 a_2 a_3$, $\ldots$ for some elements a_2, a_3, $\ldots$ in S. Let F be the free S-system freely generated by x_1, x_2, $\ldots$, x_n, $\ldots$ and let ρ be the congruence on F generated by the relation

$$\sigma = \{(x_i, a_i x_{i+1}) : i = 1, 2, \ldots\}.$$

Then ρ is the transitive closure of the relation θ where

$$\theta = \{(sx_i, sa_i x_{i+1}), (sa_i x_{i+1}, sx_i), (sx_i, sx_i) : s \in S, \quad i = 1, 2, \ldots\}.$$

We show first that for elements s, t of S, $sx_n \rho t x_n$ if and only if there is a non-negative integer m such that $sa_n \ldots a_{n+m} = ta_n \ldots a_{n+m}$. If we have

$$sa_n \ldots a_{n+m} = ta_n \ldots a_{n+m},$$

then clearly $sx_n\rho tx_n$. Suppose that $sx_n\rho tx_n$. Then $s'x_h\rho t'x_k$ where if $s \notin Sa_{n-1}$, then $h = n$, $s' = s$ and if $s \in Sa_{n-1}$, then h is the smallest positive integer such that $s \in Sa_h...a_{n-1}$ and $s' \in S$ is such that $s = s'a_h...a_{n-1}$ and k, t' are obtained similarly from t. Thus there are elements y_1, ..., y_p of F such that

$$s'x_h = y_1, \quad t'x_k = y_p \text{ and } y_r\theta y_{r+1} \ (r = 1, ..., p-1).$$

From the definitions of θ, s', t' we have that for some r,

$$y_r = s'a_h...a_i x_{i+1}, \quad y_{r+1} = t'a_k...a_j x_{j+1}$$

and one of $i = j+1$, $i = j$ or $i+1 = j$ holds. Whichever one holds we certainly obtain $s'a_h...a_{i+1} = t'a_k...a_{i+1}$ and the desired conclusion follows.

Now for each positive integer n let ρ_n be the congruence on F generated by the relation

$$\sigma_n = \{(x_1, a_1x_2), ..., (x_n, a_nx_{n+1})\}.$$

If $i, j > n$ and $sx_i\rho_n tx_j$, then $i = j$ and $s = t$. If $i \leq n$, then $x_i\rho_n a_i...a_n x_{n+1}$. Since $\{x_i\rho_n: i = 1, 2, ...\}$ is clearly a set of generators for F/ρ_n, it follows that $\{x_i\rho_n: i = n+1, n+2, ...\}$ is a free set of generators for F/ρ_n. Clearly $\rho_i \subseteq \rho_{i+1} \subseteq \rho$ for all $i = 1, 2, ...$ Thus for $m < n$, $\alpha_{m,n}: F/\rho_m \to F/\rho_n$ which takes $x\rho_m$ to $x\rho_n$ is a surjective S-homomorphism, as is $\alpha_m: F/\rho_m \approx F/\rho$ which takes $x\rho_m$ to $x\rho$, and $\alpha_n\alpha_{m,n} = \alpha_m$. Furthermore, if x, $y \in F$, say $x = sx_k$, $y = tx_j$, $j \leq k$ and $x\rho_m$, $y\rho_n$ have the same image in F/ρ, then $x\rho y$ so that $sx_k\rho ta_j...a_{k-1}x_k$ and so $sa_k...a_{k+q} = ta_j...a_{k+q}$ for some q giving $x\rho_{k+q}y$. Hence for some $r \geq m, n$ we have $x\rho_r = y\rho_r$ and it follows that F/ρ is the direct limit of the free S-systems F/ρ_n. By Result 3, F/ρ is weakly flat and so by assumption is projective. There is, therefore, an S-homomorphism $\mu: F/\rho \approx F$ with $\nu\mu$ the identity map on F/ρ, where ν is the natural S-homomorphism $F \to F/\rho$. Let $\mu(x_1\rho) = cx_k(c \in S)$ and for $i > 1$ let $\mu(x_i\rho) = c_ix_k|_i|$. Then

$$cx_k = \mu(x_1\rho) = \mu(a_1...a_{i-1}(x_i\rho)) = a_1...a_{i-1}c_ix_{k(i)}.$$

Hence $k = k(i)$ and $c = a_1...a_{i-1}c_i \in a_1...a_{i-1} S$ so that $cS \subseteq a_1...a_nS$ for all n. But $x_1\rho = \nu\mu(x_1\rho) = (cx_k)\rho$ and $x_1\rho a_1...a_{k-1}x_k$ so that by the first part of the proof $a_1...a_{k+m} = ca_k...a_{k+m}$ for some m. Thus $a_1...a_{k+m}S \subseteq cS$ and our descending chain terminates.

The implication (v)$\Rightarrow$(iii) now follows since by (4), condition A follows from (v).

We next turn our attention to the implications (ii)$\Rightarrow$(v) and (iii)$\Rightarrow$(v). Our first objective is to show that in the presence of condition A, every indecomposable weakly flat S-system is cyclic. Call a generating set X for an S-system A independent if for x, $x' \in X$ and $x \in Sx'$ we have $x = x'$. For an arbitrary S, not every S-system has an independent generating set; for example, a non-cyclic, locally cyclic S-system does not. However, we have

Lemma 2. *Let A be an S-system which satisfies the ascending chain condition for cyclic subsystems. If X is a set of generators for A, then X contains an independent set of generators for A.*

Proof. Call an element x in X basic if whenever $Sx \subseteq Sx'$ with $x' \in X$ we have $Sx = Sx'$. Denote by X' the set of basic elements in X. If $x \in X$, then there is a maximal member Sx' among the cyclic subsystems of A which contain Sx. Now $x' \in X'$ by the maximality of Sx'. Also $x = sx'$ for some $s \in S$ and consequently X' generates A.

If x, $x' \in X'$ and $x = sx'$, then $Sx \subseteq Sx'$ so that $Sx = Sx'$ and $x' = tx$ for some $t \in S$. It follows that the relation $\sim$ on X' defined by $x \sim x'$ if and only if $x = sx'$ for some $s \in S$ is an equivalence relation. Now let X'' be a set of representatives of the $\sim$-classes.

If $x \in X'$, then $x \sim x'$ for some $x' \in X''$ so that $x = sx'$ ($s \in S$) and hence X'' generates A. Finally, X'' is independent.

Lemma 3. *Let A be a weakly flat S-system which satisfies the ascending chain condition for cyclic subsystems. If A is indecomposable, then A is cyclic.*

Proof. By Lemma 2, A has an independent set of generators X. If $x \in X$, let $a \in A$ be such that Sa is maximal among those cyclic subsystems of A which contain Sx. Put $X' = \{a\} \cup Y$, where $Y = X \setminus \{y \in X : y \in Sa\}$. Clearly X' generates A and is independent. If X' contains elements other than a, then since A is indecomposable we have $Sa \cap SX'' \neq \varnothing$ where $X'' = X' \setminus \{a\}$. Hence $sa = ty$ for some $y \in X''$, s, $t \in S$. But A is weakly flat and so by Result 4, there is an element $z \in A$ and elements s', $t' \in S$ such that $a = s'z$, $y = t'z$ and $ss' = tt'$. Thus $Sa \subseteq Sz$ and $Sa = Sz$ by the maximality of Sa, so that $z = s''a$ for some $s'' \in S$. This gives $y = t's''a$ contradicting the independence of X'. Thus $X' = \{a\}$ and $A = Sa$ is cyclic.

Corollary. *If S satisfies condition A, then every weakly flat S-system is a disjoint union of weakly flat cyclic S-systems.*

Proof. Immediate from Lemma 3 and the remark following Result 4.

In view of this corollary and Result 2, to prove the implications (iii)$\Rightarrow$(v) and (ii)$\Rightarrow$(v), it suffices to show that if S satisfies M_R or condition D, then weakly flat cyclic S-systems are projective.

Lemma 4. *If S satisfies M_R, then every weakly flat cyclic S-system is projective.*

Proof. Let C be a weakly flat cyclic S-system. By the Corollary of Result 4, $C \approx S/\rho$, where ρ is a weakly flat left congruence. If $B = 1\rho$, then B is a subsemigroup of S by Result 5. Let $c \in B$ be such that cS is minimal in the collection of principal right ideals of S generated by elements of B. If $a \in B$, then $a \rho c$ so that $au = cu$ and $uv \in B$ for some elements $u, v \in S$. Hence $d = auv = cuv \in B$ and $dS \subseteq aS \cap cS$. The minimality of cS gives $cS = dS \subseteq aS$.

Now define $\alpha: S/\rho \to Sc$ by $\alpha(s\rho) = sc$. If $s\rho = t\rho$, then for some elements u, v in S, $su = tu$ and $uv \in B$. Hence $cS \subseteq uvS$, so that $c = uvw$ ($w \in S$) and

$sc = suvw = tuvw = tc$ giving that α is well-defined. Clearly α is a surjective S-homomorphism. Also α is injective, because if $sc = tc$, then

$$sp = s(1p) = s(cp) = (sc)p = (tc)p = tp.$$

Thus $S/\rho \approx Sc$. Since $1\rho c$, it follows that c is idempotent so that, by Result 2, Sc is projective.

Lemma 5. *If S satisfies condition D, then every weakly flat cyclic S-system is projective.*

Proof. As in Lemma 4, a weakly flat cyclic S-system is isomorphic to S/ρ for some weakly flat left congruence ρ and $B = 1\rho$ is a left unitary sub-semigroup of S. Condition D ensures that B has a minimal left ideal Be generated by an idempotent e. By (3, Lemma 8.12), eB is a minimal right ideal of B. Since ρ is weakly flat, if $a \in B$, then there are elements $u, v \in S$ such that $au = eu$ and $uv \in B$. Thus $auvB \subseteq aB \cap eB$ and the minimality of eB gives $eB = auvB \subseteq aB$. Hence $eS \subseteq aS$ for all $a \in B$ and $S/\rho \approx Se$ as in the proof of Lemma 4.

Finally, to show that the implication (iii)$\Rightarrow$(iv) holds we have

Lemma 6. *If the right S-system A satisfies the descending chain condition for cyclic subsystems, then it also satisfies the descending chain condition for finitely generated subsystems.*

Proof. Since A contains minimal cyclic subsystems, the family of subsystems of A which satisfy the descending chain condition for finitely generated sub-systems is not empty. This family is partially ordered by inclusion and Zorn's lemma gives a maximal member K. If $K \neq A$, then the set of cyclic subsystems of A not contained in K has a minimal member xS. Consider the subsystem $K \cup xS$ and let $K_1 \supseteq K_2 \supseteq \ldots$ be a descending chain of finitely generated sub-systems of $K \cup xS$. If for any n, $K_n \subseteq K$, then clearly the chain terminates. We may assume, therefore, that for all n there is an element $y_n \in K_n \setminus K$. Clearly $y_n \in xS$ so that $y_n S \subseteq xS$ but $y_n S$ is not contained in K. The minimality of xS gives $y_n S = xS$ and consequently $K_n \cap xS = xS$ for all n. Given a finite set X of generators for K_n, we define K_n' to be the subsystem generated by $X \setminus xS$. Clearly $K_n' \subseteq K$ and $K_n = K_n' \cup xS$. Further, $K_{n+1}' \subseteq K_n'$ for if y is one of the generators of K_{n+1}', then $y \in K_{n+1} \setminus xS$ so that $y \in K_n \setminus xS$ and hence $y \in K_n'$. The chain $K_1' \supseteq K_2' \supseteq \ldots$ consists of finitely generated subsystems of K and so terminates. It follows that the chain $K_1 \supseteq K_2 \supseteq \ldots$ terminates so that $K \cup xS$ satisfies the descending chain condition for finitely generated subsystems, contradicting the maximality of K. Thus $K = A$ and the Lemma is proved.

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