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COMPLEX BERWALD MANIFOLDS WITH VANISHING HOLOMORPHIC SECTIONAL CURVATURE*

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Abstract. In this paper, we prove that a strongly convex and Kähler-Finsler metric is a complex Berwald metric with zero holomorphic sectional curvature if and only if it is a complex locally Minkowski metric.

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1. Preliminaries.

DEFINITION 1. A *Riemann-Finsler metric* on a real smooth manifold M is a function $F : TM \to R^+$ satisfying the following properties:

(i) $G = F^2$ is smooth on $\tilde{M}(= TM - \{0\})$;
(ii) $F(u) > 0$ for all $u \in \tilde{M}$;
(iii) $F(\lambda u) = \lambda F(u)$ for all $u \in TM$ and $\lambda \geq 0$;
(iv) for any $p \in M$, the indicatrix $I_F(p) = \{u \in T_pM | F(u) < 1\}$ is strongly convex.

A manifold M endowed with a Riemann-Finsler metric will be called a Riemann-Finsler manifold.

DEFINITION 2. A *strongly pseudoconvex complex Finsler metric* (we shall simply call it complex Finsler metric below) on a complex manifold M is a continuous function $F : T^{1,0}M \to R^+$ satisfying:

(i) $G = F^2$ is smooth on $\tilde{M}(= T^{1,0}M - \{0\})$;
(ii) $F(v) > 0$ for all $v \in \tilde{M}$;
(iii) $F(\zeta v) = |\zeta| F(v)$ for all $v \in T^{1,0}M$ and $\zeta \in C$;
(iv) for any $p \in M$, the F-indicatrix $I_F(p) = \{v \in T^{1,0}_p M | F(v) < 1\}$ is strongly pseudoconvex.

A complex manifold M endowed with a complex Finsler metric will be called a complex Finsler manifold.

Let M be a complex manifold of complex dimension n. Let $\{z^1, \ldots, z^n\}$ be a set of local complex coordinates, with $z^\alpha = x^\alpha + ix^{n+\alpha}$, so that $\{x^1, \ldots, x^n, x^{n+1}, \ldots, x^{2n}\}$ are local real coordinates. Lowercase Greek indices will run from 1 to n, whereas lowercase Roman indices will run from 1 to $2n$. Let T_RM denote the real tangent bundle of M, and $T^{1,0}M$ denote the holomorphic tangent bundle of M. $\tilde{M}$ will denote either $T^{1,0}M$ or T_RM minus the zero section, depending on the actual situation.

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A local frame over R for $T_R\tilde{M}$ is given by $\{\partial^\circ_1, \ldots, \partial^\circ_{2n}, \dot{\partial}^\circ_1, \ldots, \dot{\partial}^\circ_{2n}\}$, where $\partial^\circ_a = \frac{\partial}{\partial x^a}$ and $\dot{\partial}^\circ_a = \frac{\partial}{\partial u^a}$; analogously, a local frame over C for $T^{1,0}\tilde{M}$ is given by $\{\partial_1, \ldots, \partial_n, \dot{\partial}_1, \ldots, \dot{\partial}_n\}$, where $\partial_\alpha = \frac{\partial}{\partial z^\alpha}$ and $\dot{\partial}_\alpha = \frac{\partial}{\partial v^\alpha}$; and, $z^\alpha = x^\alpha + ix^{n+\alpha}$ and $v^\alpha = u^\alpha + iu^{n+\alpha}$.

To proceed, we need some notation. We shall denote by indices after G the derivatives with respect to the v (or u, we use Greek and Roman indices respectively)-coordinates; for instance,

$$G_{\alpha\bar{\beta}} = \frac{\partial^2 G}{\partial v^\alpha \partial \bar{v}^\beta} \quad or \quad G_a = \frac{\partial G}{\partial u^a}.$$

On the other hand, the derivatives with respect to the z (or x)-coordinates will be denoted by indexes after a semicolon; for instance,

$$G_{;\mu\bar{\nu}} = \frac{\partial^2 G}{\partial z^\mu \partial \bar{z}^\nu} \quad or \quad G_{a;b} = \frac{\partial^2 G}{\partial u^a \partial x^b}.$$

For our aims, we focus on some classes of special real or complex Finsler metrics. Such metrics are important Finsler metrics, on which many discussions have been made.

DEFINITION 3. A Riemann-Finsler manifold (M, F) is said to be *locally Minkowskian* if, at every point $x \in M$, there is a local coordinate system (x^i), with induced tangent space coordinates (u^i), such that F has no dependence on the x^i. Equivalently speaking, G_{ab} has no dependence on the x^i.

DEFINITION 4. A complex Finsler manifold (M, F) is said to be *complex locally Minkowskian* if, at every point $z \in M$, there is a local coordinate system (z^α), with induced holomorphic tangent space coordinates (v^α), such that F has no dependence on the z^α. Equivalently speaking, $G_{\alpha\bar{\beta}}$ has no dependence on the z^α.

DEFINITION 5. A Riemann-Finsler structure F is said to be of *Berwald type* if the Chern connection coefficients $\hat{\Gamma}^i_{jk}$ in natural coordinates have no u dependence, where

$$\hat{\Gamma}^i_{jk} = \frac{G^{is}}{2}\left(\frac{\delta G_{sj}}{\delta x^k} - \frac{\delta G_{jk}}{\delta x^k} + \frac{\delta G_{ks}}{\delta x^j}\right);$$

(G^{ij}) is the inverse matrix of (G_{ij}), and $\frac{\delta}{\delta x^j}$ are vectors on TM which can be found in [**1**, **3**].

DEFINITION 6. A complex Finsler metric F is said to be a *complex Berwald metric* if the Chern-Finsler connection coefficients $\Gamma^\alpha_{\beta;\gamma}$ in natural coordinates have no v dependence, where

$$\Gamma^\alpha_{\beta;\gamma} = G^{\bar{\tau}\alpha}\frac{\delta G_{\beta\bar{\tau}}}{\delta z^\gamma};$$

$(G^{\bar{\tau}\alpha})$ is the inverse matrix of $(G_{\alpha\bar{\tau}})$, and $\frac{\delta}{\delta z^\mu}$ are vectors on $T^{1,0}M$ which can be found in [**1**].

DEFINITION 7. In local coordinates, a complex Finsler metric is called *strongly-Kähler* if and only if $\Gamma^\alpha_{\mu;\nu} = \Gamma^\alpha_{\nu;\mu}$; it is called *Kähler* if and only if $\Gamma^\alpha_{\mu;\nu}v^\mu = \Gamma^\alpha_{\nu;\mu}v^\mu$; it is called *weakly-Kähler* if and only if $G_\alpha[\Gamma^\alpha_{\mu;\nu} - \Gamma^\alpha_{\nu;\mu}]v^\mu = 0$.

MAIN THEOREM. *Let F be a strongly convex and Kähler-Finsler metric on a complex manifold M. Then it is a complex Berwald metric with vanishing holomorphic sectional curvature if and only if it is a complex locally Minkowski metric.*

The definitions of a strongly convex metric and the holomorphic sectional curvature can be found in the following sections.

2. A Riemann-Finsler metric induced by a complex Finsler metric. Let $F : T^{1,0}M \to R^+$ be a complex Finsler metric on a complex manifold M. To F we may associate a function $F^\circ : T_R M \to R^+$ just by setting

$$\forall u \in T_R M, F^\circ(u) = F(u_\circ) \tag{1}$$

where $_\circ : T_R M \to T^{1,0}M$ is given by

$$\forall u \in T_R M, u_\circ = \frac{1}{2}(u - iJu). \tag{2}$$

Notice that J is the complex structure.

DEFINITION 8 ([**1**]). We shall say that the complex Finsler metric F is *strongly convex* if F° is a Riemann-Finsler metric.

Now we assume F is a strongly convex complex Finsler metric. Thanks to (1), the F-length of any curve in M is the same as its F°-length; in particular, F and F° have the same geodesics and induce the same distance function on M.

We continue to compare these two metrics. There is a natural isomorphism, still denoted by $^\circ : T^{1,0}\tilde{M} \to T_R\tilde{M}$, given by

$$\forall X \in T^{1,0}\tilde{M}, X^\circ = X + \bar{X},$$

with inverse $_\circ : T_R\tilde{M} \to T^{1,0}\tilde{M}$ given again by (2). It is easy to check that

$$(\partial_\alpha)^\circ = \partial^\circ_\alpha, \quad (i\partial_\alpha)^\circ = \partial^\circ_{\alpha+n}, \quad (\dot{\partial}_\alpha)^\circ = \dot{\partial}^\circ_\alpha, \quad \text{and} \quad (i\dot{\partial}_\alpha)^\circ = \dot{\partial}^\circ_{\alpha+n}.$$

Recall that Greek indices run from 1 to $n = \dim_C M$, Latin indices run from 1 to $2n = \dim_R M$, and we use the following convention: if in an equality there is a free Latin index on one side and a free Greek index on the other side, the Greek index is equal to the corresponding Latin index taken mod n. For example,

$$\alpha = \begin{cases} a & \text{if } 1 \le a \le n \\ a - n & \text{if } n+1 \le a \le 2n \end{cases}$$

Now we compare the Hessian and the Levi form of G. It is easily found that

$$G_a = \begin{cases} G_\alpha + G_{\bar{\alpha}} & \text{if } 1 \le a \le n \\ i(G_\alpha - G_{\bar{\alpha}}) & \text{if } n+1 \le a \le 2n \end{cases}$$

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and

$$G_{ab} = \begin{cases} G_{\alpha\beta} + G_{\bar\alpha\beta} + G_{\alpha\bar\beta} + G_{\bar\alpha\bar\beta} & \text{if } 1 \le a, b \le n \\ i(G_{\alpha\beta} + G_{\bar\alpha\beta} - G_{\alpha\bar\beta} - G_{\bar\alpha\bar\beta}) & \text{if } 1 \le a \le n \text{ and } n+1 \le b \le 2n \\ i(G_{\alpha\beta} - G_{\bar\alpha\beta} + G_{\alpha\bar\beta} - G_{\bar\alpha\bar\beta}) & \text{if } n+1 \le a \le 2n \text{ and } 1 \le b \le n \\ -(G_{\alpha\beta} - G_{\bar\alpha\beta} - G_{\alpha\bar\beta} + G_{\bar\alpha\bar\beta}) & \text{if } n+1 \le a, b \le 2n \end{cases}$$

Using this and [**1**, Proposition 2.6.2], J.Xiao and the author [**6**] give the relation of the Cartan connection and the Chern-Finsler connection for weakly K*ä*hler-Finsler metrics:

for $1 \le a \le n$,

$$\hat\Gamma^b_{ca} = \begin{cases} Re\left(\Gamma^\beta_{\gamma;\alpha} + \Gamma^\beta_{\bar\gamma;\alpha}\right) & \text{if } 1 \le b, c \le n \\ Im\left(\Gamma^\beta_{\gamma;\alpha} + \Gamma^\beta_{\bar\gamma;\alpha}\right) & \text{if } n+1 \le b \le 2n \text{ and } 1 \le c \le n \\ Im\left(-\Gamma^\beta_{\gamma;\alpha} + \Gamma^\beta_{\bar\gamma;\alpha}\right) & \text{if } 1 \le b \le n \text{ and } n+1 \le c \le 2n \\ Re\left(\Gamma^\beta_{\gamma;\alpha} - \Gamma^\beta_{\bar\gamma;\alpha}\right) & \text{if } n+1 \le b, c \le 2n \end{cases} \tag{3}$$

and for $n+1 \le a \le 2n$,

$$\hat\Gamma^b_{ca} = \begin{cases} Im\left(-\Gamma^\beta_{\gamma;\alpha} - \Gamma^\beta_{\bar\gamma;\alpha}\right) & \text{if } 1 \le b, c \le n \\ Re\left(\Gamma^\beta_{\gamma;\alpha} + \Gamma^\beta_{\bar\gamma;\alpha}\right) & \text{if } n+1 \le b \le 2n \text{ and } 1 \le c \le n \\ Re\left(-\Gamma^\beta_{\gamma;\alpha} + \Gamma^\beta_{\bar\gamma;\alpha}\right) & \text{if } 1 \le b \le n \text{ and } n+1 \le c \le 2n \\ Im\left(-\Gamma^\beta_{\gamma;\alpha} + \Gamma^\beta_{\bar\gamma;\alpha}\right) & \text{if } n+1 \le b, c \le 2n \end{cases} \tag{4}$$

where $\Gamma^\beta_{\bar\gamma;\alpha} = \frac{\partial \Gamma^\beta_{;\alpha}}{\partial \bar v^\gamma}$ and $\Gamma^\beta_{;\alpha} = G^{\bar\tau\beta} G_{\bar\tau;\alpha}$. In fact, $\Gamma^\beta_{\gamma;\alpha} = \frac{\partial \Gamma^\beta_{;\alpha}}{\partial v^\gamma}$ and $\Gamma^\beta_{;\alpha} = \Gamma^\beta_{\gamma;\alpha} v^\gamma$.

If F is a complex Berwald metric, $\Gamma^\beta_{\bar\gamma;\alpha} = 0$. (3) and (4) reduce to: for $1 \le a \le n$,

$$\hat\Gamma^b_{ca} = \begin{cases} Re\left(\Gamma^\beta_{\gamma;\alpha}\right) & \text{if } 1 \le b, c \le n \\ Im\left(\Gamma^\beta_{\gamma;\alpha}\right) & \text{if } n+1 \le b \le 2n \text{ and } 1 \le c \le n \\ Im\left(-\Gamma^\beta_{\gamma;\alpha}\right) & \text{if } 1 \le b \le n \text{and } n+1 \le c \le 2n \\ Re\left(\Gamma^\beta_{\gamma;\alpha}\right) & \text{if } n+1 \le b, c \le 2n \end{cases} \tag{5}$$

and for $n+1 \le a \le 2n$,

$$\hat\Gamma^b_{ca} = \begin{cases} Im\left(-\Gamma^\beta_{\gamma;\alpha}\right) & \text{if } 1 \le b, c \le n \\ Re\left(\Gamma^\beta_{\gamma;\alpha}\right) & \text{if } n+1 \le b \le 2n \text{ and } 1 \le c \le n \\ Re\left(-\Gamma^\beta_{\gamma;\alpha}\right) & \text{if } 1 \le b \le n \text{ and } n+1 \le c \le 2n \\ Im\left(-\Gamma^\beta_{\gamma;\alpha}\right) & \text{if } n+1 \le b, c \le 2n \end{cases} \tag{6}$$

Hence a direct conclusion from this is:

THEOREM 1 ([**6**]). *Let F be a strongly convex and weakly Kähler-Finsler metric on M. F° is a Riemann-Finsler metric induced by F. If F is also a complex Berwald metric, F° must be a real Berwald metric.*

3. A characterization of complex locally Minkowski spaces. In this section, we will finish our proof of the main Theorem.

From now on, we assume F is a strongly convex and K$\ddot{a}$hler-Berwald metric on a complex manifold M. F° is Riemann-Finsler metric induced by F. By Theorem 1, F° is a real Berwald metric.

Using the coefficients $\hat{\Gamma}^i_{jk}$, we can define a linear connection $\hat{D}$ directly on the underlying manifold M, which we call the Berwald connection:

$$\hat{D}W := \left(\frac{\partial W^i}{\partial x^k} + W^j \hat{\Gamma}^i_{jk}\right)\frac{\partial}{\partial x^i} \otimes dx^k,$$

for a vector field $W = W^i \frac{\partial}{\partial x^i}$.

Similarly, we define the complex Berwald connection by $\Gamma^\alpha_{\beta;\gamma}$:

$$DV = \left(\frac{\partial V^\alpha}{\partial z^\gamma} + V^\beta \Gamma^\alpha_{\beta;\gamma}\right)\frac{\partial}{\partial z^\alpha} \otimes dz^\gamma,$$

for a holomorphic vector field $V = V^\alpha \frac{\partial}{\partial z^\alpha}$.

The curvature forms of D and $\hat{D}$ are

$$\Omega^\beta_\alpha = d\omega^\beta_\alpha - \omega^\gamma_\alpha \wedge \omega^\beta_\gamma, \quad \Theta^j_i = d\theta^j_i - \theta^k_i \wedge \theta^j_k,$$

respectively, where $\omega^\beta_\alpha = \Gamma^\beta_{\gamma;\alpha} dz^\gamma$ and $\theta^j_i = \hat{\Gamma}^j_{ki} dx^k$. From (5) and (6), we easily have

$$\Omega^\beta_\alpha = \Theta^\beta_\alpha + \sqrt{-1}\Theta^{n+\beta}_\alpha$$

and

$$\Theta^{n+\beta}_{n+\alpha} = \Theta^\beta_\alpha \quad \Theta^\beta_{n+\alpha} = -\Theta^{n+\beta}_\alpha.$$

Under local coordinate system, we write

$$\Omega^\beta_\alpha = \frac{1}{2} K^\beta_{\alpha\gamma\bar{\delta}} dz^\gamma \wedge d\bar{z}^\delta$$

and

$$\Theta^j_i = \frac{1}{2} R^j_{ikl} dx^k \wedge dx^l,$$

where $K^\beta_{\alpha\gamma\bar{\delta}} = -2\frac{\partial \Gamma^\beta_{\alpha;\gamma}}{\partial \bar{z}^\delta}$ and $R^j_{ikl} = \frac{\partial \hat{\Gamma}^j_{il}}{\partial x^k} - \frac{\partial \hat{\Gamma}^j_{ik}}{\partial x^l} + \hat{\Gamma}^j_{ks}\hat{\Gamma}^s_{il} - \hat{\Gamma}^s_{ik}\hat{\Gamma}^j_{ls}$.

Furthermore, we can have

$$K^\beta_{\alpha\gamma\bar{\delta}} = \left(R^\beta_{\alpha\gamma\delta} + R^{\beta+n}_{\alpha\gamma+n\delta}\right) + \sqrt{-1}\left(R^\beta_{\alpha\gamma\delta+n} + R^{\beta+n}_{\alpha\gamma+n\delta+n}\right).$$

We know from [**1**,**2**, **3**, **9**] that the flag curvature of (M, F°) is

$$K(P, y) = \frac{G_{is}(y) R^s_{jkl} y^j y^l w^i w^k}{G(y)G(w) - G_{ij}(y) y^i w^j},$$

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where the flag P is described by one edge along the flag pole $y = y^i \frac{\partial}{\partial x^i}$ and another transverse edge $w = w^i \frac{\partial}{\partial x^i}$. The holomorphic sectional curvature of (M, F) is

$$K(X) = -\frac{G_{\alpha\bar{\beta}} K^{\alpha}_{\sigma\gamma\bar{\delta}} y^{\sigma} \bar{y}^{\beta} y^{\gamma} \bar{y}^{\delta}}{2G^2(y)},$$

where $X \in T_pM, p \in M$, and $X = y + \bar{y}, y \in T^{1,0}_p M, y = y^{\alpha} \frac{\partial}{\partial z^{\alpha}}$.

If (M, F) has vanishing holomorphic sectional curvature, $\Omega^{\beta}_{\alpha} = 0$. Hence $\Theta^j_i = 0$. that is, (M, F°) has vanishing flag curvature. It has been known (for example, in [**2**]) that a real Berwald space with zero flag curvature is in fact a locally Minkowski space. This means G_{ab} is independent of (x^i), so $G_{\alpha\bar{\beta}}$ must be independent of (z^{γ}). And (M, F) is a complex locally Minkowski space.

Suppose (M, F) is a complex locally Minkowski space. Obviously $\Gamma^{\beta}_{;\alpha}$ vanishes in some priviledged coordinate charts and so $\Gamma^{\beta}_{\gamma;\alpha}$. This means (M, F) is a complex Berwald space with vanishing holomorphic sectional curvature. Thus we finish our proof of the main Theorem.

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