Systems of Spheres connected with the Tetrahedron.

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[References in square brackets are to my paper on Systems of Circles analogous to Tucker Circles.]

1. If A_1 and B_3 (Fig. 17), B_1 and C_3, C_1 and D_3, D_1 and A_3, A_2 and C_2, B_2 and D_2 be pairs of points on the edges

AB, BC, CD, DA, AC and BD

respectively of a tetrahedron ABCD, such that the two points on any edge are concyclic with the two points on any other edge—a manifestly possible condition of things—then the twelve points lie on a sphere.

Demonstration. Since $A_1A_2B_3C_2$, $A_2A_3C_2D_1$, and $A_3A_1B_3D_1$ are concyclic quartets, it is obvious that the six points $A_1A_2A_3B_3C_2D_1$ lie on a sphere cutting the edges of the trihedral angle A in three pairs of points. Again since $A_1A_2B_3C_2$, $B_1B_3C_3A_1$, and $C_2C_3A_2B_1$ are quartets of concyclic points, the six points $A_1A_2B_1B_3C_2C_3$ are concyclic [1]. Therefore B_1 and C_3 lie on the same section of the sphere $A_1A_2A_3B_3C_2D_1$ as the quartet $A_1A_2B_3C_2$. Similarly it might be shown that C_1 and D_3 lie on the same section of this sphere as the quartet $A_2A_3C_2D_1$ and that B_2 and D_2 lie on the same section as the quartet $A_3A_1B_3D_1$. Therefore all the twelve points lie on the same sphere.

The foregoing theorem is an easy extension of that given in Art. 1 of my paper on *Systems of Circles analogous to Tucker Circles.* It is obvious from the demonstration given that it might be enunciated also in the following way: If two points be taken on each edge of a tetrahedron, and if (i) the six points lying on the three edges of each solid angle lie on a sphere, or (ii) the six points lying on each face be concyclic, then all the twelve points lie on a sphere.

2. In the figure of the preceding article (Fig. 17) let S_1 be the centre of perspective of the triangle ABC and the triangle determined by the connectors A_1A_2, B_1B_3, C_2C_3; S_2 the centre of

https://doi.org/10.1017/S0013091500029163 Published online by Cambridge University Press

perspective of the triangle BCD and the triangle determined by the connectors B_1B_2, C_1C_3, D_2D_3; and S_3, S_4 the corresponding points on the faces CDA and DAB respectively. Also let P be the point of intersection of B_1B_3 and C_2C_3.

Since $B_1C_3B_3C_2$, $B_3C_2B_2C_1$, and $B_2C_1B_1C_3$ are quartets of concyclic points, it follows, by an obvious extension of the principle of the concurrency of the three radical axes of three circles taken in pairs, that the connectors B_3C_2 and B_2C_1 meet in a point on BC, say Y. Again if AS_1 meets BC in X_1, since the opposite sides of the complete quadrangle AB_3PC_2 meet the transversal BC in three pairs of points in involution, $X_1YBC_3B_1C$ form an involution. Similarly if DS_2 meets BC in X'_1 it is evident that $X'_1YBC_3B_1C$ form an involution. Therefore X'_1 is the same point as X_1.

Similarly it might be shown that AS_3, BS_2 meet on CD, say in X_2; that BS_4, CS_3 meet on DA, say in X_3; that CS_1, DS_4 meet on AB, say in X_4; that BS_1, DS_3 meet on AC, say in X_5; and that AS_4, CS_2 meet on BD, say in X_6.

The planes ADX_1, BCX_3 obviously intersect on the line X_1X_3; the planes ABX_2, CDX_4 on the line X_2X_4; and the planes BDX_5, ACX_6 on the line X_5X_6.

The lines X_1X_3, X_2X_4, X_5X_6 are concurrent. This may be proved as follows: From the harmonic properties of quadrilaterals (the quadrilateral $BX_1S_1X_4$ being considered) it is evident that the connector X_1X_4 meets AC in a point which is the harmonic conjugate of X_5 with respect to A and C. Similarly it is evident that the connector X_2X_3 meets AC in a point which is the harmonic conjugate of X_5 with respect to A and C. Thus X_1X_4 and X_2X_3 meet in a point. Therefore X_1, X_2, X_3, X_4 lie in a plane. Therefore X_1X_3 and X_2X_4 meet in a point. Similarly it may be shown that X_2X_4 and X_5X_6 meet in a point, and also X_5X_6 and X_1X_3. Therefore, unless the six points X_1, X_2, X_3, X_4, X_5, X_6 lie in one plane, which is impossible, X_1X_3, X_2X_4, and X_5X_6 meet in the same point.

Hence the six planes (ADX_1, etc.) determined by the six points X_1, X_2, X_3, X_4, X_5, X_6 and the opposite edges of the tetrahedron have a common point, say Σ. These planes may be called for convenience the Σ-planes. They are, of course, determinate when the point Σ is given.

The twelve points A_1, B_1, C_1, etc., in which the sphere meets

https://doi.org/10.1017/S0013091500029163 Published online by Cambridge University Press

the edges of the tetrahedron, may be regarded as determined by four planes $A_1A_2A_3$, $B_1B_2B_3$, $C_1C_2C_3$ and $D_1D_2D_3$ which cut off the solid angles A, B, C, D respectively (thus determining an octahedron inscribed in the sphere). These four planes, which are analogous to the directive chords of a coaxaloid system of circles [3] intersect two and two on the six Σ-planes. This may be demonstrated as follows: B_1B_3 and C_2C_3 intersect on AX_1; and B_1B_2 and C_1C_3 intersect on DX_1. Therefore the planes $B_1B_2B_3$ and $C_1C_2C_3$ intersect on the plane ADX_1, *i.e.*, $AD\Sigma$. Similarly it might be shown that each of the other five pairs in which the four planes $A_1A_2A_3$, etc., may be taken intersect on one of the five remaining Σ-planes (a different plane for each pair). Thus the four planes $A_1A_2A_3$, etc., determine a tetrahedron, such that its edges lie with the edges of the original tetrahedron ABCD in six planes which meet in the point Σ.

3. From the results established in the preceding article it is readily deducible that the following construction is possible, namely, to draw other four planes parallel and corresponding as regards the intersection of the edges, to $A_1A_2A_3$, $B_1B_2B_3$, $C_1C_2C_3$ and $D_1D_2D_3$, so as to intersect two and two on the same Σ-planes as their correspondents; for if the vertices of the tetrahedron formed by the four planes $A_1A_2A_3$, etc., be a, b, c, d lying on the lines ΣA, ΣB, ΣC, ΣD respectively, it is clearly possible to find any number of quartets of corresponding points a', b', c', d' lying on the same lines such that the ratios $\Sigma a/\Sigma a'$, $\Sigma b/\Sigma b'$, $\Sigma c/\Sigma c'$ and $\Sigma d/\Sigma d'$ shall be equal.

If now the construction indicated be made, the twelve points determined by the four new planes on the edges of the tetrahedron also lie on a sphere.

Demonstration. Three of the new planes determine on the face ABC three lines. These lines are parallel to A_1A_2, B_1B_3, C_2C_3 and intersect two and two in the same manner on CS_1, AS_1, BS_1. Therefore [3] the six points which they determine on the edges of the face ABC, are concyclic. Similar statements apply to the other faces. Therefore by Art. 1, the twelve points which the four new planes determine on the edges of the tetrahedron lie on a sphere.

By drawing a system of parallel planes in the manner indicated, we obtain a system of twelve-point spheres connected with the tetrahedron.

4. The centres of the spheres of such a system are collinear.

Demonstration. Consider three spheres of the system. Let M_1, M_2, M_3 (Fig. 18) be the centres, known to be collinear [8], of the three circular sections of these made by one face of the tetrahedron, and K_1, K_2, K_3 the centres, also collinear, of the corresponding sections made by another face. Perpendiculars drawn from M_1, M_2, M_3 to the face in which they lie are coplanar, and meet the corresponding perpendiculars from K_1, K_2, K_3, also coplanar, in three points, Γ_1, Γ_2, Γ_3 which are the centres of the spheres. Γ_1, Γ_2, Γ_3 lie on the common section of the planes determined by the two triads of perpendiculars, and are therefore collinear.

5. The following properties are easy inferences from the corresponding properties of coaxaloid circles connected with the triangle.

(i) If a system of parallel planes be drawn through the centres of the spheres of a system such as has been described, the circumferences of the great circles thus determined lie on a hyperboloid of one sheet.

This is evident from the following considerations. When the diametral planes are perpendicular to a face of the tetrahedron, the great circles pass through the extremities, known to lie on different branches of a hyperbola [16], of a system of parallel diameters of the coaxaloid system of circles on that face. Hence, since their centres are collinear, the great circles, in this case, lie on a hyperboloid of one sheet. Hence the extremities of any system of parallel diameters of these great circles lie on opposite branches of a hyperbola. Therefore each of these diameters is to the distance of its mid-point from a certain fixed point in a constant ratio. If now these parallel diameters be turned round their mid-points through any angle in the plane in which they lie, the ratio referred to is unaltered and therefore [*note*] their extremities lie on opposite branches of a hyperbola. This proves the proposition.

By varying the direction of the diametral planes a system of hyperboloids associated with the spheres is obtained.

(ii) Since the circumferences of a system of parallel great circles of the spheres lie on a hyperboloid of one sheet, these circles would, if turned round their centres into one plane, form a coaxaloid

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system. Hence the spheres also form a coaxaloid system, *i.e.*, they can be obtained from a coaxal system of the intersecting species by increase or diminution, in a constant ratio, of the radii of the latter.

(iii) If the spheres be cut by any plane through their line of centres, the great circles so obtained are, as we have seen, coaxaloid; hence their envelope and that of their associated hyperbolas is a conic. By rotating this plane round the line of centres as axis we see that the envelope of the spheres and their associated hyperboloids is a quadric of revolution.

This quadric touches the edges of the tetrahedron at

$$X_1,\ X_2,\ X_3,\ X_4,\ X_5,\ X_6;$$

for the envelopes of the coaxaloid systems of circles on the faces of the tetrahedron do so, and these envelopes are obviously sections of the quadric.

(iv) The spheres of the system are related to any other tetrahedron whose edges touch their envelope, in the same way that they are related to ABCD.

"La Perspective d'une Conique est une Conique." Démonstration Élémentaire.

By M. L. Leau.

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