ON SURFACES WHOSE CANONICAL SYSTEM IS HYPERELLIPTIC

PATRICK DU VAL

1. Generalities. On a surface F of genus $p_g = p_a = p$ and linear genus $p^{(1)} = n + 1$ whose canonical system is irreducible, and which we shall ordinarily think of as simple and free from exceptional curves, the characteristic series of the canonical system is a semicanonical g_n^{p-2}, since the adjoint system of the canonical system is its double, so that the canonical series on a curve of the canonical system is its characteristic series doubled. This is in general free from fixed points, so that the actual grade of the canonical system is n, and the canonical model of the surface is of order n in $[p - 1]$ (by which we indicate space of $p - 1$ dimensions). The bicanonical model is a surface of order $4n$ in $[P_2 - 1]$ where, by a formula derived from the Riemann-Roch theorem (**9**, p.159),

$$P_2 - 1 \geqslant p_a + p^{(1)} - 1 = n + p.$$

Equality will hold in general, and we shall shortly see that it holds in all cases we are going to discuss.

On a hyperelliptic curve of genus $n + 1$ however, every semicanonical g_n^{p-2} consists of $p - 2$ variable pairs of the unique g_2^1 on the curve together with $n - 2p + 4$ fixed points, which are a subset of the $2n + 4$ jacobian points of the g_2^1; for since any two sets of the series together form a canonical set, consisting of $p - 1$ pairs of the g_2^1, the variable part of the series must consist of whole pairs of g_2^1 and each fixed point must be a half pair, i.e., a jacobian point. As an obvious corollary, $n \geqslant 2p - 4$, that is,

$$p^{(1)} \geqslant 2p - 3,$$

which is a classical formula (**9**, p.294).

Hence if the general curve of the canonical system on F is irreducible and hyperelliptic, the canonical system has $n - 2p + 4$ unassigned base points at simple points of F, and its actual grade is $2p - 4$. As the projective model of g_n^{p-2} is a double normal rational curve of order $p - 2$, the canonical model of the surface is a double surface of order $p - 2$ in $[p - 1]$ with rational hyperplane sections, i.e.,[1] either a normal rational ruled surface or (for $p = 6$ only) the Veronese surface V_2^4. We shall denote the ruled surface by R_2^{p-2}; for $p = 3$, of course, it is a plane, and for $p = 2$ it can hardly be held to exist. On V_2^4 or R_2^{p-2} the $n - 2p + 4$ base points P_i of the canonical system appear as ex-

Received December 21, 1950; in revised form September 17, 1951.

[1]For standard properties of rational surfaces reference may be made to (**3**); for the present result, pp. 271 ff., 298.

https://doi.org/10.4153/CJM-1952-019-3 Published online by Cambridge University Press

ceptional lines λ_i ($i = 1, \ldots, n - 2p + 4$) which, moreover, are constituents of the branch curve of the double surface, since each P_i is a jacobian point of the g_2^1 on each curve of the canonical system, i.e., contributes a branch point to the general hyperplane section of the double surface; and as the general hyperplane section has $2n + 4$ branch points altogether, there is a residual branch curve[2] of order $n + 2p$. We note as an obvious corollary that if for $p = 6$ the canonical model is the double Veronese surface, we must have $n = 8$, i.e., $p^{(1)} = 9$, its lowest value for $p = 6$, since there are no lines on the surface; in this case the branch curve is of order 20, and the surface is equivalent to a double plane with general branch curve of order 10 (see exceptional case (i) below).

The bicanonical model is a double rational surface Φ^{2n} on which the base points P_i appear as points, since the bicanonical system traces on each curve of the canonical system its canonical series, compounded with g_2^1, and has no base points at P_i. The projection of Φ^{2n} from these points is a surface Ψ^{4p-8}, projective model of the system of all quadric sections of R_2^{p-2} (or of all conics in the plane for $p = 2$). The ambient space of this latter is $[3p - 4]$, as the freedom of quadrics in the ambient of R_2^{p-2} is $\frac{1}{2}(p-1)(p+2)$, and R_2^{p-2} is itself the intersection of $\frac{1}{2}(p-2)(p-3)$ linearly independent quadrics, its equations being the vanishing of all quadratic minors in a matrix of 2 rows and $p - 2$ columns whose elements are linear in the coordinates. The projections of the points P_i are the images of the lines λ_i, i.e., they are conics S_i, so that the points P_i are isolated branch points at conical nodes of Φ^{2n}; on F, all curves of the bicanonical system that pass through a point P_i have a double point there. Φ^{2n} also has a branch curve of order $2n + 4p$ not passing through the points P_i, which projects into a curve of the same order on Ψ^{4p-8}, image of f^{n+2p}, the residual branch curve of the double R_2^{p-2}. Since the bicanonical system is of genus[3] $3n + 1$, and consists of doubled hyperplane sections of Φ^{2n} with $2n + 4p$ branch points, the section genus π of Φ^{2n} is given by

$$3n = 2(\pi - 1) + \tfrac{1}{2}(2n + 4p),$$

that is,

$$\pi = n - p + 1.$$

The general hyperplane section of Φ^{2n} is a curve of order $2n$ and genus π, non-special since $2n > 2\pi - 2$, and its ambient is therefore of dimensions at most $2n - \pi = n + p - 1$; but we have seen that the ambient of Φ^{2n} is of dimensions at least $n + p$, consequently Φ^{2n} is in $[n + p]$ precisely. A further consequence of this is that, as the difference in dimensions between the ambients of Φ^{2n} and Ψ^{4p-8} is just $n - 2p + 4$, the $n - 2p + 4$ points P_i, from which the former is projected into the latter, are all linearly independent, i.e., their join Ω is an $[n - 2p + 3]$ precisely; and further, as the difference in the order of the two

[2]The cases $n = 2p - 4$ are briefly treated by Enriques (**9**, p. 296). His case (II) is our exceptional case (i) ($p = 7$ is a misprint for $p = 6$).

[3]See the formulae for the genera of the bicanonical system in (**9**, p. 61) putting $p^{(1)} = n + 1$.

surfaces is just $2(n - 2p + 4)$, Ω does not meet Φ^{2n} except in the double points $P_1, \ldots, P_{n-2p+4}$.

The lines λ_i on R_2^{p-2} and conics S_i on Ψ^{4p-8} are thus of virtual grade -2 with respect to the base points of the system $|\phi|$ which represents, on either surface, the hyperplane sections of Φ^{2n}; they are fundamental to $|\phi|$, and also have no intersections outside of these base points either with each other or with the residual branch curve. They satisfy $|\phi| \equiv |\psi + \Sigma\lambda_i|$, where $|\psi|$ represents the quadric sections of R_2^{p-2}, or hyperplane sections of Ψ^{4p-8}.

An s-ple base point A of $|\phi|$ on Ψ^{4p-8} arises in projection from a curve a^s of order s on Φ^{2n}; a^s cannot have a multiple point at any P_i since its multiplicity at P_i is equal to that of S_i, an irreducible conic, in A. a^s thus passes simply through precisely s of the points P_i, say

$$P_{i_1}, \ldots, P_{i_s},$$

and the corresponding conics S_i intersect in A; the corresponding s lines λ_i meet in the image of A on R_2^{p-2}, which we may likewise call A. These s points P_i are joined to A by an $[s]$ which is properly the ambient of a^s since the latter passes through these s points and is not contained in the $[s-1]$ joining them; a^s is thus a rational curve.

If A is a simple point of Ψ^{4p-8}, that is, of R_2^{p-2}, a^s is of virtual grade -1 on Φ^{2n}, and the corresponding hyperelliptic curve on F is of virtual grade -2, with respect to the points

$$P_{i_1}, \ldots, P_{i_s},$$

and hence of virtual grade $s - 2$ without regard to any base points; since the latter curve has s intersections with a general curve of the canonical system, its own canonical series is of order $2s - 2$, i.e., its genus is s, and as the double a^s has branch points at

$$P_{i_1}, \ldots, P_{i_s},$$

it has $s + 2$ elsewhere, i.e., a^s meets the branch curve of Φ^{2n} in $s + 2$ points, and A is an $(s + 2)$-ple point on the residual branch curve of R_2^{p-2} or of Ψ^{4p-8}. If A is a multiple point of Ψ^{4p-8}, it must also be multiple on R_2^{p-2}, that is, R_2^{p-2} is a cone, and A is its vertex; this special case will be considered later.

The case $p = 2$ is somewhat peculiar in that the canonical system is a pencil, its characteristic series has no variable part, there is no canonical model, and no surface Ψ^{4p-8}. We shall find it possible however to study the bicanonical models in this case also.

2. The standard case. An obviously possible arrangement of lines λ_i, and the only one possible for high values of p or n, is for them all to be generators of R_2^{p-2}, or (for $p = 3$) lines of a pencil in the double plane. In this case they must, to be of virtual grade -2, each contain two simple base points of $|\phi|$, say A_{2i-1}, A_{2i} on λ_i (in addition to the vertex A_0 of the pencil for $p = 3$, which is an $(n - 2p + 4 = n - 2)$-ple base point). The curves of $|\phi|$ are coresidual to a

https://doi.org/10.4153/CJM-1952-019-3 Published online by Cambridge University Press

quadric section of R_2^{p-2} (conic in the plane) together with $n - 2p + 4$ generators (lines of the pencil), i.e., they are curves of order n meeting each generator in two variable points, and having the points $A_1, \ldots, A_{2n-4p+8}$ as simple base points, since each lies on just one of the lines λ_i. The residual branch curve f^{n+2p} has thus triple points at $A_1, \ldots, A_{2n-4p+8}$, and since these absorb all its intersections with the lines λ_i, it must meet each generator in six points (and for $p = 3$ have also an n-ple point at A_0). Φ^{2n} is accordingly a surface with hyperelliptic (or elliptic or rational) hyperplane sections of genus $\pi = n - p + 1$, belonging to the series studied classically by Castelnuovo (**1**; **3**, p.464), but special in having $n - 2p + 4$ conical nodes. In particular, for $n = 2p - 4$, Φ^{2n} coincides with Ψ^{4p-8} and is the projective model of all quadric sections of R_2^{p-2}, its section genus being $\pi = p - 3$ and its order $4p - 8 = 4\pi + 4$, the highest possible for surfaces with hyperelliptic sections of this genus (**9**, p. 296, Case III). For $p = 3$, $n = 2$, this gives the Veronese surface with 16-ic branch curve, corresponding to the familiar canonical double plane with octavic branch curve (**9**, p.311; p.296, Case I), and for $p = 4$, $n = 4$, the supernormal octavic del Pezzo surface (on which are no lines, and two pencils of conics) with 24-ic branch curve, corresponding to the canonical double quadric surface branching along a general sextic section (**9**, p.270).

Φ^{2n} has on it a pencil of conics, corresponding to the generators of R_2^{p-2} (or to the lines of the pencil), which trace on each hyperplane section its quadratic involution, and the pencil is accordingly unique for $p \geqslant 2$. The planes of these conics generate a rational threefold R_3^{n+p-2}, normal because the surface Φ^{2n} on it is normal and is obviously not coresidual to a hyperplane section or any part of one (the hyperplane sections of R_3 being rational ruled surfaces), and hence of order $n + p - 2$, since it is in $[n + p]$, and R_3^s is normal in $s + 2$. Φ^{2n} is the residual section of R_3^{n+p-2} by a quadric through $2p - 4$ of these planes, since every surface on R_3^s which meets the general plane in a t-ic curve is coresidual to a t-ic section plus or minus a suitable number of planes to make its order right. In the same way, since the branch curve of Φ^{2n} meets each conic of the pencil in six points, passes through none of the nodes, and is of order $2n + 4p$, it is the residual section of Φ^{2n} by a cubic through $2(n - p) = 2(\pi - 1)$ conics. If $\pi = 1$, Φ^{2n} is a del Pezzo surface, on which the pencil of conics is not unique, and the branch curve is a complete cubic section not residual to any conics; the values $n = p = 4, 3, 2$ respectively give for Φ^{2n}, the supernormal del Pezzo surface of order 8 just referred to, that of order 6 with a double point which is not base point of any pencil of conics on the surface (represented on a plane by cubics with three colinear base points) and that of order 4 with two double points, intersection of two quadrics in [4] one of which is a cone with line vertex. The first and last of these are mentioned by Enriques (**9**, pp. 270, 314); the other corresponds to a canonical double plane whose branch curve consists of a line λ together with a curve of order 9 with three triple points A_0, A_1, A_2 lying in λ. If $\pi = 0$ of course the pencil of conics is replaced by a larger system, and the branch curve is coresidual to a cubic section together with two conics. The only possible cases are $p = 3$, $n = 2$, which gives the double Veronese surface above, with 16-ic branch curve and $p = 2$, $n = 1$, which gives a double quadric cone in [3], branching along a general quintic section and having an isolated branch point at its vertex (**9**, p. 304).

It is worth remarking that the surfaces Φ^{2n} of section genus π $(n \leqslant 2\pi + 2)$ with $n - 2p + 4 = 2\pi + 2 - n$ conical nodes fall into a sequence for diminishing values of n, in which each is the projection of the preceeding one from a tangent line, i.e., is obtained from it by imposing two simple base points, say X_1, X_2 on the hyperplane sections, of which X_2 is in the neighbourhood of X_1. This gives on the projected surface a new conical node corresponding to the neighbourhood of X_1, through which pass two lines, one corresponding to the neighbourhood of X_2, and the other to that conic of the pencil on the original surface which passes through X_1, the two together forming a degenerate conic of the pencil on the new surface. At the same time, for the double surface to be bicanonical, the branch curve must have triple points at X_1, X_2, i.e., what Enriques calls a [3,3] point. In fact, applying, as we evidently can, Enriques' study (**9**, pp. 77-79) of the behaviour of the canonical and bicanonical curves at singular points of the branch curve of a double plane to a general point of any double surface, we see that if any double surface Σ has a branch curve variable in a linear system, as long as the branch curve acquires no extra singularity, the canonical and bicanonical models remain unchanged, except of course for the variation of the branch curve. But when the branch curve acquires a new [3,3] point X_1,X_2, the canonical system acquires a simple base point at X_1 and the bicanonical system a [1,1] point, i.e., simple base points at both X_1 and X_2. The new canonical model is thus the projection of the old from a point, and the new bicanonical model is the projection of the old from the tangent line $X_1 X_2$. On the canonical model the line arising from the neighbourhood of X_1 is a constituent of the branch curve, and the residual branch curve has a triple point at X_2 on this line. On the bicanonical model the node arising from the neighbourhood of X_1 is an isolated branch point, and the line arising from the neighbourhood of X_2 is not part of the branch curve but meets the branch curve in three points distinct from the node. When, as in the present case, there is a conic on Σ passing through X_1 and meeting the branch curve in six (or more generally in s) points, the line arising from this meets the branch curve in three (or $s - 3$) points, distinct from the node. There is thus unit diminution both in the genus of the surface and in its linear genus (since by the coincidence of three of its branch points in X_1 the general curve of the canonical system effectively loses two of them). To sum up, by projecting a bicanonical surface Φ^{2n} of genus p from a tangent line, we obtain a bicanonical surface $\Phi^{2(n-1)}$ of genus $p - 1$, provided that the branch curve is at the same time so specialized that the virtual difference between the branch curve and a complete cubic section of $\Phi^{2(n-1)}$ is the projection of the similarly defined system on Φ^{2n}. For, the hyperplane sections of $\Phi^{2(n-1)}$ are represented on Σ by the same system as those of Φ^{2n}, with the imposition of the [1,1] base point, and thus the cubic sections of $\Phi^{2(n-1)}$ are represented by the same system as those of Φ^{2n}, with the imposition of a [3,3] point. Since the branch curve simultaneously acquires the same singularity, their virtual difference is unchanged.

Hitherto, we have tacitly assumed R_2^{p-2} to be the general normal rational ruled surface of this order. We may now consider the possibility of its being one of the more special types, i.e., having a directrix (curve unisecant to its generators) of lower order than in the general case, or even in the extreme case of its being a cone. For this investigation it is convenient to map R_2^{p-2} on a plane, as we always can, so that its hyperplane sections correspond to curves of order $p - 1$ with a $(p - 2)$-ple base point X, and $p - 1$ simple base points $Y_1, \ldots, Y_{p-1}$. The generators correspond to the pencil of lines through X; thus the images of A_{2i-1}, A_{2i} (which we can conveniently indicate by the same symbols) are collinear with X. The branch curve f^{n+2p} is represented by a curve of order $n + 2p + 6$, with an $(n + 2p)$-ple point at X, sextuple points at $Y_1, \ldots, Y_{p-1}$, and triple points at $A_1, \ldots, A_{2n-4p+8}$. With the addition of the $n - 2p + 4$ lines $XA_{2i-1}A_{2i}$, this gives a total branch curve for the double plane, of order $2n + 10$, with a $(2n + 4)$-ple point at X, sextuple points at $Y_1, \ldots, Y_{p-1}$, and quadruple points at $A_1, \ldots, A_{2n-4p+8}$. The virtual canonical system is thus[4] of order $n + 2$ with $(n + 1)$-ple base point at X, double base points at $Y_1, \ldots, Y_{p-1}$, and simple base points at $A_1, \ldots, A_{2n-4p+8}$. The $n - 2p + 4$ lines $XA_{2i-1}A_{2i}$, and the $p - 1$ lines XY_i (which are pairs of coincident exceptional lines on the double plane) have negative virtual intersection numbers with this system and separate out, leaving, as we expect, the system representing the hyperplane sections of R_2^{p-2}.

The general R_2^{p-2} has (if p is odd) a single minimum directrix of order $\frac{1}{2}(p - 3)$, or (if p is even) a pencil of minimum directrices of order $\frac{1}{2}(p - 2)$. We can give it a minimum directrix of order $k \leqslant \frac{1}{2}(p - 4)$ (which includes making the surface a cone if $k = 0$) by letting all but k of the points $Y_1, \ldots, Y_{p-1}$, say $Y_{k+1}, \ldots, Y_{p-1}$, lie on a line L. If

$$n + 2p + 6 \geqslant 6(p - 1 - k)$$

i.e., if $n \geqslant 4p - 12 - 6k$ (which, since always $n \geqslant 2p - 4$, will always be the case if $k \geqslant \frac{1}{3}(p - 4)$), the above argument remains valid, and the minimum directrix plays no more special role on the surface than in the general case. If $n < 4p - 12 - 6k$ the line L separates out of the branch curve f^{n+2p+6}, giving a residual branch curve of order $n + 2p + 5$ with sextuple points at $Y_1, \ldots, Y_k$, and quintuple at $Y_{k+1}, \ldots, Y_{p-1}$; thus the k lines $XY_1, \ldots, XY_k$ also separate out, leaving a curve of order $n + 2p - k + 5$ with an $(n + 2p - k)$-ple point at X, and quintuple points at $Y_1, \ldots, T_{p-1}$. In this case, moreover, one of each pair A_{2i-1}, A_{2i}, say A_{2i}, must be on L, and the residual branch curve then has triple point at A_{2i-1} and a double point at A_{2i}. On R_2^{p-2} the branch curve, besides the $n - 2p + 4$ exceptional generators λ_i, contains the minimum di-

[4]Using again the rules for finding the canonical system of a double plane given e.g. by Enriques (**9.** pp. 77-79).

rectrix as part, with residual part of order $n + 2p - k$; the intersection of each λ_i with the directrix is double on this residual branch curve, which has also a triple point elsewhere on each λ_i. This case is only possible however if

$$n + 2p - k + 5 \geqslant 5(p - k - 1) + 2(n - 2p + 4),$$

i.e., if $n \leqslant p + 4k + 2$, otherwise L will separate out a second time leaving an effective total branch curve of order $2n + 8$ with a $(2n + 4)$-ple point at X, so that the canonical system is compounded with the pencil of lines through X. Thus if $n \leqslant 4p - 6k - 13$ we must have also $n \leqslant p + 4k + 2$; in particular, if

$$2p - 4 \leqslant p + 4k + 2 \leqslant 4p - 6k - 14,$$

i.e., if

$$\tfrac{1}{4}(p - 6) \leqslant k \leqslant \tfrac{1}{10}(3p - 16),$$

there is a gap in the values of n for which the double R_2^{p-2} with minimum directrix of order k can be a canonical surface of genus p and linear genus $n + 1$. For

$$2p - 4 \leqslant n \leqslant p + 4k + 2$$

and for

$$n \geqslant 4p - 6k - 12$$

the surface exists (the minimum directrix being a part of the branch curve in the former case), but for

$$p + 4k + 3 \leqslant n \leqslant 4p - 6k - 13$$

there is no such surface.

In particular let us consider the case $k = 0$, i.e., that in which R_2^{p-2} is a cone.

For $p = 4$ every value of $n \geqslant 2p - 4 = 4$ is possible, the residual branch curve of order $n + 8$ passing $n - 4$ times through the vertex, and meeting each generator in six points, so that the total branch curve of order $2n + 4$ passes $2n - 8$ times through the vertex, and a general curve passing through the vertex does not branch there.

For $p = 5$, again every value of $n \geqslant 2p - 4$ is possible, but for $n = 6, 7$ the vertex is a branch point on the general curve through it; for $n = 6$ the branch curve of order 16 passes simply through the vertex and meets each generator elsewhere in 5 points; for $n = 7$ there is a single branch generator, and the residual branch curve of order 17 passes twice through the vertex, its two branches touching the branch generator (in which it has elsewhere a triple point) and meets the general generator in five variable points; for $n \geqslant 8$ there are $n - 6$ branch generators, and the residual branch curve of order $n + 10$ passes $n - 8$ times through the vertex and meets each generator in six further points.

For $p = 6$ we have the gap referred to above. For $n = 8$ the branch curve of order 20 is a quintic section, and there is an isolated branch point at the vertex; for $n = 9, 10, 11$ the surface does not exist; while for $n \geqslant 12$ the general curve through the vertex does not branch there, as there are $n - 8$ branch generators, and the residual branch curve of order $n + 12$ has $n - 12$ branches through the vertex; as in the general case, it meets each generator in six points, and has two triple points (distinct from each other and from the vertex) in each of the branch generators λ_i. For higher values of p, the canonical surface can only be a cone if $n \geqslant 4p - 12$.

3. The exceptional cases. It is clear that the standard case just considered, in which the lines λ_i are generators of R_2^{p-2}, is the only one possible for $p \geqslant 7$; since the general R_2^{p-2} has no line on it except its generators, and even if R_2^{p-2} is specialized to have a directrix line, this is of grade $4 - p$, and cannot, by the imposition of any base points, be made of grade -2 (as it must be to be a line λ_i) unless $4 - p \geqslant -2$, that is, $p \leqslant 6$. We shall consider the possible cases for values of p in descending order.

(i) $p = 6$, $n = 8$. In this case, as we have seen, and in this case only, the canonical model may be a double Veronese surface instead of a ruled surface R_2^{p-2}. The branch curve f^{20} is its section by a general quintic, and corresponds to a general curve of order 10 in the standard plane mapping of the Veronese surface (**9**, p. 296, case II). Φ^{16} is thus the projective model of all quartics in the plane, and its branch curve f^{40} is its residual section by a cubic through a rational curve of order 8, image of a conic in the plane.

If now R_2^4 has a directrix line, this is already of grade -2, and needs no points A_i in it to make it so. Thus if this directrix is a line λ it is the only one, since if there were any other it could only be a generator, and its intersection with the directrix would have to be a base point A_i. Thus we have the single case:

(ii) $p = 6$, $n = 9$. R_2^4 has a directrix line which is the unique branch line λ. The residual branch curve f^{21} does not meet λ, and hence meets each generator in seven points, and is the residual section of R_2^4 by a septimic through seven generators. R_2^4 is the projective model of the complete system of rational cubics on a quadric cone in [3], and Φ^{18} is accordingly the projective model of twice this system together with the neighbourhood of the vertex (which is the image of λ), i.e., of the complete system of cubic sections of the cone. The branch curve f^{42} is the image of a septimic section of the cone, and is the residual section of Φ^{18} by a cubic through an elliptic 12-ic curve, image of a quadric section of the cone.

Turning to the case $p = 5$, we have to consider the possibility of the directrix line of R_2^3 being a line λ_i; since its grade, without base points, is -1, it must have one base point A in it to reduce the grade to -2, and as the intersection of any two lines λ_i, λ_j must be a base point, any other line λ_j can only be the generator through A. We thus have two cases:

(iii) $p = 5$, $n = 7$. The unique branch line λ is the directrix of R_2^3, and contains the unique simple base point A of $|\phi|$, which is also a triple point of the residual branch curve f^{17}. $|\phi|$ consists of curves of order 7 meeting each generator in three points and λ only in A, i.e., is the complete system of residual sections of R_2^3 by cubics through two generators and A; similarly f^{17} meets λ only in the triple point A, and each generator consequently in 7 points, and is thus the residual section by a septimic through four generators. If R_2^3 is mapped on a

https://doi.org/10.4153/CJM-1952-019-3 Published online by Cambridge University Press

plane by conics with a simple base point X, A corresponds to a point in the neighbourhood of X, so that $|\phi|$ corresponds to the complete system of quartics with a [1,1] point in X and A, and f^{17} to a curve of order 10 with a [3,3] point there.

(iv) $p = 5$, $n = 8$. λ_1 is the directrix and λ_2 a generator of R_2^3; their intersection A_1 is a double base point of $|\phi|$, and there is also a simple base point A_2 on λ_2. $|\phi|$ consists of octavic curves trisecant to the generators; that is, $|\phi|$ is the complete system of residual sections of R_2^3 by cubics through one generator and A_2, and touching the surface in A_1. The branch curve meets each generator in 7 points, and has a quadruple point in A_1 and a triple point in A_2. If R_2^3 is projected into a quadric cone in [3] from A_1, f^{18} becomes a septimic section with a [3,3] point at the images of (λ_2, A_2), and $|\phi|$ the complete system of cubic sections of the cone with a [1,1] base point there.

For $p = 4$ we have to consider the possibility that the lines λ_i include generators of both systems of R_2^2; in this case they must be not more than two in each system, since the intersections of any one with all those of the other system must be included in the two base points A_i which are needed on the generator to reduce its grade to -2. There are thus the following three cases:

(v) $p = 4$, $n = 6$. λ_1, λ_2 are generators of opposite systems of R_2^2; their intersection A_0 is a double base point of $|\phi|$, which consists of cubic sections of R_2^2, and has also two simple base points A_1 on λ_1 and A_2 on λ_2. f^{14} is a complete septimic section with a quadruple point at A_0 and triple points at A_1, A_2. $|\phi|$ and f^{14} project from A_0 into the complete system of plane quartics with [1,1] base points at the images of (λ_1, A_1) and (λ_2, A_2), and a curve of order 10 with [3,3] points at the same points.

(vi) $p = 4$, $n = 7$. λ_1, λ_2 belong to one system of generators and λ_3 to the other; $|\phi|$ meets each generator of the former system in 3 and of the latter in 4 points, and has two double base points A_i at the intersection of λ_i, λ_3, and two simple base points A_{i+2} lying on λ_i $(i = 1, 2)$. f^{15} meets generators of the former system in 7 and of the latter in 8 points, and has quadruple points in A_1, A_2 and triple points in A_3, A_4. The projective model of rational cubics (bisecant to the latter system) through A_1, A_2 is a quadric cone whose vertex is the image of λ_3; on this $|\phi|$ appears as the complete system of cubic sections with [1,1] base points in the images of (λ_1, A_3) and (λ_2, A_4), and f^{15} as a septimic section with [3.3] points in the same places.

(vii) $p = 4$, $n = 8$. λ_1, λ_2 belong to one system and λ_3, λ_4 to the other. $|\phi|$ consists of quartic sections, with double base points at the four points $A_1, \ldots, A_4$ of intersection of λ_1, λ_2 with λ_3, λ_4. f^{16} is an octavic section with quadruple points in $A_1, \ldots, A_4$. The projective model of the elliptic quartice through $A_1, \ldots, A_4$ is a four-nodal Segre (or quartic del Pezzo) surface, intersection of two quadric cones with line vertices in [4], on which $|\phi|$ appears as the complete system of quadric sections and f^{16} as a quartic section. f^{32} is thus a complete quadric section of Φ^{16}.

https://doi.org/10.4153/CJM-1952-019-3 Published online by Cambridge University Press

In the case $p = 3$ the canonical model is a double plane, and Ψ^4 is the Veronese surface. The lines $\lambda_1, \ldots, \lambda_{n-2}$ and base points $A_1, \ldots, A_r$ satisfy the conditions that precisely three points lie on each line, at least one line passes through each of the points, and the intersection of every two of the lines is a point of the set. It is easily verified that, apart from the standard case in which all the lines belong to one pencil, the only possibilities are the following:

(viii) $p = 3$, $n = 5$, $r = 6$. $\lambda_1, \lambda_2, \lambda_3$ are the sides and A_1, A_2, A_3 the vertices of a triangle, and A_{i+3} lies in λ_i only $(i = 1,2,3)$. The system $|\phi|$ and residual branch curve are

$$\phi^5(A_1^2, A_2^2, A_3^2, A_4^1, A_5^1, A_6^1); \quad f^{11}(A_1^4, A_2^4, A_3^4, A_4^3, A_5^3, A_6^3).$$

The quadratic transformation based on A_1, A_2, A_3 transforms these into quartics with three [1,1] base points at the images of (λ_i, A_{i+3}), and f into a curve of order 10 with [3,3] points at the same points.

(ix) $p = 3$, $n = 6$, $r = 7$. $\lambda_2, \lambda_3, \lambda_4$ meet in A_1, λ_1 meets λ_i in A_i and A_{i+3} is on λ_i only $(i = 2,3,4)$. $|\phi|$ and the residual branch curve are

$$\phi^6(A_1^3, A_2^2, A_3^2, A_4^2, A_5^1, A_6^1, A_7^1); \quad f^{12}(A_1^5, A_2^4, A_3^4, A_4^4, A_5^3, A_6^3, A_7^3).$$

The projective model of the cubics with base points $(A_1^2, A_2^1, A_3^1, A_4^1)$ is a quadric cone whose vertex corresponds to λ_1, on which $|\phi|$ appears as the complete system of cubic sections with [1,1] base points at the images of (λ_i, A_{i+3}), and f as a septimic section with [3,3] points at these same points.

(x) $p = 3$, $n = 6$, $r = 6$. $\lambda_1, \ldots, \lambda_4$ are the sides and $A_1, \ldots, A_6$ the vertices of a complete quadrilateral. $|\phi|$ and the residual branch curve are

$$\phi^6(A_1^2, \ldots, A_6^2); \quad f^{12}(A_1^4, \ldots, A_6^4).$$

The projective model of the cubics with base points $(A_1^1, \ldots, A_6^1)$ is a four nodal cubic surface, on which $|\phi|$ appears as the complete system of quadric sections, and f as a complete quartic section. The branch curve f^{24} is thus a complete quadric section of Φ^{12}.

(xi) $p = 3$, $n = 7$, $r = 7$. $\lambda_1, \ldots, \lambda_4$ are the sides and $A_1, \ldots, A_6$ the vertices of a complete quadrilateral; λ_5 joins the opposite vertices $A_1 A_2$, and A_7 lies in λ_5 only. $|\phi|$ and the residual branch curve are

$$\phi^7(A_1^3, A_2^3, A_3^2, A_4^2, A_5^2, A_6^2, A_7^1); \quad f^{13}(A_1^5, A_2^5, A_3^4, A_4^4, A_5^4, A_6^4, A_7^3).$$

The projective model of the quartics with base points $(A_1^2, A_2^2, A_3^1, A_4^1, A_5^1, A_6^1)$ is a four nodal Segre surface in [4] on which $|\phi|$ appears as the complete system of quadric sections with a [1,1] base point at the image of (λ_5, A_7), and f as a complete quartic section with a [3,3] point at the same point.

(xii) $p = 3$, $n = 8$, $r = 7$. $\lambda_1, \ldots, \lambda_6$ are the sides, $A_1, \ldots, A_4$ the vertices, and A_5, A_6, A_7 the diagonal points of a complete quadrangle. $|\phi|$ and the residual branch curve are

$$\phi^8(A_1^3, A_2^3, A_3^3, A_4^3, A_5^2, A_6^2, A_7^2); \quad f^{14}(A_1^5, A_2^5, A_3^5, A_4^5, A_5^4, A_6^4, A_7^4).$$

The quadratic transformation based on A_2,A_3,A_4 changes these into the system of septimics with a triple point A and three [2,2] points, the fixed tangents in which are concurrent in the triple base point; and into a 13-ic with quintuple point at A and [4,4] points at the other base points. There is a pencil of rational quartics on Φ^{16} corresponding to conics through $A_1, \ldots, A_4$ in the first plane mapping, and to lines through A in the second, and the branch curve of Φ^{16} is the residual section by a quadric through one curve of this pencil.

It is to be remarked that these twelve bicanonical models fall into series, like those in the standard case, each member of the series being obtained from the previous one by imposing a [3,3] point on the branch curve, and correspondingly a simple base point on the canonical and a [1,1] base point on the bicanonical system, so that the canonical models are obtained by repeated projection from simple points and the bicanonical by repeated projection from tangents. These are:

$\pi = 3$, $p \leqslant 6$, $n \leqslant 8$: Projective model of the plane system of quartics with $6 - p = 8 - n$ [1,1] base points, the branch curve being mapped by a 10-ic curve with [3,3] points at the same points. $|3\phi - f|$ is mapped by all conics in the plane.

$\pi = 4$, $p \leqslant 6$, $n \leqslant 9$: Projective model of the cubic sections of a quadric cone (or of plane sextics with a [3,3] base point) having $6 - p = 9 - n$ [1,1] base points. The branch curve is mapped on the cone by a septimic section (or on the plane by a 14-ic with [7,7] point) having [3,3] points at the same points. $|3\phi - f|$ is mapped by all quadric sections of the cone (or by quartics with a [2,2] base point).

$\pi = 4$, $p \leqslant 3$, $n \leqslant 6$: Projective model of the quadric sections of a four-nodal cubic surface, or of plane sextics with six double base points which are the vertices of a complete quadrilateral. We may add $3 - p = 6 - n$ [1,1] base points to cover the case $p = 2$ which we have not yet considered. The branch curve is mapped by a quartic section of the cubic, or by a 12-ic curve with quadruple points at the vertices of the quadrilateral (and [3,3] points at any [1,1] base points of $|\phi|$). $|3\phi - f|$ is mapped by all quadric sections of the cubic, i.e., by $|\phi|$ without the [1,1] base points.

$\pi = 5$, $p \leqslant 4$, $n \leqslant 8$: Projective model of the quadric sections of the four nodal Segre (quartic del Pezzo) surface, or of the plane octavics with quartic base points at two opposite vertices and double base points at the remaining vertices of a complete quadrilateral, and $4 - p = 8 - n$ [1,1] base points. The branch curve is mapped by a quartic section of the Segre surface with [3,3] points at the [1,1] base points of $|\phi|$, i.e., $|3\phi - f|$ is mapped by quadric sections of the Segre surface, or by $|\phi|$ without its [1,1] points.

$\pi = 6$, $p \leqslant 3$, $n \leqslant 8$: Projective model of the plane system of septimic curves with one triple and three [2,2] base points, the fixed tangents at the latter all passing through the former; and of course $3 - p = 8 - n$ [1,1] points. The branch curve is mapped by a 13-ic curve with a quintuple point, and three [4,4] points, and $3 - p = 8 - n$ [3,3] points, at the base points of $|\phi|$; so that $|3\phi - f|$

is the system $|k + \phi|$ without its [1,1] base points, where $|k|$ is the pencil of quartics mapped by lines through the triple base point of $|\phi|$.

As the last surface is less known than the others we may remark that (for $p = 3$, $n = 8$) it is a surface of order 16 in [13], and the ambient [4]s of the quartics $|k|$ generate a locus R_5^9. Three of the quartics consist of a conic repeated, the tangent planes at whose points lie in the corresponding [4] and which passes through two of the six nodes. The system $|\phi|$ can be represented as the residual sections of a sextic surface with hyperplane sections of genus 2, by quadrics through one conic; the sextic surface being special in having six nodes lying by pairs on three torsal lines which, counted twice, form conics of the unique pencil of conics on the surface.

4. The surfaces of genus 2. The case $p = 2$ presents some difficulty, on account of the absence of the canonical model and of the surface Ψ^{4p-8}. The bicanonical model however must still be a double rational surface Φ^{2n} in $[n + 2]$, having n isolated branch points at conical nodes, which are base points of the canonical pencil. The canonical pencil consists of normal rational curves of order n, whose ambient $[n]$s clearly generate a quadric cone Γ_{n+1}^2 with $[n - 1]$ vertex Ω_{n-1}, which is the join of the n nodes, since any two curves of the pencil together form a hyperplane section of the surface. The hyperplane sections of Φ^{2n} have genus $\pi = n - 1$.

Now there is clearly a surface Φ^{2n} to be obtained from each of those obtained in the case $p = 3$, by imposing one further [1,1] base point on $|\phi|$ and the corresponding [3,3] point on the branch curve. In particular the standard case leads to a surface Φ^{2n}, intersection of a rational normal three-fold R_3^n generated by ∞^1 planes with the quadric cone Γ_{n+1}^2, the nodes being the intersection of R_3^n with the vertex Ω_{n-1}. We may also list immediately the exceptional cases (xiii), . . . , (xvii), obtained by imposing a [3,3] point on the branch curve of each of the double planes (viii), . . . , (xii), i.e., the representatives for $p = 2$ of the five sequences of exceptional cases already formed.

It is not immediately obvious what further cases we may expect to find. Let us suppose however that Φ^{2n} is mapped on a plane by a linear system $|\phi|$, of grade $\nu = 2n$, genus $\pi = n - 1$, and freedom $\rho = n + 2$, having ϵ base points $X_1, \ldots, X_\epsilon$, of multiplicities $i_1, \ldots, i_\epsilon$ respectively, the curves of $|\phi|$ being of order m. It is clear that we need consider only systems none of whose base points are simple since, whatever the branch curve of a multiple plane, the bicanonical system cannot have an isolated simple base point; and if it has a [1,1] point corresponding to a [3,3] point of the branch curve, the surface belongs to one of the series already enumerated for $p \geqslant 3$. Thus all the base points of $|\phi|$ are likewise base points of the adjoint system $|\phi'|$, and the conditions imposed by them are of course independent for $|\phi'|$. We need also consider only systems for which $\pi \geqslant 3$, i.e., $n \geqslant 4$, since if the curves of $|\phi|$ are rational, elliptic, or hyperelliptic, the surface is included in the standard case. The virtual intersection number ξ of $|\phi|$ with the system of cubics through all the base points is given by

$$\xi = \nu - 2\pi + 2 = 4.$$

Now I have shewn elsewhere (**5**) that the grade ν' and genus π' of $|\phi'|$ and the quantity $\xi' = \nu' - 2\pi' + 2$ satisfy

$$\pi - \pi' = \xi', \quad \nu - \nu' = \xi + \xi', \quad \xi - \xi' = 9 - \epsilon.$$

From the last of these we have $\xi' = \epsilon - 5$, and putting this value with $\nu = 2n$, $\pi = n - 4$, $\xi = 4$ into the other two relations we have

$$\nu' = 2n - \epsilon + 1, \quad \pi' = n - \epsilon + 4.$$

The freedom ρ' of $|\phi'|$ is of course

$$\rho' = \pi - 1 = n - 2,$$

and consequently satisfies

$$\rho' - \pi' - \epsilon + 6 = 0.$$

It need hardly be said that these characters of $|\phi'|$ are calculated with respect only to those base points which are imposed by those of $|\phi|$. If $|\phi'|$ happens to have any other base points they are to be regarded simply as fixed points of its characteristic series of order ν'.

Now let us consider the order m of a linear system of curves, and its multiplicities $i_1, \ldots, i_\epsilon$ at the base points $X_1, \ldots, X_\epsilon$, as the components of a vector in a real affine space of $\epsilon + 1$ dimensions; and interpret the grade

$$\nu = m^2 - \sum i^2$$

as the square of the length of the vector, so that the intersection number

$$mm' - \sum ii'$$

of two systems represented by vectors $(m, i_1, \ldots, i_\epsilon)$, $(m', i'_1, \ldots, i'_\epsilon)$ is to be regarded as the scalar product of the two vectors. (I have used this device elsewhere (**7**) at some length.) This metric is of the same kind as that introduced by Minkowski[5] for the special relativity theory, a system of positive grade corresponding to a "time-like" and one of negative grade to a "space-like" vector. It is obvious that not more than $\epsilon + 1$ vectors can all be mutually perpendicular, and that of any such maximal set of perpendicular vectors just one must be time-like and the rest space-like. The vector $(3,1, \ldots, 1)$ representing the system of cubics through the base points is space-like if $\epsilon > 9$, time-like if $\epsilon < 9$, since the virtual grade of this system is $9 - \epsilon$. But the vectors representing the n fundamental curves of $|\phi|$ are all perpendicular to this latter vector, since, for a curve of virtual grade $\nu = -2$ and genus $\pi = 0$,

$$3m - \sum i = \nu - 2\pi + 2 = 0;$$

and they are perpendicular to each other, since two fundamental curves have no intersection outside of the base points. It follows that $n \leqslant \epsilon$, or if $\epsilon > 9$ then $n \leqslant \epsilon - 1$, since in this case one of the space-like vectors in a maximal

[5]For this geometry see, e.g., (**12**).

https://doi.org/10.4153/CJM-1952-019-3 Published online by Cambridge University Press

perpendicular set must be $(3,1,\ldots,1)$, which is not a fundamental curve; if $\epsilon = 9$ there can still be only $\epsilon - 1$ space-like vectors all perpendicular to each other and to $(3,1,\ldots,1)$, as one of the vectors perpendicular to this latter is itself,[6] so that in this case also $n \leqslant \epsilon - 1$. Since $\pi' = n - \epsilon + 4$ however, this means that

$$\pi' \leqslant 4, \quad \text{or} \quad \pi' \leqslant 3 \quad \text{if} \quad \epsilon \geqslant 9.$$

We have thus to seek a regular system $|\phi|$ satisfying

$$\nu' = 2n - \epsilon + 1, \quad \pi' = n - \epsilon + 4 \leqslant 4 \ (\pi' \leqslant 3 \ \text{if} \ \epsilon \geqslant 9), \quad \rho' = n - 2 \geqslant 2,$$

and accordingly

$$\rho' - \pi' - \epsilon + 6 = 0.$$

All regular linear systems of genus $\leqslant 4$ and freedom $\geqslant 2$ are known; those of genus 0, 1 are classical and may be found in many standard works, e.g. (**3**, pp. 280, 320); those of genus 2, 3 were studied by Castelnuovo[7] (**1**), (**2**), and myself (**5**); those of genus 4 by Roth (**13**) and myself (**4**), (**5**). They are listed in various places, but the most convenient references are perhaps (**11**) for $\pi' = 3$ and (**10**) for $\pi' = 4$. In the table below are listed the forms to which all linear

π'	Symbol		ρ'	ϵ	$\rho' - \pi' - \epsilon + 6$	
0	$m \ (m-1, 1^k)$	$0 \leqslant k \leqslant m-1$	$2m - k$	$k+1$	$2m + 5 - 2k$	
	1		2	0	8	
	2		5	0	11	
1	$3 \ (1^k)$	$0 \leqslant k \leqslant 7$	$9 - k$	k	$14 - 2k$	A
	$4 \ (2^2)$		8	2	11	
2	$4 \ (2, 1^k)$	$0 \leqslant k \leqslant 9$	$11 - k$	$k+1$	$14 - 2k$	B
	$6 \ (2^8, 1^k)$	$0 \leqslant k \leqslant 1$	$3 - k$	$k+8$	$-1 - 2k$	
3	$5 \ (3, 1^k)$	$0 \leqslant k \leqslant 12$	$14 - k$	$k+1$	$16 - 2k$	C
	$4 \ (1^k)$	$0 \leqslant k \leqslant 12$	$14 - k$	k	$17 - 2k$	
	$6 \ (2^7, 1^k)$	$0 \leqslant k \leqslant 4$	$6 - k$	$k+7$	$2 - 2k$	D
4	$6 \ (4, 1^k)$	$0 \leqslant k \leqslant 7$	$17 - k$	$k+1$	$18 - 2k$	
	$6 \ (3^2)$		15	2	15	
	$5 \ (2^2, 1^k)$	$0 \leqslant k \leqslant 6$	$14 - k$	$k+2$	$14 - 2k$	
	$6 \ (2^6, 1^k)$	$0 \leqslant k \leqslant 3$	$9 - k$	$k+6$	$5 - 2k$	
	$9 \ (3^8)$		6	8	0	E

[6]I have discussed the peculiarities of the metric in this special case $\epsilon = 9$ at some length in (**7**).

[7]Strictly, he studied the corresponding rational surfaces, but the classification of linear systems originated with these investigations.

https://doi.org/10.4153/CJM-1952-019-3 Published online by Cambridge University Press

systems satisfying the above inequalities can be reduced by Cremona transformation. In the column headed "symbol" the order m is written for clarity outside the parenthesis, and the numbers within are the multiplicities $(i_1, \ldots, i_\epsilon)$, except that s consecutive i's within the parentheses are abbreviated as i^s.

It will be seen on inspection of the last column that the relation

$$\rho' - \pi' - \epsilon + 6 = 0$$

is only satisfied by the five systems marked A, B, C, D, E in the margin, for $k = 7, 7, 8, 1$ respectively in the first four cases. From each of these, increasing the order m by 3, and each of the base multiplicities $(i_1, \ldots, i_\epsilon)$ by 1, we obtain the symbol for the system $|\phi|$ to which the given system $|\phi'|$ is adjoint; and these we can tabulate as follows:

	Symbol	ν	π	n
A	$6\ (2^7)$	8	3	4
B	$7\ (3, 2^7)$	12	5	6
C	$8\ (4, 2^8)$	16	7	8
D	$9\ (3^7, 2)$	14	6	7
E	$12\ (4^8)$	16	7	8

These systems all have the required relations between their numerical characteristics; but of course they still need to be investigated as to the possibility of choosing the base points in such a way as to make actual the required set of n rational curves of virtual grade -2. In the last two cases this is impossible, as can easily be seen from the following considerations:

Each of the systems D, E has 8 base points, and they require respectively 7 and 8 actual rational curves of grade -2. It is easy of course to find a set of 8 virtual systems of genus 0 and grade -2, every two of which have virtual intersection number 0; but not more than six of these can be made actual by any configuration of the base points. For the system $6(2^8)$ with the same base points has as its projective model a double quadric cone in [3], branching along a sextic curve of genus 4, intersection of the cone with a cubic surface which does not pass through its vertex (the latter, which corresponds to the ninth associated point of the base points, being also an isolated branch point) (**3**, pp. 364-365). Every actual rational curve of grade -2 whose intersection number with every other such curve is 0, corresponds to a conical node on the surface, i.e. to a double point of the branch curve (**6**, p. 457); and this curve can clearly have six double points (by degenerating into three plane sections of the cone) and no more.

Cases A, B, C, on the other hand give the following specializations respectively:

(xviii). The system $6(2^7)$ requires four actual curves of grade -2. This can be achieved by letting six of the base points be the vertices of a complete quadrilateral; the branch curve is 12-ic with quadruple points at all seven base points, and the canonical pencil consists of the lines through the seventh base point. Alternatively, A_2A_3, A_4A_5 may coincide in [2,2] points, the lines A_2A_3, A_4A_5 meeting in A_1; the branch curve is again 12-ic with quadruple points at all seven base points, and the canonical pencil consists of the conics through A_2, A_4, A_6, A_7. The two mappings are Cremona equivalent. The sextics with seven double base points are well known in general to represent the quadric section of the cone V_3^4 in [6] projecting a Veronese surface from a point; this specialization makes the secant quadric the cone Γ_5^2 with [3] vertex Ω_3, the intersections of the latter with V_3^4 being the four nodes of Φ^8, of which f^{16} is a general quadric section.

(xix) The system $7(3,2^7)$ requires six actual curves of grade -2. This can be achieved by making the double base points A_2A_3, A_4A_5, A_6A_7, coincide in [2,2] points, with the lines A_2A_3, A_4A_5, A_6A_7 all passing through the triple base point A_1. The branch curve is 13-ic with quintuple point at A_1 amd quadruple at all seven double base points; the canonical pencil consists of conics through A_2, A_4, A_6, A_8. On Φ^{12}, the branch curve f^{20} is a quadric section residual to one rational quartic of the pencil represented by lines through A_1.

(xx) The system $8(4,2^8)$ requires eight actual curves of grade -2, which are obtained by making the double base points A_2A_3, A_4A_5, A_6A_7, A_8A_9 coincide in four [2,2] points, the lines A_2A_3, A_4A_5, A_6A_7, A_8A_9 all passing through the quadruple base point A_1. The branch curve is 14-ic with sextuple point at A_1 and quadruple at all eight double base points; the canonical pencil consists of conics through A_2, A_4, A_6, A_8. On Φ^{16} the branch curve f^{24} is a quadric section, residual to two curves of the pencil of rational quartics represented by lines through A_1.

It is interesting to note that all the exceptional cases we have found for $p \geqslant 2$, twenty in all, with a solitary exception, fall under a single formula. We observe that the cubics $3[1^6]$ whose base points are the vertices of a complete quadrilateral is Cremona transformable into the system $3[1^6]$, two pairs of whose base points coincide in [1,1] points, the lines joining these both passing through the same fifth base point; and that the system of cubic sections of a quadric cone, with a [1,1] base point, projects from this point into the plane system of quintics with a [2,2] point and a simple base point in the same line. Thus the systems $|\phi|$ we have considered, save that for $p = 6$, $n = 9$, are equivalent to plane systems of curves of order $m + 4$ $(m \geqslant 0)$ with

(i) An m-ple base point A;

(ii) $m[2,2]$ points B_{2i-1}, B_{2i} $(i = 1, \ldots, m)$, the lines $B_{2i-1}\, B_{2i}$ all passing through A;

(iii) h double base points $C_1, \ldots, C_h$;

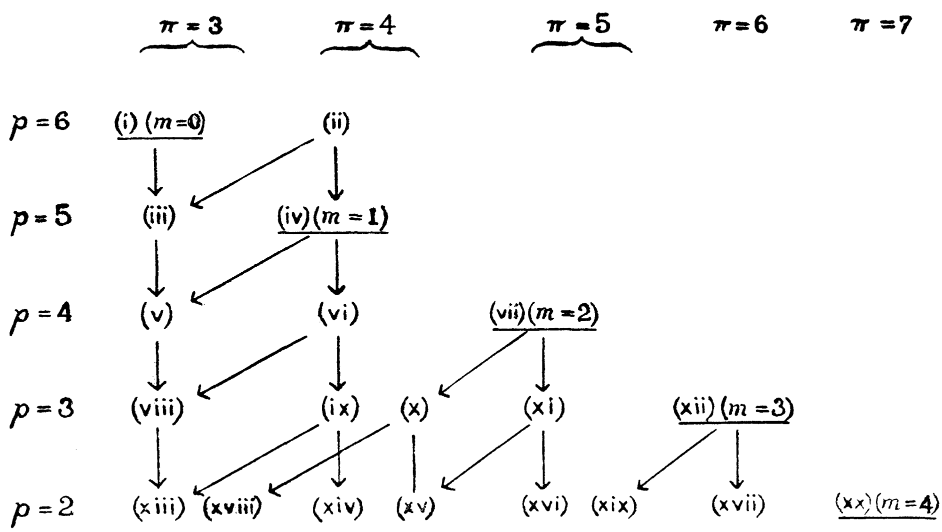
(iv) j [1,1] base points $D_{2i-1}\, D_{2i}$ $(i = 1, \ldots, j)$.

The branch curve is of order $m + 10$, with an $(m + 2)$-ple point at A, [4,4] points at $B_{2i-1}\, B_{2i}$, quadruple points at C_i, and [3,3] points at $D_{2i-1}\, D_{2i}$. There are $2m + j$ nodes on the surface Φ, model of $|\phi|$, represented by the m lines $AB_{2i-1}\, B_{2i}$, the neighbourhoods of the m points B_{2i-1}, and those of the j points

D_{2i-1}; and these are to be isolated branch points, so that the total branch curve of the double plane is of order $2m + 10$, with $(2m + 2)$-ple point at A, [5,5] points at $B_{2i-1}\, B_{2i}$, quadruple points at C_i, and [3,3] points at $D_{2i-1}\, D_{2i}$. The canonical system is thus of order $m + 2$, with m-ple point at A, double at B_{2i-1}, and simple at B_{2i}, C_i, D_{2i-1}; from this system the m lines $AB_{2i-1}\, B_{2i}$ separate out leaving that of conics with $m + h + j$ simple base points at B_{2i-1}, C_i, D_{2i-1}, of which any two (with the m lines) form a curve of $|\phi|$.

$|\phi|$ has grade $2n$ where $n = 8 - 2h - j$, and genus $\pi = m - n + 3$; and the double Φ^{2n} with the defined branch curve and isolated branch points at the $2m + j$ nodes is a bicanonical surface of genus $p = 6 - m - h - j$ and linear genus $n + 1$. There is on Φ^{2n} a pencil (for $m = 0$ a homoloidal net) of rational quartics represented by the lines through A (which for $m = 0$ is absent), and for $j = 0$ the branch curve f is coresidual to a quadric section together with $2 - m$ of these curves. For $m \geqslant 2$, $j \geqslant 1$ the $n - 2p + 4 = 2m + j$ nodes fall into two sets; for m curves of the pencil (represented by the neighbourhoods of B_{2i}) consist of a repeated conic joining two nodes, while j curves of it consist of two conics (represented by the neighbourhood of D_{2i} and the line AD_{2i-1}) meeting in a node. For $m = 1$ the nodes are of two kinds again, but the distinction is not quite the same, as the pencil of quartics is not unique; here however the representation by cubic sections of a quadric cone, with $j + 1$ [3,3] points, makes it clear that there is a unique pencil of rational cubics, passing through one of the nodes, and of which $j + 1$ members break up into a conic through this node and a line, meeting in one of the other $k + 1 = n - 2p + 3$ nodes.

Our 20 exceptional cases can now be tabulated as is shown below, values of p reading downwards and those of $\pi = n - p + 1$ across. A vertical arrow indicates the imposition of a [3,3] point on the branch curve and a [1,1] base point



on $|\phi|$, with unit decrease of n and p; an oblique arrow indicates the imposition of a quadruple point on the branch curve and a double base point on $|\phi|$ with unit decrease of p and decrease of 2 in n. The cases corresponding to $m = 0$, 1, 2, 3, 4 respectively, with $h = j = 0$, are underlined and the values of m indicated. We note that the sequences for $m = 1$, $h \geqslant 1$ are the same as for $m = 0$, with h diminished by 1 and j increased by 2; and those for $m = 0$, $h \geqslant 1$ (and of course $m = 1$, $h \geqslant 2$) are included in the standard case. Apart from this the sequences for different values of m are wholly distinct.

We could of course continue this table a line further, obtaining surfaces of genus 1 whose unique canonical curve (when deprived of exceptional constituents) is irreducible and hyperelliptic; and the standard case would give similar surfaces for all values of the linear genus. There is however no reason to suppose that we should obtain a complete list of such surfaces, since so far as I know there is no reason to suppose that the involution on the unique canonical curve would be contained in an involution on the whole surface.

Similar considerations apply to the surfaces of genus zero which would be obtained by continuing the above table, or the specification of the standard case, a stage further again. It is interesting to note however that we can obtain in this way, if not a complete enumeration, at least samples of surfaces of genus zero and arbitrarily high bigenus.

References

1. G. Castelnuovo, *Sulle superficie algebriche le cui sezioni piane sono curve iperellittiche*, Rendiconti Circ. Mat. Palermo, vol. 4 (1890), 73-88.
2. ———, *Sulle superficie algebriche le cui sezioni sono curve di genere* 3, Atti Accad. Torino, vol. 25 (1890), 465-484.
3. F. Conforto, *Le superficie razionali* (Bologna, 1939).
4. P. Du Val, *On regular surfaces in space of three dimensions whose plane sections are of genus* 4, J. London Math. Soc., vol. 9 (1933), 11-18.
5. ———, *On the discovery of linear systems of plane curves of given genus*, J. London Math. Soc., vol. 10 (1934), 7-11.
6. ———, *On isolated singularities of algebraic surfaces which do not affect the conditions of adjunction* (III), Proc. Cambridge Phil. Soc., vol. 30 (1934), 483-491.
7. ———, *On the Kantor group of a set of points in a plane*, Proc. London Math. Soc. (2), vol. 32 (1936), 18-51.
8. ———, *On absolute and non-absolute singularities of algebraic surfaces*, Rev. Fac. Sci. Istanbul (A), vol. 9 (1944), 159-215.
9. F. Enriques, *Le superficie algebriche* (Bologna, 1949).
10. F. Jongmans, *Mémoire sur les surfaces et les variétés algébriques à courbes sections de genre quatre*, Mém. Acad. R. Belg. (cl. sci.) (2), vol. 23, no. 4 (1949).
11. F. Jongmans and L. Nollet, *Classification des systèmes linéaires de courbes algébriques planes de genre trois*, Mém. Acad. R. Belg. (cl. sci.) (2), vol. 24, no. 6 (1949).
12. H. Minkowski, *Raum und Zeit*, Jber. dtsch. MatVer., vol. 18 (1909), 75-88.
13. L. Roth, *On surfaces of sectional genus four*, Proc. Cambridge Phil. Soc., vol. 29 (1933), 184-194.

University of Georgia