Can. J. Math., Vol. XXIX, No. 2, 1977, pp. 295–298

A LEBESGUE DECOMPOSITION FOR VECTOR VALUED ADDITIVE SET FUNCTIONS DEFINED ON A LATTICE

THOMAS P. DENCE

I. Introduction. Our aim is to establish the Lebesgue decomposition for s-bounded vector valued additive functions defined on lattices of sets in both the finitely and countably additive setting. Strongly bounded (s-bounded) set functions were first studied by Rickart [**15**], and then by Rao [**14**], Brooks [**1**] and Darst [**5**; **6**]. In 1963 Darst [**6**] established a result giving the decomposition of s-bounded elements in a normed Abelian group with respect to an algebra of projection operators. Consequently, one can decompose s-bounded additive functions defined on an algebra of sets. The corresponding restrictions of additive set functions defined on a lattice of sets corresponds to a lattice of projection operators [**7**]. Changing the setting from an algebra to a lattice represents a significant change, and although a decomposition exists, it is not necessarily unique, and it may split f very differently from the way it splits the extension of f. One recent trend in mathematics is the development of integration theory with respect to lattices of sets; cf. [**2**-**4**; **7**-**9**; **12**].

The notation and terminology of Darst [**6**] and his decomposition result [**7**] will be followed throughout the remainder of this article.

II. Decomposing set functions. We now focus our attention on finitely additive vector valued functions defined on a lattice. Under appropriate conditions we can apply the result from [**7**] to get a decomposition. To set the scene, let $(X, \|\ \|)$ be a Banach space, $\mathscr{M} \subset \mathscr{A}$ a lattice of sets that contains $\emptyset$ and $\mathscr{A}$, and $f : \mathscr{M} \to X$ is a modular function, where by this we mean $f(E \cup F) + f(E \cap F) = f(E) + f(F)$ for all E and F in $\mathscr{M}$. A result due to Pettis [**13**] asserts that a modular function f has a unique additive extension $\hat{f}$ to the ring $R(\mathscr{M})$ generated by $\mathscr{M}$. A function $\gamma : \mathscr{M} \to [0, \infty)$ is a submeasure if $\gamma(\emptyset) = 0$ and if γ is monotone and subadditive. The notions of f being continuous ($f \ll \gamma$) and singular ($f \perp \gamma$) with respect to γ are known. But we define f to be *weakly singular* with respect to γ, denoted by $f \perp\perp \gamma$, if given $\epsilon > 0$ there exists $E^* \in \mathscr{M}$ such that $\gamma(E^*) < \epsilon$ and $\|f(E^{*\prime} \cap E)\| < \epsilon$ whenever $E^{*\prime} \cap E \in \mathscr{M}$. Our first result can now be stated.

THEOREM 1. *Let $f : \mathscr{M} \to X$ be modular, and let $\hat{f} : R(\mathscr{M}) \to X$ be continuous with respect to some finitely additive non-negative mapping $u : R(\mathscr{M}) \to$ Reals. Let $\gamma : \mathscr{M} \to [0, \infty)$ be a submeasure. Then there exist unique modular functions*

Received January 13, 1976 and in revised form, October 15, 1976.

$g, h : \mathcal{M} \to X$ such that $f = g + h$ and $g \ll \gamma$, $h \perp \gamma$. The decomposition is not unique if $h \perp \gamma$ is replaced by $h \perp\perp \gamma$.

Proof. The proof involves a direct application of Darst's result [**7**]. Let $G = \{\hat{f} : R(\mathcal{M}) \to X, f \text{ is modular}\}$, and define a norm on G by $|||\hat{f}||| = \sup \{||\hat{f}(E)|| : E \in R(\mathcal{M})\}$. The elements of $\mathcal{U}$ are the projection operators induced by the elements of $R(\mathcal{M})$, i.e., $t \in \mathcal{U}$ corresponds to some set $E \in R(\mathcal{M})$, say $t = t_E$, so that $t_E(\hat{f})(F) = \hat{f}(E \cap F)$. Furthermore let $M = \{t_E \in \mathcal{U} : E \in \mathcal{M}\}$ and $M_x = \{t_E \in M : \gamma(E) \leqq x\}$. The conditions on f imply $\hat{f}$ is bounded and s-bounded. Applying the earlier result gives unique elements $\hat{g}, \hat{h} \in G$ such that $\hat{f} = \hat{g} + \hat{h}$, $S(\hat{g}) = 0$ and given $\epsilon > 0$ there exists $t \in M_\epsilon$ such that $|||t'(\hat{h})||| < \epsilon$. Here and below $S(h) = \limsup_{x \to 0} \{||Ah|| : A \in \mathcal{M}_x\}$ and t' is the complement of t in $\mathcal{U}$. The functions g and h then establish the theorem. One can conjure up simple examples to show that uniqueness does not hold in the second part of the theorem.

With regard to the first part of the theorem (which turns out to be a special case of a result by Drewnowski [**10**]), the condition that f be continuous with respect to u is essential and can not be removed.

We now consider the more involved case when $f : \mathcal{M} \to X$ and $\mathcal{M}$ is a sigma lattice and f is at least countably additive. One is then interested in knowing when f has a unique extension $\hat{f}$ to the generated sigma ring $S(\mathcal{M})$. A partial result is due to Pettis [**13**], and requires the introduction of new terms. Let $\mathcal{A}$ be a sigma complete Boolean algebra, $L \subset \mathcal{A}$ a lattice with $\emptyset \in L$ and let $f : L \to R^+$ be monotone and modular. Let U, V and W be subsets of $\mathcal{A}$.

Definition. $C(W) = \{a \in \mathcal{A}$: any countable covering of a by elements of W has a finite subcovering$\}$; $U/V = \{u - v : u \in U, v \in V\}$; $U//V = \{u - v : u \in U, v \in V, u > v\}$; $H_0(L) = \{x - y : x, y \in L, x \neq y\}$; $L(U, V) = \{l \in L$: given $\epsilon > 0$ there exists $l', l'' \in L$, $u \in U$, $v \in V$ such that $l' \leqq u \leqq l \leqq v \leqq l''$ and $f(l'') - \epsilon \leqq f(l) \leqq f(l') + \epsilon\}$.

The result due to Pettis can now be stated.

THEOREM (Pettis). *Suppose there are subsets U and V of $\mathcal{A}$ such that $U/V \subset C(V//U)$ and $H_0(L) \subset L(U, V)//L(U, V)$. Then there exists a unique sigma-finite measure $\hat{f} : S(L) \to R^+ \cup \{\infty\}$ where $S(L)$ is the sigma ring generated by L and $\hat{f}$ is the extension of f.*

We must make some definitions that are consistent with those made earlier regarding finite additivity. Actually, the only new definitions are in regard to singularity. Let $f : \mathcal{M} \to R^+ \cup \{\infty\}$ be countably additive and modular where $\mathcal{M}$ is a sigma lattice, and let γ be an outer measure (sub-measure together with sigma subadditivity) on $\mathcal{M}$. We say f is *singular* with respect to γ ($f \perp \gamma$) if there exists $E^* \in \mathcal{M}$ such that $\gamma(E^*) = 0$ and $\hat{f}(E^{*\prime} \cap E) = 0$ for all $E \in S(\mathcal{M})$ where $\hat{f}$ is an extension of f. We say f is *weakly singular* with respect to γ

https://doi.org/10.4153/CJM-1977-032-1 Published online by Cambridge University Press

($f \perp\perp \gamma$) if there exists $E^* \in \mathscr{M}$ such that $\gamma(E^*) = 0$ and $f(E^{*\prime} \cap E) = 0$ whenever $E^{*\prime} \cap E \in \mathscr{M}$. The final decomposition result is the following.

THEOREM 2. *Let $\mathscr{A}$ and $\mathscr{M}$ be as above with $\emptyset \in \mathscr{M}$. Let $f : \mathscr{M} \to R^+$ be countably additive, modular and monotone, and let γ be an outer measure on $\mathscr{M}$. Suppose there are subsets U, V of $\mathscr{A}$ such that*

1) $U/V \subset C(V//U)$
2) $H_0(\mathscr{M}) \subset \mathscr{M}(U, V)//\mathscr{M}(U, V)$.

Let $\hat{f} : S(\mathscr{M}) \to R^+ \cup \{\infty\}$ denote the unique measure that extends f to the generated sigma ring. Then there exists unique functions g, $h : \mathscr{M} \to R^+$ that are countably additive, modular, monotone and where $f = g + h$ and

3) $g \ll \gamma$
4) $h \perp \gamma$.

The decomposition is not unique if $h \perp \gamma$ is replaced by $h \perp\perp \gamma$.

Proof. Following the proof of Theorem 1, we let $G = \{\hat{f} : S(\mathscr{M}) \to R^+ \cup \{\infty\}$ where f is finite, countably additive, modular and monotone$\}$, and a norm is defined on G by $|||\hat{f}||| = \sup \{\hat{f}(E) : E \in S(\mathscr{M})\}$. The elements of $\mathscr{U}$ are the projection operators induced by the elements of $S(\mathscr{M})$. Then $M = \{t_E \in \mathscr{U} : E \in \mathscr{M}\}$ is a lattice containing 0 and $M_x = \{t_E \in M : \gamma(E) \leqq x\}$ possess the desired two properties. Applying Darst's result [**7**] gives unique elements $\hat{g}$, $\hat{h} \in G$ with $\hat{f} = \hat{g} + \hat{h}$, $S(\hat{g}) = 0$ and given $\epsilon > 0$ there exists $t \in M_\epsilon$ such that $|||t'(h)||| < \epsilon$. Showing $g \ll \gamma$ follows from knowing $S(\hat{g}) = 0$, but showing singularity is more involved. From the previous sentence we know given $\epsilon_n = 2^{-n}$ there exists $t_{\hat{E}_n} \in M_{\epsilon_n}$ such that $|||t_{\hat{E}_n'}(\hat{h})||| < 2^{-n}$. Thus $\hat{E}_n \in \mathscr{M}$, $\gamma(\hat{E}_n) < 2^{-n}$ and $\sup \{\hat{h}(\hat{E}_n' \cap E) : E \in S(\mathscr{M})\}$ is less than 2^{-n}. Define $E_n^* = \bigcup_{i=1}^{n}[\bigcap_{j=i}^{\infty} \hat{E}_j']$. Then $E_n^* \in S(\mathscr{M})$ and $\hat{h}(E_n^*) < 2^{-n}$. Letting $E^* = \bigcup E_n^* = \lim E_n^*$ implies $E^* \in S(\mathscr{M})$ and $\hat{h}(E^*) = \lim \hat{h}(E_n^*) \leqq \lim 2^{-n} = 0$. Now let $E \in S(\mathscr{M})$ be arbitrary. To show $\hat{h}(E) = \hat{h}(E \cap \hat{E})$ where $\hat{E}$ is the desired set needed to prove h is γ-singular, it suffices to show $\hat{h}(E \cap \hat{E}') = 0$. Our set $\hat{E}$ will be $\hat{E} = E^{*\prime}$. Then $E \cap \hat{E}' = E \cap E^* = \lim (E \cap E_n^*)$, so $\hat{h}(E \cap \hat{E}') \leqq \lim 2^{-n} = 0$. And $t_{\hat{E}_n} \in M_{\epsilon_n}$ implies $\gamma(\hat{E}_n) \leqq 2^{-n}$ for all n, so $\gamma(\bigcup_{k \geqq n} \hat{E}_k) \leqq 2^{1-n}$. Then

$$\gamma(\hat{E}) = \gamma(E^{*\prime}) = \gamma\left(\bigcap_{n=1}^{\infty}\left[\bigcup_{i \geqq n} \hat{E}_i\right]\right) \leqq \gamma\left(\bigcup_{i \geqq n} \hat{E}_i\right) \quad \text{for all } n$$

$$\leqq \sum_{i=n}^{\infty} \gamma(\hat{E}_i) \quad \text{for all } n$$

$$\leqq 2^{1-n} \quad \text{for all } n.$$

Thus $\gamma(\hat{E}) = 0$, and it makes sense to apply γ to $\hat{E}$ because $E_n^{*\prime} \in \mathscr{M}$ implies $\hat{E} \in \mathscr{M}$. This establishes $h \perp \gamma$. To verify the uniqueness, suppose $f = g_1 + h_1$ with $g_1 \ll \gamma$ and $h_1 \perp \gamma$ and we will show $S(\hat{g}_1) = 0$ and that given $\epsilon > 0$ there exists $t \in M_\epsilon$ such that $|||t'(\hat{h}_1)||| < \epsilon$. This will imply $g = g_1$ and $h = h_1$. Since $h_1 \perp \gamma$ then there exists $E^* \in \mathscr{M}$ with $\gamma(E^*) = 0$ and $\hat{h}_1(E^{*\prime} \cap E) = 0$

for all E. Hence let $t = t_{E^*}$ and then $t \in M_\epsilon$ and $|||t'(\hat{h}_1)||| < \epsilon$ for all $\epsilon > 0$. Finally, since $S(\hat{g}_1) = \lim_{x\to 0}$ [sup [sup $\{\hat{g}(E \cap E^*) : E^* \in S(\mathcal{M}) : t_E \in M_x]]$ it suffices to show that $\{E_n\} \subset \mathcal{M}$, $\gamma(E_n) \to 0$, $F_n \subset E_n$, $F_n \in S(\mathcal{M})$ implies $\hat{g}_1(F_n) \to 0$. One can show that if $R_n \subset E_n$ with $R_n \in R(\mathcal{M})$ then $\hat{g}_1(R_n) \to 0$. But the ring $R(\mathcal{M})$ is dense (with respect to $\hat{g}_1$ measure) in $S(\mathcal{M})$ [**11**]. Hence $\hat{g}_1(F_n) \to 0$ and $S(\hat{g}_1) = 0$.

References

1. James Brooks, *On the existence of a control measure for strongly bounded vector measures*, Bull. Amer. Math. Soc. *77* (1971), 999–1001.

2. H. D. Brunk, *Best fit to a random variable by a random variable measurable with respect to a σ-lattice*, Pac. J. Math. *11* (1961), 785–802.

3. ——— *Conditional expectation given a σ-lattice and applications*, Ann. Math. Statist. *36* (1965), 1339–1350.

4. H. D. Brunk and S. Johansen, *A generalized Radon-Nikodym derivative*, Pac. J. Math. *34* (1970), 585–617.

5. R. B. Darst, *A decomposition for complete normed Abelian groups with applications to spaces of additive set functions*, Trans. Amer. Soc., *103* (1962), 545–559.

6. ——— *The Lebesgue decomposition*, Duke Math. J. *30* (1963), 553–556.

7. ——— *The Lebesgue decomposition for lattices of projection operators*, Advances in Math. *17* (1975), 30–33.

8. R. B. Darst and G. A. DeBoth, *Two approximation properties and a Radon-Nikodym derivative for lattices of sets*, Indiana U. Math. J. *21* (1971), 47–58.

9. ——— *Norm convergence of martingales of Radon-Nikodym derivatives given a σ-lattice*, Pac. J. Math. *40* (1972), 547–552.

10. L. Drewnowski, *Decompositions of set functions*, Studia Math. *48* (1973), 218–231.

11. Paul Halmos, *Measure theory* (Van Nostrand, 1950).

12. S. Johansen, *The descriptive approach to the derivative of a set function with respect to a σ-lattice*, Pac. J. Math. *21* (1967), 49–58.

13. B. J. Pettis, *On the extension of measures*, Annals of Math. *54* (1951). 186–197.

14. M. M. Rao, *Decomposition of vector measures*, Proc. Nat. Acad. Sci. USA *51* (1964), 771–774.

15. C. E. Rickart, *Decomposition of additive set functions*, Duke Math. J. *10* (1943), 653–665.

Bowling Green State University,
Huron, Ohio