ON THE PSEUDO-EUCLIDEAN GEOMETRY DUE TO G. HESSENBERG

PAUL SZÁSZ

To Professor H. S. M. Coxeter on the occasion of his sixtieth birthday

Introduction. G. Hessenberg **(2)** showed that euclidean plane geometry can be realized on the surface of the sphere without assuming the parallel axiom. This geometry will be called the *pseudo-euclidean geometry due to G. Hessenberg*. In the present paper we give a slightly different treatment, which is perhaps simpler than that of Hessenberg and which has a greater transparency. As an application we shall prove the basic formula of spherical trigonometry, namely the formula

$$\cot \alpha = \cot a \sin b$$

for the right-angled spherical triangle, from which follows the entire trigonometry on the sphere. Here a and b denote the sides, while α designates the angle opposite the side a. The trigonometric functions are defined by means of pseudo-euclidean geometry. This intrinsic presentation of spherical trionometry is independent of continuity, contrary to both H. Liebmann's **(5)** and P. Mansion's **(6)** treatment.

1. The pseudo-euclidean geometry due to G. Hessenberg. Let U be a fixed point on the sphere. Let the other points of the surface be called *pseudo-points*, the spherical circles through U (the point U excluded) *pseudo-lines*. These pseudo-points and pseudo-lines together form the fundamental elements of the pseudo-geometry to be realized. The notions of incidence and of order are defined as usual. Then it may be seen immediately that Hilbert's groups of axioms I, II of incidence and order in the plane **(3**, pp. 3–5**)** are all valid. Let us call two pseudo-angles *pseudo-equal* or *pseudo-congruent* if and only if the corresponding angles on the sphere are equal. Then Hilbert's III 4 axiom **(3**, pp. 13–14**)** is evidently also valid.

To define the congruence of two pseudo-intervals some preparation is necessary.

Two pseudo-lines without a common pseudo-point are called parallel. This obviously means that the corresponding circles on the sphere touch each other at the point U. The parallel axiom is evidently also valid even in the stronger

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form, according to which *one and only one parallel to a given pseudo-line goes through a given pseudo-point not on the pseudo-line.* Further, the validity of the following theorem is at once obvious.

THEOREM 1. *If a pseudo-line cuts two parallels, the corresponding angles are equal, and vice versa.*

COROLLARY. *If a pseudo-line cuts two parallels, the alternate angles are equal, and vice versa.*

On the basis of the parallel axiom (cf. Theorem 1, or its Corollary), we obtain, by a familiar argument, the following:

THEOREM 2. *The angles of a pseudo-triangle together form two right angles.*

Now, by means of the next theorem the treatment will become appreciably simpler than that of Hessenberg **(2)**.

THEOREM 3. *If the point U belongs to the exterior* of a circle k on the sphere, then a pseudo-inscribed angle $\gamma = \angle ACB$ upon the pseudo-chord $\overline{AB}$ is equal to the opposite angle ϕ made by the circle k and the pseudo-chord $\overline{AB}$* (Fig. 1).

Proof. Let us denote the angles made by the pseudo-chords $\overline{AC}$, $\overline{BC}$, and the circle k opposite to the vertices B, A by α', β' respectively. Denoting by α, β the angles of the pseudo-triangle ABC with the vertices A, B respectively, we have in the sense of Theorem 2

$$\alpha + \beta + \gamma = \pi$$

(π denoting the straight angle). Consequently

$$\beta + \gamma = \alpha' + \phi$$

because α completes both sides to π. On the other hand,

$$\alpha' + \gamma = \beta + \phi$$

since β' completes both sides to π. Thus, by addition,

$$\alpha' + \beta + 2\gamma = \beta + \alpha' + 2\phi;$$

hence

$$\gamma = \phi.$$

COROLLARY 1. *If the point U belongs to the exterior of a circle on the sphere, then the two pseudo-inscribed angles upon the same pseudo-chord are equal.*

COROLLARY 2. *Suppose that the point U lies in the exterior of a circle on the sphere. If the points A, C separate B, D on the periphery of this circle, then in the pseudo-quadrilateral $ABCD$ the opposite angles are supplementary.*

*The exterior of k is defined, when k does not pass through U, as the spherical cap bounded by k and not containing U.

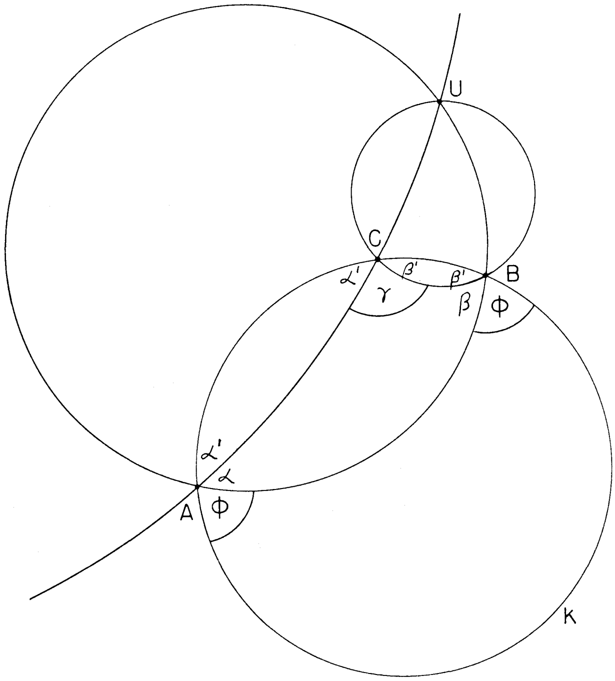

FIGURE 1

Together with these, we have the following converses:

THEOREM 4. *If in the pseudo-convex quadrilateral ABCD we have*

$$\angle B + \angle D = \pi,$$

then the circle ABC on the sphere goes through the point D.

THEOREM 5. *If in the pseudo-convex quadrilateral ABCD we have*

$$\angle ADB = \angle ACB,$$

then the circle ABC on the sphere goes through the point D.

We modify the definition of "*antiparallel lines*" due to Hessenberg (**2**, p. 182) as follows. Let l, l' be distinct pseudo-lines. If l, l' have a pseudo-point in common, then let this point be denoted by S. If l, l' are parallel, then instead of S we take the point U, which is not a pseudo-point. Further, let AA' and BB' be different pseudo-lines not passing through S (if S exists) and intersecting l, l' in A, A' and B, B', respectively. Now, instead of G. Hessenberg's definition, we make the following equivalent one:

DEFINITION 1. *AA' and BB' are "antiparallel" pseudo-lines with respect to the base pseudo-lines l, l' if and only if either*

$$\angle A'AS = \angle BB'S,$$

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when S exists, or

$$\angle A'AU = \angle BB'U,$$

where the parallel pseudo-rays AU, $B'U$ lie on the same side of the pseudo-line AB' (Figs. 2 a, 2 b).

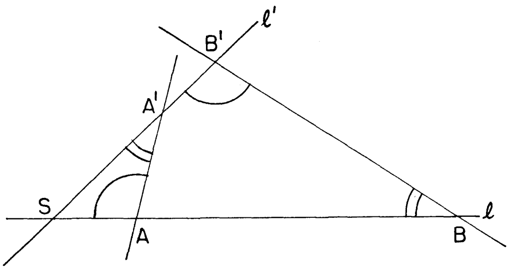

FIGURE 2a

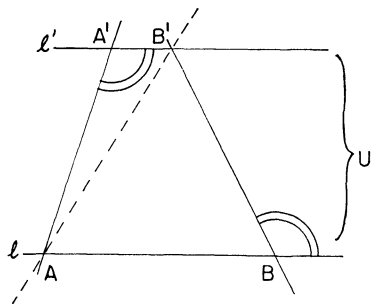

FIGURE 2b

These relations are equivalent to

$$\angle AA'S = \angle B'BS, \qquad \angle AA'U = \angle B'BU$$

respectively, in consequence of the above theorems. Thus, neither l nor l' is here distinguished.

For the statement "*AA' and BB' are antiparallels*" we write, following H. G. Forder (**1**, p. 154),

$$AA' \wedge BB' \quad \text{or} \quad BB' \wedge AA'.$$

Applying the above theorems with this definition, we have immediately:

THEOREM 6. *If $AA' \wedge BB'$ and $BB' \parallel CC'$ with C on l, C' on l', then*

$$AA' \wedge CC'.$$

THEOREM 7. *If $AA' \wedge BB'$ and $BB' \wedge CC'$, where CC' is different from AA', then $AA' \parallel CC'$.*

THEOREM 8. *$AA' \wedge BB'$ implies that $AB' \wedge BA'$.*

Concerning antiparallel pseudo-lines we also need the following:

THEOREM 9. *Given AA', through a given point B on l (different from A and also from S if the latter exists) there exists a BB' such that $AA' \wedge BB'$.*

Proof. First we suppose that l, l' cut each other in the pseudo-point S. Let C' be any pseudo-point on the ray SA' and the pseudo-line $C'P$ parallel to l, the pseudo-point P lying on the side of l' opposite to the pseudo-point A (Fig. 3). Then $\angle ASA' = \angle PC'S$ because these are alternate angles (Theorem 1, Corollary). But by Theorem 2

$$\angle ASA' + \angle A'AS < \pi,$$

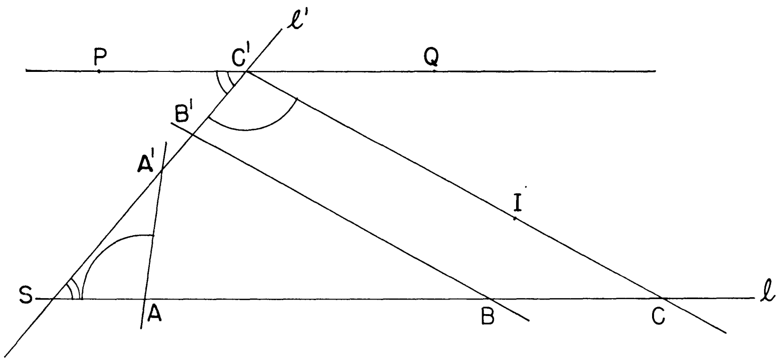

FIGURE 3

and so

$$\angle PC'S + \angle A'AS < \pi.$$

Consequently, Q being a pseudo-point for which C' lies between P and Q, the pseudo-angle $\angle SC'Q$ has an interior point I with $A'AS = IC'S$. But $C'P$ being parallel to l, the pseudo-line $C'I$ cuts l in a pseudo-point C, by the parallel axiom. By the construction, $\angle A'AS = \angle CC'S$; thus by Definition 1 we have $AA' \wedge CC'$. Now, in the case $B \neq C$, when $BB' \parallel CC'$, we obtain $AA' \wedge BB'$ (Theorem 6).

If $l \parallel l'$, then for any pseudo-point C' on l' a pseudo-point C exists on l for which $\angle A'AU = \angle CC'U$ (Fig. 4); thus $AA' \wedge CC'$ and the statement follows by Theorem 6 again.

On the basis of Theorems 6–9 and using ideas due to Hessenberg (**2**, pp. 182–183) the affine form of Pascal's theorem can be proved for the present pseudo-geometry, that is to say the following:

THEOREM 10. *Let l, l' be distinct pseudo-lines. Further, let A, B, C and A', B', C' be two triplets of pseudo-points situated on l, l' respectively and different from their*

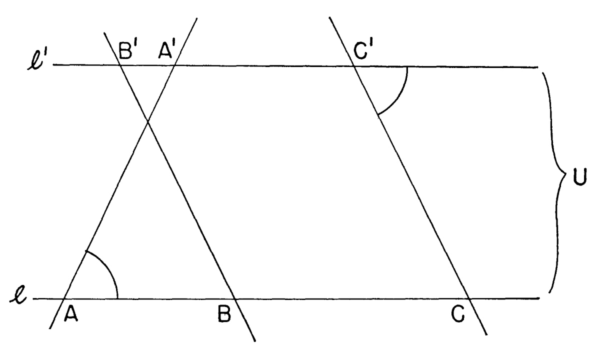

FIGURE 4

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intersection S, if this exists. If the pseudo-lines AB', BA' are parallel and if AC', CA' are also parallel, then the pseudo-lines BC', CB' are parallel.

Applying this last theorem, we prove the following:

THEOREM 11. *If $BB' \parallel CC'$ and the pseudo-lines BC, $B'C'$ meet each other at the pseudo-point P, and $AB \parallel A'B'$, $AC \parallel A'C'$, then the pseudo-line AA' intersects the parallels BB' and CC'.*

(It is not this theorem that was proved by Hessenberg (**2**, p. 184), but its converse: if AA' cuts the parallels BB' and CC', then BC and $B'C'$ meet each other.)

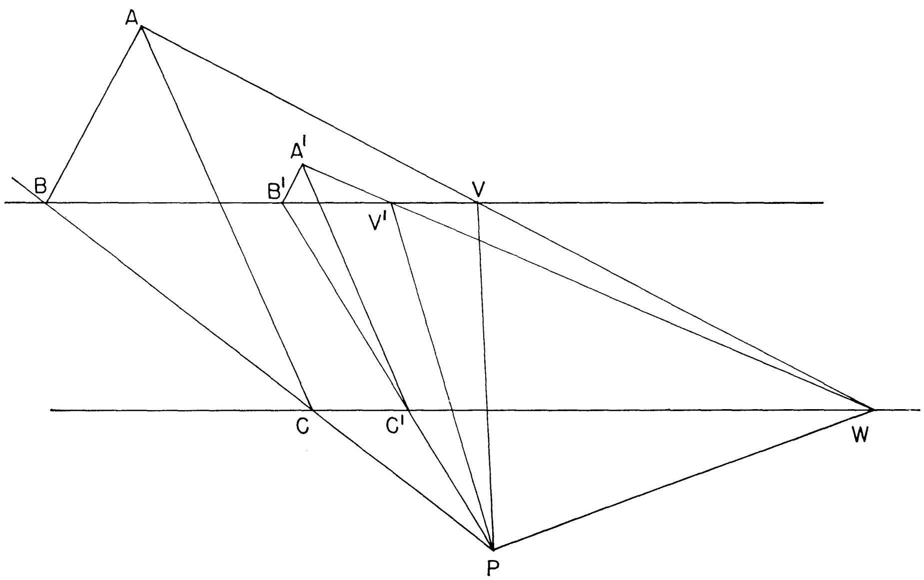

FIGURE 5

Proof. Call W the pseudo-point at which the parallel to the pseudo-line AB through P meets CC' (Fig. 5). Let us denote by V, V' the pseudo-points at which the pseudo-lines AW, $A'W$ cut BB' respectively. Applying Theorem 10 to the triplets W, A, V and B, P, C, we have

$$AC \parallel VP \tag{1}$$

since $WP \parallel AB$ and $WC \parallel VB$. Similarly, applying Theorem 10 to the triplets W, A', V' and B', P, C', we have

$$A'C' \parallel V'P, \tag{2}$$

since $WP \parallel A'B'$ and $WC' \parallel V'B'$. But by assumption $AC \parallel A'C'$. Thus, by (1), (2), the pseudo-lines VP and $V'P$ coincide, because in this pseudo-geometry the parallel axiom is valid. This means that $V = V'$; consequently, the pseudo-lines AW and $A'W$ coincide, that is to say the pseudo-line AA' intersects BB', CC' at the pseudo-points V, W respectively.

From Theorem 11 the *small Desargues theorem* for the present pseudo-geometry follows:

THEOREM 12. *If AA', BB', CC' are parallel pseudo-lines with $AB \parallel A'B'$ and $AC \parallel A'C'$, then $BC \parallel B'C'$.*

With the aid of this theorem we can proceed to the definition of the congruence of two pseudo-intervals.

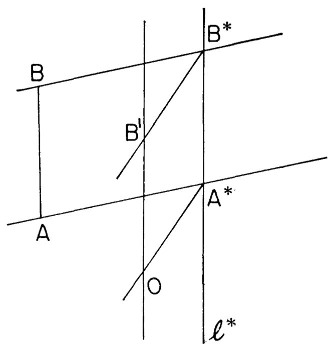

FIGURE 6

Let us denote by O the point opposite to U on the sphere, which is a pseudo-point. We make correspond to any directed pseudo-interval $\overline{AB}$ (with the initial and end points A, B respectively) a great-circle arc OB' (as *spherical characteristic*) in the following way. Let l^* be any pseudo-line not through O and parallel to AB. Further, let AA^*, BB^* be two parallels which cut l^* at the pseudo-points A^*, B^* respectively. Finally, B' marks the pseudo-point at which the parallel to OA^* through B^* intersects the parallel to l^* through O (Fig. 6). By employing Theorem 12 it is easy to prove that B' is independent of the particular construction. *The great-circle arc OB' not containing the point U, which is determined in this way, will be called the "spherical characteristic" of the directed pseudo-interval $\overline{AB}$.* This is at the same time a pseudo-interval. Now we introduce the pseudo-congruence for directed pseudo-intervals by the following:

DEFINITION 2. *Two directed pseudo-intervals will be called "pseudo-equal," or "pseudo-congruent," if and only if the corresponding spherical characteristics are equal as great-circle arcs.*

COROLLARY 1. *The pseudo-congruence of directed pseudo-intervals is reflexive, symmetric, and transitive.*

COROLLARY 2. *If* $AB \parallel A_1B_1$ *and* $AA_1 \parallel BB_1$ (*i.e.* $\overline{A_1B_1}$ *arises by parallel translation from* $\overline{AB}$), *then* $\overline{AB} = \overline{A_1B_1}$.

COROLLARY 3. *If* B *is an interior point of the pseudo-interval* $\overline{AC}$, *we have* $\overline{AB} \neq \overline{AC}$.

COROLLARY 4. *If* B', B'' *are interior points of the pseudo-intervals* $\overline{OC'}$, $\overline{OC''}$ *respectively, where* $\overline{OB'} = \overline{OB''}$ *and* $\overline{OC'} = \overline{OC''}$, *then* $\overline{B'C'} = \overline{B''C''}$.

THEOREM 13. *For any directed pseudo-interval* $\overline{AB}$, *we have* $\overline{AB} = \overline{BA}$.

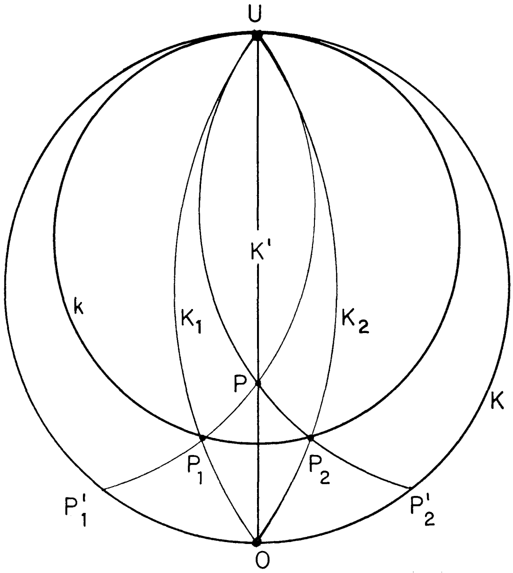

FIGURE 7

Proof. Let the great-circle arc OP_1' be the spherical characteristic of $\overline{AB}$ and let K be the great-circle which contains this great-circle arc (Fig. 7). Thus K goes through the point U. We make the arc OP_2' = the arc OP_1' on this great-circle K, P_2' being different from P_1'. Further, let K' be the great-circle through U and O perpendicular to K and let P be any point of it distinct from U, O. The arcs UPP_1', UPP_2' being touched at the point U by the great-circles K_2, K_1 respectively, K_1 cuts the spherical circle UPP_1' for the second time at P_1, while K_2 intersects the circle UPP_2' again at P_2, say. Then the spherical circle k determined by the points U, P_1, P_2 touches the great-circle K at U, as is obvious by the symmetry on the sphere. Now, the great-circle arc OP_2' is the spherical characteristic of the directed pseudo-interval $\overline{P_1'O}$, since by construction the pseudo-lines P_1P_2, $P_1'O$ and likewise $P_1'P_1$, OP_2, and P_1O, P_2P_2' are parallel. But the arc OP_1' = the arc OP_2' as effected above; thus, the spherical characteristics of $\overline{AB}$, $\overline{P_1'O}$ are equal. Consequently, in the sense of Definition 2 we have $\overline{AB} = \overline{P_1'O}$. On the other hand, $P_1'O = BA$, because (Corollary 2 above) $\overline{P_1'O}$ arises from $\overline{BA}$ by a sequence of parallel translations in the construction of the spherical characteristic OP_1' of $\overline{AB}$. Thus, to summarize, we have $\overline{AB} = \overline{P_1'O} = \overline{BA}$; consequently (Corollary 1 above) $\overline{AB} = \overline{BA}$.

On the basis of this theorem we can explain the congruence for undirected pseudo-intervals by the following:

DEFINITION 3. *Two undirected pseudo-intervals are called "pseudo-equal" or "pseudo-congruent" if and only if by direction they are equal as directed pseudo-intervals.*

It is not difficult to convince oneself of the fact that by this definition the axioms of congruence due to Hilbert (**3**, pp. 11–14) are all valid for the present pseudo-geometry. Thus, for this pseudo-geometry, besides the groups of axioms of incidence, of order, and of parallels of plane I, II, IV of Hilbert, the group III of congruence axioms also subsists. So, *the pseudo-geometry described is truly euclidean plane geometry without the axioms of continuity.*

In this pseudo-euclidean plane geometry we can calculate by directed pseudo-intervals according to the algebra of intervals due to Hilbert (**3**, pp. 60–64; see also **4**, §§34, 35), and the trigonometric functions can be defined in the usual way.

2. The basic formula of spherical trigonometry. Using Hessenberg's pseudo-geometry treated above, we define the trigonometric functions of a spherical angle as those of the corresponding pseudo-angle that is equal to it, independent of the continuity. To derive the basic formula of spherical trigonometry we still need three lemmas.

LEMMA 1. *A great-circle not passing through the point U is a pseudo-circle whose pseudo-centre is the spherical reflection of U in this great-circle.*

This is obvious when U coincides with a pole of the great-circle. In this case the pseudo-centre is the point O and the pseudo-radius is a quadrant joined from O as pseudo-interval.

Now suppose that U is not a pole of the great-circle K. Let U' be the spherical reflection of U in K (Fig. 8). Then U and U' are not opposite points. The great-circle K' through U and U', which is evidently perpendicular to K, intersects K in opposite points A, B say. If P is a fixed point of K, distinct from A, B, consider the arc AP of the spherical circle APU lying on the side of K opposite to U. In the pseudo-triangle APU' we have

$$\angle PAU' = \angle APU'$$

because the spherical circle UPU' is obviously perpendicular to K just as K' and the spherical angles made by K and the arc AP at A, P are equal. Consequently, by the pseudo-geometry,

$$\overline{U'P} = \overline{U'A}.$$

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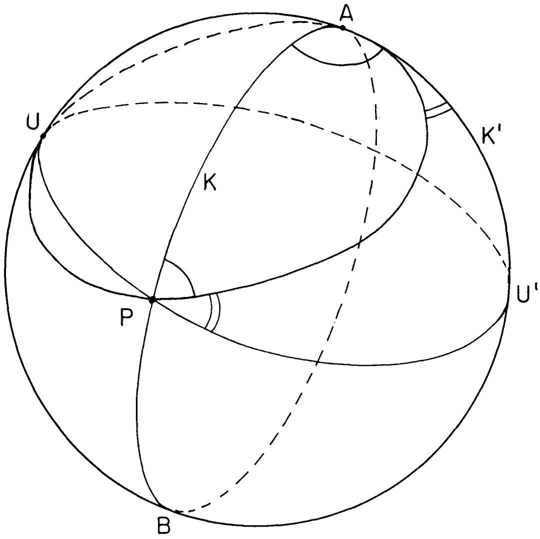

FIGURE 8

By the same argument we get

$$\overline{U'P} = \overline{U'B}$$

and so a fortiori

$$\overline{U'B} = \overline{U'A}.$$

Thus,

$$\overline{U'P} = \overline{U'A} = \overline{U'B},$$

and since any pseudo-line passing through U' intersects the great-circle K (this being a spherical circle which passes through U and U'), Lemma 1 follows.

LEMMA 2. *If P,P' are opposite points on the sphere different from U and we choose the pseudo-radius of the great-circle with the poles U, O as pseudo-unit, then for the pseudo-intervals $\overline{OP}$, $\overline{OP'}$*

$$\overline{OP}.\overline{OP'} = 1.$$

We prove this as follows. Since the points P, P' are opposite (as are O, U), the four points P, O, P', U lie on a great-circle K (Fig. 9). Consider the great-circle K' passing through O, U and perpendicular to K. This cuts the great-circle K_0 with the poles O, U in the opposite points X, X' which are the two poles of K. Consequently, the great-circle $PXP'X'$ is perpendicular to K. By Lemma 1 this great-circle $PXP'X'$ is a pseudo-circle, and so, by the above, is perpendicular to the pseudo-line PP'. Thus, the pseudo-interval $\overline{PP'}$ containing the point O is a diameter of this pseudo-circle $PXP'X'$. Since the spherical angle $\angle POX$ is a right angle, the pseudo-interval $\overline{OX}$ is perpendicular to this diameter $\overline{PP'}$, and so by a familiar theorem of euclidean geometry we have

$$\overline{OP}.\overline{OP'} = \overline{OX}^2.$$

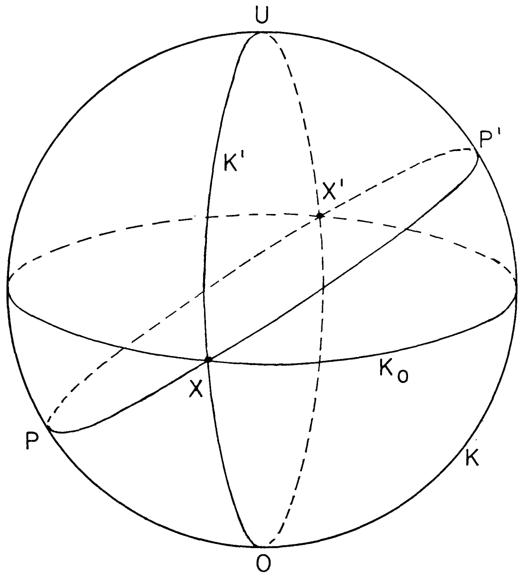

FIGURE 9

But the great-circle K_0, by Lemma 1, is also a pseudo-circle and has the pseudo-centre O. Thus, the pseudo-interval $\overline{OX}$ is the pseudo-radius of K_0 (and the spherical radius as well) and so by our choice $\overline{OX} = 1$; consequently, the last formula completes the proof of Lemma 2.

LEMMA 3. *If we choose the pseudo-radius of the great-circle with the poles U, O as pseudo-unit, then, for any pseudo-interval $\overline{OB}$ corresponding to the great-circle arc $OB = a$,*

$$\overline{OB} = \tan \tfrac{1}{2}a.$$

For, let $\overline{OB}$ as a pseudo-interval be $\overline{OB} = u$ and let the two poles of the great-circle OBU be Y, Y'. Then the great-circle K_0 with the poles U, O evidently passes through Y, Y' and so by our choice, $\overline{OY} = 1$; further, the arc $OB = a$ means that, as a spherical angle, $\angle BYO = a$ (Fig. 10). Let B' be the spherical point opposite to B. Since $\overline{OB} = u$, by Lemma 2 we have $\overline{OB'} = 1/u$. The great-circle $BYB'Y'$ is, by Lemma 1, a pseudo-circle, and it is perpendicular to the great-circle $BOB'U$ because it goes through the poles Y, Y' of the latter. Consequently, the pseudo-interval $\overline{BB'}$ is a diameter of this pseudo-circle $BYB'Y'$. Further, this makes by the pseudo-interval $\overline{OY}$ the pseudo-angle a, and $\overline{OY}$ is perpendicular to the pseudo-diameter $\overline{BB'}$. Hence, if r designates the pseudo-radius of the pseudo-circle $BYB'Y'$, then obviously

$$\sin a = 1/r$$

in all of the three cases $a < \frac{1}{2}\pi$, $a = \frac{1}{2}\pi$, $a > \frac{1}{2}\pi$, where $\frac{1}{2}\pi$ denotes the spherical right-angle (Figs. 11a, 11b, 11c). But we have

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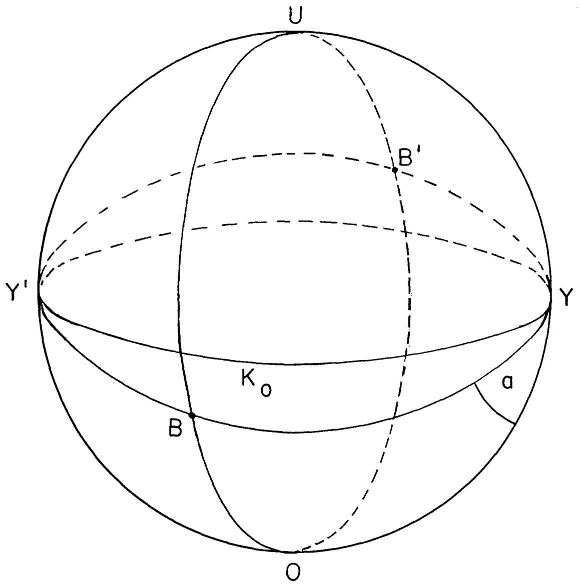

FIGURE 10

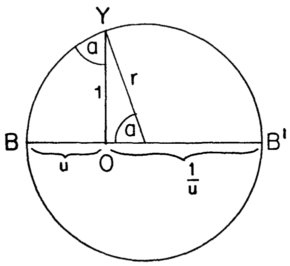

FIGURE 11a

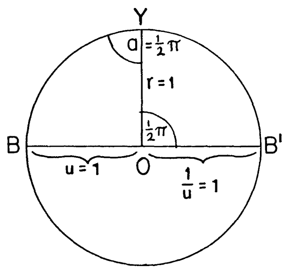

FIGURE 11b

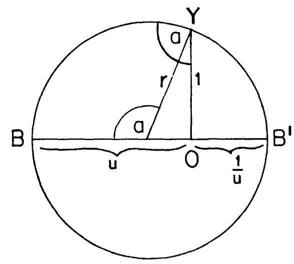

FIGURE 11c

$$2r = \overline{BB'} = \overline{BO} + \overline{OB'} = u + 1/u,$$

and so

$$u + 1/u = 2/(\sin a).$$

This means that u and $1/u$ are the roots of the quadratic equation

$$x^2 - \frac{2}{\sin a}x + 1 = 0,$$

which has, on the other hand, the roots

$$x_1 = \tan \tfrac{1}{2}a, \qquad x_2 = \cot \tfrac{1}{2}a.$$

But we have $\tan \frac{1}{2}a < 1$, $\tan \frac{1}{2}a = 1$, $\tan \frac{1}{2}a > 1$ according as $a < \frac{1}{2}\pi$, $a = \frac{1}{2}\pi$, $a > \frac{1}{2}\pi$ respectively and so $u < 1$, $u = 1$, $u > 1$ respectively. Thus we obtain

$$u = \tan \tfrac{1}{2}a,$$

i.e., $\overline{OB} = \tan \frac{1}{2}a$, and Lemma 3 is proved.

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Now, we are able to derive the basic formula of spherical trigonometry, as expressed by the following:

THEOREM. *If a, b are the sides of a right-angled spherical triangle, then, for the angle α opposite to the side a,*

$$\cot \alpha = \cot a \sin b. \tag{1}$$

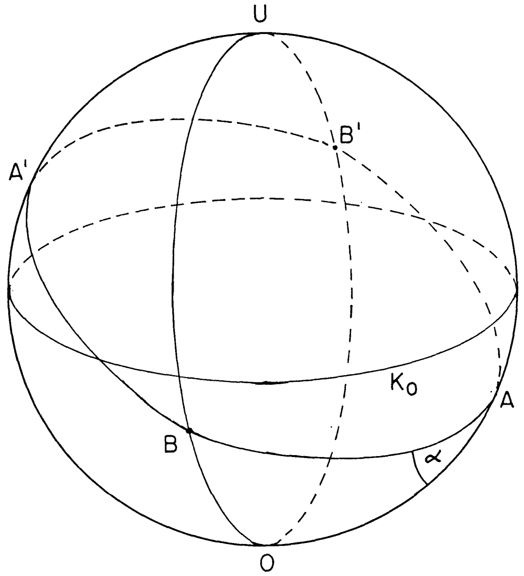

FIGURE 12

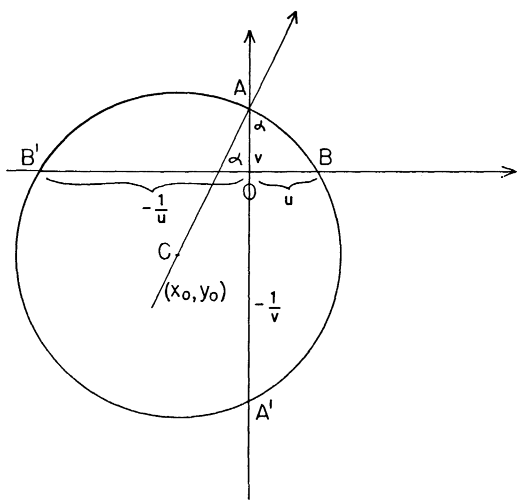

FIGURE 13

Proof. Let OAB be a right-angled spherical triangle in which $\angle AOB$ is the right angle and

$$\text{the arc } OB = a, \qquad \text{the arc } OA = b, \qquad \angle OAB = \alpha.$$

Let u, v be the pseudo-intervals corresponding to the sides a, b respectively, that is to say

$$\overline{OB} = u, \qquad \overline{OA} = v, \tag{2}$$

Further let A', B' be the spherical points opposite to A, B (Fig. 12). By Lemma 1 the great-circle $ABA'B'$ is a pseudo-circle. Let us consider this with respect to the pseudo-coordinate-system, the axes of which are the pseudo-lines OB, OA directed from O towards B, A respectively (Fig. 13) and having as pseudo-unit the pseudo-radius of the great circle K_0 with the poles U, O. By (2), we have

$$\overline{OB'} = -1/u, \qquad \overline{OA'} = -1/v, \tag{3}$$

from Lemma 2. Let x_0, y_0 be the pseudo-coordinates of the pseudo-centre C of this pseudo-circle $ABA'B'$. Since on the sphere we have $\angle OAB = \alpha$, by a simple argument we obtain that the pseudo-line CA directed towards A has the slope angle α. This pseudo-line passes through the pseudo-points $C(x_0, y_0)$, $A(0, v)$; consequently

$$\cot\alpha = x_0/(y_0 - v). \tag{4}$$

But the pseudo-centre $C(x_0, y_0)$ lies on the perpendicular bisectors of the pseudo-chords $\overline{BB'}$, $\overline{AA'}$ and so, by (2), (3),

$$2x_0 = u - 1/u, \qquad 2y_0 = v - 1/v.$$

Thus, by (4), we get

$$\cot\alpha = \frac{u - \dfrac{1}{u}}{v - \dfrac{1}{v} - 2v} = \frac{\dfrac{1}{u} - u}{v + \dfrac{1}{v}}. \tag{5}$$

According to Lemma 3 we have

$$u = \tan \tfrac{1}{2}a, \qquad v = \tan \tfrac{1}{2}b;$$

therefore

$$\frac{1}{u} - u = \cot \tfrac{1}{2}a - \tan \tfrac{1}{2}a = \frac{\cos^2\frac{1}{2}a - \sin^2\frac{1}{2}a}{\sin \frac{1}{2}a \cos \frac{1}{2}a} = 2\cot a,$$

while

$$v + \frac{1}{v} = \tan \tfrac{1}{2}b + \cot \tfrac{1}{2}b = \frac{\sin^2\frac{1}{2}b + \cos^2\frac{1}{2}b}{\cos \frac{1}{2}b \sin \frac{1}{2}b} = \frac{2}{\sin b}$$

and so (5) implies (1).

Formula (1) is the basis of spherical trigonometry.

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Loránd Eötvös University,
Budapest, Hungary