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SUBEXPONENTIAL DISTRIBUTION FUNCTIONS

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Abstract

A distribution function F on $[0, \infty)$ belongs to the subexponential class $\mathcal{S}$ if and only if $1-F^{(2)}(x) \sim 2(1-F(x))$, as $x \to \infty$. For an important class of distribution functions, a simple, necessary and sufficient condition for membership of $\mathcal{S}$ is given. A comparison theorem for membership of $\mathcal{S}$ and also some closure properties of $\mathcal{S}$ are obtained.

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1. Introduction

Throughout this paper all distribution functions will be distribution functions F on $[0, \infty)$ such that $F(0) = 0$, $F(x) < 1$ for all $x > 0$, $F(\infty) = 1$. F is said to belong to the *subexponential class* $\mathcal{S}$ if

$$\lim_{x \to \infty} \frac{1-F^{(2)}(x)}{1-F(x)} = 2,$$

where $F^{(2)}$ is the convolution of F with itself. Subexponential distribution functions are of interest in the theory of branching processes, and in queueing theory; see Athreya and Ney (1972), Chover, Ney and Wainger (1974), Pakes (1975) and Teugels (1975).

We define the function F^c by $F^c(x) = 1-F(x)$. It will sometimes be convenient to denote the convolution of the distribution functions F_X, F_Y by F_{X+Y}, and the convolution of F_X with itself by F_{X+X}. We have then

$$F_{X+Y}(x) = \int_0^\infty F_X(x-y)\,dF_Y(y) = \int_0^\infty F_Y(x-y)\,dF_X(y),$$

and therefore

$$F^c_{X+Y}(x) = \int_0^\infty F^c_X(x-y)\,dF_Y(y) = \int_0^\infty F^c_Y(x-y)\,dF_X(y).$$

Thus

$$\frac{F^c_{X+X}(x)}{F^c_X(x)} = \int_0^\infty \frac{F^c_X(x-y)}{F^c_X(x)}\,dF_X(y) = \int_0^x + \int_x^\infty \frac{1}{F^c_X(x)}\,dF_X(y)$$
$$= \int_0^x \frac{F^c_X(x-y)}{F^c(x)}\,dF_X(y) + 1,$$

and so

(1) $$F \in \mathscr{S} \quad \text{if and only if} \quad \lim_{x\to\infty} \int_0^x \frac{F^c(x-y)}{F^c(x)}\,dF(y) = 1.$$

It is well known (Athreya and Ney (1972), p. 148) that if $F \in \mathscr{S}$,

(2) $$\lim_{x\to\infty} \frac{F^c(x+y)}{F^c(x)} = 1 \quad \text{for all } y.$$

The class of distribution functions for which (2) is true is denoted by $\mathscr{L}$, and so $\mathscr{L} \supset \mathscr{S}$. If $F \in \mathscr{L}$, $F^c(\log x)$ is a slowly varying function of x at ∞, because for $k > 0$, $F^c(\log kx)/F^c(\log x) \to 1$ as $x \to \infty$. Hence for $\alpha > 0$, $x^\alpha F^c(\log x) \to \infty$ as $x \to \infty$. Replacing x by e^x, we obtain $e^{-\alpha x}/F^c(x) \to 0$. It is this property that suggested the name subexponential; but as all members of $\mathscr{L}$ possess it, it would be logical to call all distribution functions in $\mathscr{L}$ subexponential. However, the name has been restricted to the subclass $\mathscr{S}$. Note that if we define the tail function G by

$$G(x) = F(x), \quad x < 0,$$
$$= 1 - F(x), \quad x \geqslant 0,$$

we may write (1) as

$$F \in \mathscr{S} \quad \text{if and only if} \quad \lim_{x\to\infty} \int_0^\infty \frac{G(x-y)}{G(x)}\,dF(y) = 1.$$

Since

$$\int_0^\infty \left| \frac{G(x-y)}{G(x)} - 1 \right| dF(y) = \int_0^x \frac{G(x-y)}{G(x)}\,dF(y) - \int_0^x dF(y) + \int_x^\infty dF(y),$$

it is evident that $F \in \mathscr{S}$ if and only if $G(x-y)/G(x) \to 1$ in mean F, as $x \to \infty$. The requirement for membership of $\mathscr{L}$ is the weaker $G(x-y)/G(x) \to 1$ everywhere as $x \to \infty$. Note that

$$\frac{G_{X+X}(x)}{G_X(x)} - 1 = \int_0^x \frac{G_X(x-y)}{G_X(x)}\,dF_X(y) \geqslant \int_0^x dF_X(y) = F_X(x),$$

which $\to 1$ as $x \to \infty$. Hence

$$\liminf_{x\to\infty} \frac{G_{X+X}(x)}{G_X(x)} \geqslant 2.$$

(This result can also be found in Chistyakov (1964) and in Pakes (1975), equation 8.)

2.

THEOREM I. *If $F_X \in \mathscr{L}$, and $G_Y(x)/G_X(x) \to c$, as $x \to \infty$, then*

$$\left\{\frac{G_{X+Y}(x)}{G_X(x)} - 1\right\} \Big/ \left\{\frac{G_{X+X}(x)}{G_X(x)} - 1\right\} \to c, \quad \text{as } x \to \infty, \tag{3}$$

and, if $c > 0$,

$$\frac{G_{Y+Y}(x)}{G_Y(x)} - 2 = (c+\eta_1)\left\{\frac{G_{X+X}(x)}{G_X(x)} - 2\right\} + \eta_2, \tag{4}$$

where $\eta_1, \eta_2 \to 0$ as $x \to \infty$.

COROLLARY 1. *If $F_X \in \mathscr{S}$, $G_Y(x) \sim cG_X(x)$, $x \to \infty$, $c > 0$, then $F_Y \in \mathscr{S}$.*

COROLLARY 2. *If $F_X \in \mathscr{S}$, $G_Y(x) = o\{G_X(x)\}$, $x \to \infty$, then $G_{X+Y}(x) \sim G_X(x)$, $x \to \infty$, and $F_{X+Y} \in \mathscr{S}$.*

PROOF.

$$\frac{G_{X+Y}(x)}{G_X(x)} - 1 = \int_0^x \frac{G_Y(x-y)}{G_X(x)}\, dF_X(y) = \int_0^{x-A} + \int_{x-A}^x.$$

The last integral is

$$\leqslant \int_{x-A}^x \frac{dF_X(y)}{G_X(x)} = \frac{G_X(x-A) - G_X(x)}{G_X(x)}, \quad \text{which } \to 0 \text{ as } x \to \infty.$$

Thus

$$\frac{G_{X+Y}(x)}{G_X(x)} - 1 - \int_0^{x-A} \frac{G_Y(x-y)}{G_X(x)}\, dF_X(y) \to 0 \quad \text{as } x \to \infty.$$

If $\varepsilon > 0$, and $0 < y < x - A$, then when A is sufficiently great,

$$c - \varepsilon \leqslant \frac{G_Y(x-y)}{G_X(x-y)} \leqslant c + \varepsilon,$$

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$$(c-\varepsilon)\int_0^{x-A}\frac{G_X(x-y)}{G_X(x)}\,dF_X(y)\leqslant\int_0^{x-A}\frac{G_Y(x-y)}{G_X(x)}\,dF_X(y)$$
$$\leqslant(c+\varepsilon)\int_0^{x-A}\frac{G_X(x-y)}{G_X(x)}\,dF_X(y).$$

$$(c-\varepsilon)\left\{\frac{G_{X+X}(x)}{G_X(x)}-1-\eta_1\right\}\leqslant\frac{G_{X+Y}(x)}{G_X(x)}-1-\eta_2$$
$$\leqslant(c+\varepsilon)\left\{\frac{G_{X+X}(x)}{G_X(x)}-1-\eta_1\right\},$$

where $\eta_1,\eta_2\to 0$ as $x\to\infty$. Therefore for any $\varepsilon>0$, when x is great,

$$(c-2\varepsilon)\left\{\frac{G_{X+X}(x)}{G_X(x)}-1\right\}\leqslant\frac{G_{X+Y}(x)}{G_X(x)}-1\leqslant(c+2\varepsilon)\left\{\frac{G_{X+X}(x)}{G_X(x)}-1\right\}.$$

This proves (3).

Let $K_1,K_2,\ldots$ denote functions of x which $\to 0$ as $x\to\infty$.

$$G_{X+Y}/G_X-1=(c+K_1)(G_{X+X}/G_X-1).$$

Hence

$$\frac{G_{X+Y}-G_X-G_Y}{G_X}=(c+K_1)(G_{X+X}/G_X-2)+K_2.$$

Similarly, if $c>0$,

$$\frac{G_{X+Y}-G_X-G_Y}{G_Y}=(c^{-1}+K_3)(G_{Y+Y}/G_Y-2)+K_4.$$

Combining these, we obtain

$$\text{(5)}\qquad \frac{G_{Y+Y}}{G_Y}-2=\frac{(c+K_1)G_X}{(c^{-1}+K_3)G_Y}\left\{\frac{G_{X+X}}{G_X}-2\right\}+\frac{K_2G_X}{G_Y}-\frac{K_4}{c^{-1}+K_3},$$
$$=(c+\eta_1)\left\{\frac{G_{X+X}}{G_X}-2\right\}+\eta_2,$$

where $\eta_1,\eta_2\to 0$ as $x\to\infty$.

If $F_X\in\mathscr{S}$, the right side of (5) $\to 0$ as $x\to\infty$, and so $G_{Y+Y}(x)/G_Y(x)\to 2$. $F_Y\in\mathscr{S}$. This proves Corollary 1. This result was given in Pakes (1975), and the particular case, $c=1$, in Tengels (1975). Corollary 2 follows immediately from (3). //

3.

It turns out that the theory is simpler in terms of the logarithms of tail functions. For any tail function G, we shall write $g=-\log G$, $G=e^{-g}$. Thus g is a

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nondecreasing function of x such that $g(0) = 0$. $g(\infty) = \infty$, and we shall reserve the symbols g, g_1, etc. for such functions. The set of g functions corresponding to distribution functions in $\mathscr{L}$ will be denoted by $\mathscr{H}$.

$$\mathscr{H} = \{g;\ 1-e^{-g} \in \mathscr{L}\}.$$

We also define

$$\mathscr{K} = \{g;\ 1-e^{-g} \in \mathscr{S}\}.$$

Note that $g \in \mathscr{H}$ if and only if, for every a, $e^{-g(x)}/e^{-g(x+a)} \to 1$ as $x \to \infty$, that is if and only if

$$g(x+a)-g(x) \to 0.$$

Obviously $g \in \mathscr{H}$, $g_1(x)-g(x) \to 0$ as $x \to \infty \Rightarrow g_1 \in \mathscr{H}$. We shall say that the functions g, g_1 are *equivalent*, and write $g \leftrightarrow g_1$. It follows from Corollary 1 above that $g \in \mathscr{K}$, $g \leftrightarrow g_1 \Rightarrow g_1 \in \mathscr{K}$.

If $g \in \mathscr{H}$, and $\lim_{x\to\infty} g'(x)$ exists, this limit must be 0. Also, if g is any function in $\mathscr{H}$, we can construct a function g_1, which is equivalent to g, and which has a continuous derivative g_1' with limit 0 at ∞. Define g_0 by $g_0(x) = g(x)$ at $x = 0, 1, 2, \ldots$, and g_0 linear in $[n-1, n]$, $n = 1, 2, \ldots$. Clearly $g_0 \leftrightarrow g$, and in the set of points at which it exists, $g_0'(x) \to 0$ as $x \to \infty$. We obtain g_1 from g_0 by rounding off the corners, if any, at the points $x = 1, 2, \ldots$, by circular arcs. Thus, $\mathscr{H}$ consists of those g with a continuous derivative g' which $\to 0$ at ∞, and their equivalents. We shall therefore consider only g having a continuous derivative g' with limit 0 at ∞. If $G = e^{-g}$, the density function of the distribution is $f = -G' = e^{-g}g'$. Denoting the tail function of $F^{(2)}$ by $G^{(2)}$, we have

$$\begin{aligned}\frac{G^{(2)}(x)}{G(x)} - 1 &= \int_0^x \frac{G(x-y)}{G(x)}\,dF(y)\\ &= \int_0^x \exp\{g(x)-g(x-y)-g(y)\}\,g'(y)\,dy.\end{aligned}$$

THEOREM II. *If g has a derivative g' which eventually $\downarrow 0$, a necessary and sufficient condition for $g \in \mathscr{K}$ is*

(6) $$\lim_{x\to\infty} \int_0^x \exp\{yg'(x)-g(y)\}\,g'(y)\,dy = 1,$$

and a sufficient condition is

(7) $$\exp\{yg'(y)-g(y)\}\,g'(y)$$

integrable over $[0, \infty]$.

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PROOF. If g' is not monotonic over the whole range $[0, \infty]$, there is an equivalent g_0 with a derivative g_0' which is so. We may therefore assume that g' is nonincreasing over the whole range.

$$\frac{G^{(2)}(x)}{G(x)} - 1 = \int_0^x \exp\{g(x) - g(x-y) - g(y)\}\, g'(y)\, dy$$
$$\geqslant \int_0^x \exp\{yg'(x) - g(y)\}\, g'(y)\, dy$$
$$\geqslant \int_0^x \exp\{-g(y)\}\, g'(y)\, dy = F(x).$$

This shows that the condition (6) is necessary, since if $g \in \mathscr{K}$, the first and the last $\to 1$ as $x \to \infty$.

$$\int_0^x \exp\{g(x) - g(x-y) - g(y)\}\, g'(y)\, dy$$
$$= \int_0^{\frac{1}{2}x} + \int_{\frac{1}{2}x}^x$$
$$= \int_0^{\frac{1}{2}x} \exp\{g(x) - g(x-y) - g(y)\}\, g'(y)\, dy$$
$$+ \int_0^{\frac{1}{2}x} \exp\{g(x) - g(x-y) - g(y)\}\, g'(x-y)\, dy.$$

The first integral is $\geqslant F(\frac{1}{2}x)$ which $\to 1$ as $x \to \infty$. On the other hand, $y \leqslant \frac{1}{2}x$, and therefore $x - y \geqslant \frac{1}{2}x$, $g(x) - g(x-y) \leqslant yg'(x-y) \leqslant yg'(\frac{1}{2}x)$. Thus the first integral is $\leqslant \int_0^{\frac{1}{2}x} \exp\{yg'(\frac{1}{2}x) - g(y)\}\, g'(y)\, dy$, which $\to 1$ as $x \to \infty$ if (6) is true. The first integral then $\to 1$. Moreover, as $x \to \infty$, the first integrand $\to e^{-g(y)} g'(y) = f(y)$ everywhere, and the integral $\to 1 = \int_0^\infty f(y)\, dy$. Thus the first integrand coverges in mean to $f(y)$. The second integrand $\to 0$ everywhere. It is dominated by the first integrand since $g'(x-y) \leqslant g'(y)$. Therefore the second integral $\to 0$ as $x \to \infty$, and $G^{(2)}(x)/G(x) - 1 \to 1$; $g \in \mathscr{K}$. The second part of the theorem follows by dominated convergence, since $g'(x) \leqslant g'(y)$. //

EXAMPLE. Suppose $G(x) \sim \exp\{-x(\log x)^{-m}\}$, $m > 0$, $x \to \infty$. We may take

$$g(x) = x(\log x)^{-m} \quad \text{when } x \text{ is great,}$$
$$g'(x) = (\log x)^{-m} - m(\log x)^{-m-1}.$$

When y is great

$$\exp\{yg'(y) - g(y)\}\, g'(y) = \exp\{-my(\log y)^{-m-1}\}\, \{(\log y)^{-m} - m(\log y)^{-m-1}\}$$

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and is therefore integrable over $[0, \infty]$. Therefore $g \in \mathscr{K}$, $F \in \mathscr{S}$. Teugels (1975), p. 1001, states that $F \in \mathscr{S}$ if and only if $m > 1$.

The necessary condition (6) enables us to define distribution functions which belong to $\mathscr{L}$ but not to $\mathscr{S}$. Thus $\mathscr{S}$ is a proper subset of $\mathscr{L}$. Let (x_n) be an increasing sequence of numbers, to be defined later, with $x_0 = 0$. Define g by $g(x_0) = g(0) = 0$; g is continuous and piecewise linear so that for $x_{n-1} < x < x_n$, $g'(x) = 1/n$. Consider

$$\int_0^{x_n} \exp\{yg'(x_n) - g(y)\}\, g'(y)\, dy > \int_{x_{n-1}}^{x_n} .$$

For $x_{n-1} < y < x_n$,

$$yg'(x_n) - g(y) = y/n - \{g(x_{n-1}) + n^{-1}(y - x_{n-1})\} > -g(x_{n-1}),$$

and $g'(y) = n^{-1}$. Therefore

$$\int_{x_{n-1}}^{x_n} \exp\{yg'(x_n) - g(y)\}\, g'(y)\, dy$$

$$> \int_{x_{n-1}}^{x_n} \exp\{-g(x_{n-1})\}\, n^{-1}\, dy$$

$$= \exp\{-g(x_{n-1})\}\, (x_n - x_{n-1})/n.$$

Choose the x_n so that

$$\exp\{-g(x_{n-1})\}\, (x_n - x_{n-1})/n = 2,$$

$$(x_n - x_{n-1}) = 2n \exp\{g(x_{n-1})\}.$$

We then have

$$g(x_n) = g(x_{n-1}) + (x_n - x_{n-1})/n = g(x_{n-1}) + 2 \exp g(x_{n-1}),$$

$$x_0 = 0, \quad g(x_0) = 0, \quad x_1 = 2, \quad g(x_1) = 2, \ldots.$$

Clearly $g(x) \uparrow \infty$ as $x \uparrow \infty$. Also $g'(x) \downarrow 0$, and so $g \in \mathscr{H}$. However,

$$\int_0^{x_n} \exp\{yg'(x_n) - g(y)\}\, g'(y)\, dy > 2,$$

and so

$$\int_0^{x} \exp\{yg'(x) - g(y)\}\, g'(y)\, dy \text{ does not} \to 1 \text{ as } x \to \infty.$$

Thus $g \in \mathscr{H}$ but $g \notin \mathscr{K}$.

The following theorem shows how the general case may often be reduced to the case $g'(x) \downarrow 0$. We need to consider only distribution functions F with continuous derivatives f.

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THEOREM III. *If f_2/f_1 is bounded, and G_2/G_1 bounded away from* 0, *then*

$$F_1 \in \mathscr{L} \Rightarrow F_2 \in \mathscr{L} \quad \textit{and} \quad F_1 \in \mathscr{S} \Rightarrow F_2 \in \mathscr{S}.$$

PROOF. Suppose $f_2/f_1 < C < \infty$, $G_2/G_1 > c > 0$, then $c < G_2/G_1 < C$.

$$0 \leqslant \frac{G_2(x-y)-G_2(x)}{G_2(x)} \leqslant \frac{C}{c} \frac{G_1(x-y)-G_1(x)}{G_1(x)}.$$

If $F_1 \in \mathscr{L}$, the last $\to 0$ as $x \to \infty$. Therefore so does the other. $G_2(x-y)/G_2(x) \to 1$, and $F_2 \in \mathscr{L}$.

$$F_1 \in \mathscr{S} \Rightarrow F_1 \in \mathscr{L} \Rightarrow F_2 \in \mathscr{L}.$$

Hence

$$\frac{G_2(x-y)}{G_2(x)} f_2(y) \to f_2(y), \quad \text{as } x \to \infty.$$

Also

$$\frac{G_2(x-y)}{G_2(x)} f_2(y) \leqslant \frac{C^2}{c} \frac{G_1(x-y)}{G_1(x)} f_1(y),$$

which converges in mean to $C^2 c^{-1} f_1(y)$. Therefore $(G_2(x-y)/G_2(x)) f_2(y)$ converges in mean to $f_2(y)$, and

$$\int_0^x \frac{G_2(x-y)}{G_2(x)} f_2(y)\, dy \to 1. \qquad //$$

In terms of the g functions we may state the corollary : if $g_2 - g_1$ and g_2'/g_1' are both bounded, $g_1 \in \mathscr{H} \Rightarrow g_2 \in \mathscr{H}$, $g_1 \in \mathscr{K} \Rightarrow g_2 \in \mathscr{K}$.

EXAMPLE. Consider the case, when x is great

$$g_1(x) = x/\log x, \quad g_1'(x) = 1/\log x - 1/(\log x)^2,$$

$$g_2(x) = x/\log x + \sin(x/\log x),$$

$$g_2'(x) = \{(1/\log x - 1/(\log x)^2\}\{1+\cos(x/\log x)\}.$$

The derivative $g_2'(x)$ is zero when $x/\log x$ is an odd multiple of π and positive everywhere else. It is not monotonic in any infinite interval.

$$g_2(x) - g_1(x) = \sin(x/\log x), \quad g_2'(x)/g_1'(x) = 1 + \cos(x/\log x),$$

which are both bounded. As shown above, $g_1 \in \mathscr{K}$, and so $g_2 \in \mathscr{K}$.

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THEOREM IV. *If f_Y/f_X is bounded, then*

$$0 < p < 1, \quad F_X \in \mathscr{S} \Rightarrow pF_X+(1-p)F_Y \in \mathscr{S}, \tag{8}$$

$$F_X \in \mathscr{S} \Rightarrow F_{X+Y} \in \mathscr{S}. \tag{9}$$

PROOF. If $F = pF_X+(1-p)F_Y, f = pf_X+(1-p)f_Y$.

$$p \leqslant \frac{pf_X+(1-p)f_Y}{f_X} = \frac{f}{f_X} = p+(1-p)f_Y/f_X.$$

Thus f/f_X is bounded away from 0 and from ∞. The conditions of Theorem III are fulfilled, and $F_X \in \mathscr{S} \Rightarrow F \in \mathscr{S}$.

Suppose $f_Y \leqslant Cf_X$.

$$f_{X+Y}(x) = \int_0^x f_Y(x-y)f_X(y)\,dy \leqslant C\int_0^x f_X(x-y)f_X(y)\,dy = Cf_{X+X}(x),$$

$$f_{X+Y}(x)/f_{X+X}(x) \leqslant C.$$

If $F_X \in \mathscr{S}$, $G_{X+X}(x)/G_X(x) \to 2$ as $x \to \infty$. Therefore G_{X+X}/G_X is bounded, and G_{X+X}/G_{X+Y} also, since $G_{X+Y} \geqslant G_X$. Thus G_{X+Y}/G_{X+X} is bounded away from 0, and f_{X+Y}/f_{X+X} is bounded. $F_X \in \mathscr{S} \Rightarrow F_{X+X} \in \mathscr{S} \Rightarrow F_{X+Y} \in \mathscr{S}$. //

THEOREM V.

$$g_1, g_2 \in \mathscr{K} \Rightarrow g_1+g_2 \in \mathscr{K}. \tag{10}$$

$$m > 1, g \in \mathscr{K} \Rightarrow mg \in \mathscr{K}. \tag{11}$$

PROOF. If $0 < u < x$,

$$\begin{aligned}\int_0^u \exp\{g(x)-g(x-y)-g(y)\}\,g'(y)\,dy - \int_{x-u}^x \\ = \int_0^u \exp\{g(x)-g(x-y)-g(y)\}\,\{g'(y)-g'(x-y)\}\,dy \\ = 1-\exp\{g(x)-g(x-u)-g(u)\}.\end{aligned}$$

$$\begin{aligned}&\exp\{g(x)-g(x-u)-g(u)\} \\ &\quad = 1+\int_{x-u}^x \exp\{g(x)-g(x-y)-g(y)\}\,g'(y)\,dy - \int_0^u \\ &\quad \leqslant 1+\int_0^x \exp\{g(x)-g(x-y)-g(y)\}\,g'(y)\,dy,\end{aligned} \tag{12}$$

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which $\to 2$ as $x \to \infty$ if $g \in \mathscr{K}$, and so is bounded. Therefore $g(x)-g(x-u)-g(u)$ is bounded for all x, and all $u \leqslant x$.

Suppose $g_1, g_2 \in \mathscr{K}$, $g = g_1 + g_2$.

$$\int_0^x \exp\{g(x)-g(x-y)-g(y)\}\, g'(y)\, dy = \int_0^x I(x,y)\, dy,$$

where

$$\begin{aligned} I(x,y) = \exp\{&g_1(x)-g_1(x-y)-g_1(y)+g_2(x)-g_2(x-y)-g_2(y)\} \\ &\times \{g_1'(y)+g_2'(y)\} \leqslant \exp\{g_1(x)-g_1(x-y)-g_1(y)+C_2\}\, g_1'(y) \\ &+ \exp\{C_1+g_2(x)-g_2(x-y)-g_2(y)\}\, g_2'(y), \end{aligned}$$

which converges in mean to $\exp\{-g_1(y)+C_2\}\, g_1'(y) + \exp\{C_1 - g_2(y)\}\, g_2'(y)$, as $x \to \infty$.

Also $I(x,y) \to e^{-g(y)} g'(y)$. Hence I converges in mean, and $\int_0^x I(x,y)\, dy \to 1$. $g \in \mathscr{K}$.

If u, $x-u$ are both $\geqslant A > 0$, we have from (12)

$$\begin{aligned} &\exp\{g(x)-g(x-u)-g(u)\} \\ &\qquad \leqslant 1 + \int_A^x \exp\{g(x)-g(x-y)-g(y)\}\, g'(y)\, dy - \int_0^A \\ &\qquad = 1 + \int_0^x - 2\int_0^A, \end{aligned}$$

which $\to 2 - 2F(A) = 2G(A)$ if $g \in \mathscr{K}$. Choose A so that $G(A) < \frac{1}{2}$, then $\exp\{g(x)-g(x-u)-g(u)\} < 1$ when $x-u$, $u \geqslant A$, and x is great. Consider

$$\int_A^{x-A} \exp\{mg(x)-mg(x-y)-mg(y)\}\, mg'(y)\, dy.$$

When x is great, the integrand $< \exp\{g(x)-g(x-y)-g(y)\}\, mg'(y)$, which converges in mean. Therefore, as $x \to \infty$,

$$\begin{aligned} &\int_A^{x-A} \exp\{mg(x)-mg(x-y)-mg(y)\}\, mg'(y)\, dy \\ &\qquad\qquad \to \int_A^\infty \exp\{-mg(y)\}\, mg'(y) = \exp\{-mg(A)\}. \end{aligned}$$

As shown above, $\int_{x-A}^x \to 0$, and it may easily be shown by dominated convergence that $\int_0^A \to 1 - \exp\{-mg(A)\}$. Thus

$$\int_0^x \exp\{mg(x)-mg(x-y)-mg(y)\}\, mg'(y)\, dy \to 1,$$

so that $mg \in \mathscr{K}$. //

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If $\mathscr{S}'$ denotes the set of tail functions G corresponding to distribution functions F in $\mathscr{S}$, the above results may be written

$$G_1, G_2 \in \mathscr{S}' \Rightarrow G_1 G_2 \in \mathscr{S}',$$

$$m > 1, \quad G \in \mathscr{S}' \Rightarrow G^m \in \mathscr{S}'.$$

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