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ON THE PRIMITIVITY OF THE GROUP ALGEBRA

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Let G be a group and F a field of arbitrary characteristic. In [**4**] Kaplansky asks under what conditions is $F[G]$ primitive, where $F[G]$ is the group algebra of G over F. We give some necessary conditions on G that $F[G]$ be primitive and propose a conjecture.

Definition. A ring R is primitive if it has a faithful irreducible right module.

The above should really be considered as a definition of right primitive. One can analogously define left primitive and the two properties are not equivalent. For our purposes, the two concepts are equivalent, for the group algebra possesses a nice involution.

If we assume that $F[G]$ is primitive, there are some immediate restrictions on G. First of all G cannot be Abelian since the only primitive commutative rings are fields. (I exclude of course the case when G consists of one element.) Secondly, the group G cannot be finite since in that case the Density Theorem [**2**, Theorem 2.12] would imply that $F[G]$ be simple, but the augmentation ideal belies that. (Again I exclude the trivial case.) Our first goals will be to strengthen these two results.

It is well known that a primitive ring is prime and [**1**, Theorem 8] tells us that $F[G]$ is prime if and only if it has no nontrivial finite normal subgroups. This shows that if P is any property of G that makes $F[G]$ primitive, that property is lost upon taking the direct product with $\mathbf{Z}/2\mathbf{Z}$.

By the Density Theorem we know that if $F[G]$ is primitive, there is a division ring Δ such that for every integer m there is a subring $S_m \subset F[G]$ and an epimorphism α_m of S_m onto Δ_m, the ring of $m \times m$ matrices over Δ. But by [**2**, Lemma 6.3.1], Δ_m does not satisfy a polynomial identity of degree less than $2m$. Since m is arbitrary and any polynomial identity satisfied by $F[G]$ would also be satisfied by any subring, we see that $F[G]$ satisfies no polynomial identity. But it is easy to show that if $[G:Z(G)] = n$, where $Z(G)$ is the centre of the group G, then $F[G]$ would satisfy a standard polynomial identity. One can also see this from a result in [**7**, p. 443] which says that if $[G:Z(G)] = n$ then its commutator subgroup is finite. Theorem 1 and Theorem 2 strengthen this result in two directions.

THEOREM 1. *If $F[G]$ is primitive, then G has no Abelian subgroup of finite index.*

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Proof. If G did have an Abelian subgroup H of finite index then it would have a normal Abelian subgroup, K, of finite index. For H must only have a finite number of conjugates and so we need only let K be the intersection of those conjugates. But then [**5**, Theorem 1] tells us that G satisfies the P_n condition for some integer n, i.e., for any n elements $g_1, \ldots, g_n$ of G, the $n!/2$ products $g_{i_1}g_{i_2}\ldots g_{i_n}$ obtained from all even permutations is identical with the $n!/2$ products obtained from the odd permutations. But the fact that G has property P_n is equivalent to the group ring $F[G]$ (for F of characteristic 0) satisfying the standard polynomial identity

$$[x_1, x_2, \ldots, x_n] = 0$$

Passman has pointed out that if $[G:H] = n$, then $F[G]$ can be embedded in E_n where E is the commutative ring $F[H]$ and so $K[G]$ satisfies a standard polynomial identity, regardless of the characteristic of F. This completes the proof.

Again let us assume that $F[G]$ is primitive. Then as a consequence of the definition there exists a maximal right ideal, ρ, of $F[G]$ such that

$$(\rho:R) \equiv \{r \in R | Rr \subset \rho\} = 0.$$

Note that $(\rho:R)$ is the largest two sided ideal contained in ρ. Let

$$H = \{\sigma \in G | \sigma\rho \subset \rho\}.$$

THEOREM 2. *H is a subgroup of G and $[G:H] = \infty$.*

Proof. First we show that H is a subgroup. Clearly if $\sigma, \tau \in H$, then $\sigma\tau \in H$. Now suppose $\sigma \in H$. Hence

$$\sigma\rho \subseteq \rho$$
$$\rho \subseteq \sigma^{-1}\rho.$$

But $\sigma^{-1}\rho$ is a right ideal and thus $\rho = \sigma^{-1}\rho$, by maximality of ρ.

Now if H is of finite index, group theory tells us that there exists a $K \subset H$ with K normal of finite index. Let $\{\psi_i\}$, $1 \leqq i \leqq k$, be a set of left coset representatives of K. They will also be right coset representatives since K is normal. Let

$$A = \bigcap_{i=1}^{k} \psi_i \rho \psi_i^{-1}$$

A is clearly a right ideal contained in ρ. I claim that it is also a left ideal; for suppose $a \in A$ and $\sigma \in G$ is such that $\sigma a \notin A$. Then we can write $\sigma = k\psi_i$, $k \in K$, and $\sigma a \notin \psi_j \rho \psi_j^{-1}$, for some i, j. Since $a \in A$, we can also write $a = \psi_i^{-1}\psi_j p \psi_j^{-1}\psi_i$. Hence

$$\begin{aligned}\sigma a &= k\psi_i\psi_i^{-1}\psi_j p\psi_j^{-1}\psi_i = k\psi_j p\psi_j^{-1}\psi_i \\ &= k\psi_j p \psi_j^{-1}\psi_i\psi_j\psi_j^{-1} = k\psi_j p'\psi_j^{-1} \\ &= \psi_j k' p' \psi_j^{-1} = \psi_j p'' \psi_j^{-1}, \qquad & k, k' \in K \\ & & p, p', p'' \in \rho.\end{aligned}$$

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Hence $\sigma a \in \psi_j \rho \psi_j^{-1}$. If now we could show that $A \neq (0)$, we would have a contradiction and we would be through, but the following result on group algebras, the proof of which was suggested by D. Passman supplies that result.

LEMMA. *If $\rho_1, \ldots, \rho_n$ are maximal right ideals of the group algebra, $F[G]$, and G is infinite, then $\bigcap_{i=1}^{n} \rho_i \neq (0)$.*

Proof. Suppose $(0) = \bigcap_{i=1}^{n} \rho_i$. Then as $F[G]$ modules

$$F[G] \subseteq \oplus \sum_{i=1}^{n} F[G]/\rho_i.$$

Since the module on the right is completely reducible, this implies that $F[G]$ is completely reducible and hence, by a corollary in [**1**], G is finite. This completes the proof of the lemma, and the proof of Theorem 2.

Note that if H is of finite index in G and $F[H]$ is primitive, we cannot conclude that $F[G]$ is primitive, because the group algebra of the trivial group is primitive, being a field, but the group algebra of any other finite group cannot be primitive as we have seen. Is this the only exception? Theorem 3 is a partial answer to this question. The following result is attributed to Higman and a full proof appears in [**6**].

LEMMA. *Let G be a group and H a subgroup of finite index. If an exact sequence of $F[G]$ modules splits as a sequence of $F[H]$ modules, then it also splits as a sequence of $F[G]$ modules.*

Sketch of proof. Since H contains a normal subgroup of finite index we may assume without loss of generality that H is normal, and the result follows by the usual Maschke averaging process.

THEOREM 3. *If $[G:H] < \infty$ and $F[H]$ is primitive and G has no nontrivial finite normal subgroups then $F[G]$ is primitive.*

Proof. Let M be a faithful irreducible right $F[H]$ module. Consider the right $F[G]$ module

$$W = M \otimes_{F[H]} F[G].$$

Let $\{\sigma_i\}^n{}_{i=1}$ be a set of right coset representatives of H in G. Now $F[G]$ is a free left $F[H]$ module with $\{\sigma_i\}^n{}_{i=1}$ as a basis; so

$$W = \sum M \otimes \sigma_i.$$

Now M can be made into a left $F[H]$ module by

$$km = mk^* \quad m \in M, k \in F[H]$$

where * indicates the standard involution in the group algebra. It is easy to see that M is also faithful and irreducible on the left. Hence $M \otimes \sigma_i$ is a left $F[H]$ module. I claim that it is irreducible; for if Z is a nontrivial submodule of $M \otimes \sigma_i$,

$$\bar{Z} = \{m \in M \mid m \otimes \sigma_i \in Z\}$$

can be seen to be a nontrivial proper submodule of M.

Hence W is completely reducible as a $F[H]$ module. Therefore, if U is a $F[G]$ submodule of W, we know that the exact sequence

$$0 \to U \to W \to W/U \to 0$$

splits as a sequence of $F[H]$ modules. But now, by the above lemma, it splits as a sequence of $F[G]$ modules. Thus the lattice of $F[G]$ submodules of W is complemented, and so W is completely reducible as a $F[G]$ module.

W is also faithful as an $F[G]$ module, for if $\sum k_i\sigma_i$ annihilates W, $k_i \in F[H]$, let $m \neq 0 \in M$. Then

$$(m \otimes 1) \sum k_i\sigma_i = 0,$$
$$\sum mk_i \otimes \sigma_i = 0.$$

But this means that $mk_i = 0$ for all i, which in turn implies that $k_i = 0$ for all i, since m was arbitrary and M is faithful.

Hence $W = \bigoplus V_j$, where the V_j are irreducible $F[G]$ modules. Let $A_j = \text{Ann } V_j$. Then the A_j are two-sided ideals of $F[G]$ and $\bigcap A_j = (0)$, since W is faithful. But since we are assuming that G has no nontrivial normal subgroups we must have that $F[G]$ is prime. But then $\bigcap A_j = (0)$ implies that some $A_j = (0)$. But then V_j would be a faithful irreducible $F[G]$ module and so $F[G]$ would be primitive. This completes the proof.

Again suppose that $R = F[G]$ is primitive and let ρ be a maximal right ideal containing no nontrivial two-sided ideals. Let $M = R/\rho$. By Schur's Lemma, $\Delta = \text{End}_R\,(M)$ is a division ring and we can consider M as a right vector space over Δ. The Density Theorem tells us that R is a dense ring of $\text{End}_\Delta\,(M)$. [**3**, Theorem I, p. 25] tells us what Δ looks like. Let $S = \{r \in R | r\rho \subseteq \rho\}$. Note that $\rho \subset S$. Define a map from S into Δ by sending $s \in S$ into left multiplication by s. This is easily seen to be a homomorphism with kernel ρ. We will show that it is onto.

Suppose $\delta \in \Delta$ and let $(\bar{1})\ \delta = \bar{a}$. (Henceforth $^{-}$ will denote congruence class modulo ρ). Then if $\bar{x} \in M$

$$(\bar{x})\delta = (\bar{1}x)\delta = (\bar{1})\delta x = \bar{a}x.$$

I claim that $a \in S$; for let $p \in \rho$. Since δ is linear,

$$\bar{0} = (\bar{p})\delta = (\bar{1})\delta p = \bar{a}p = \overline{ap}.$$

That is, $a\rho \subseteq \rho$ and so

$$(\bar{x})\delta = \bar{a}x = a\bar{x}.$$

Let H be as in Theorem 2, with $\{\psi_i\}$ a set of right coset representatives of H in G. The following is an interesting result.

THEOREM 4. *Let e be the identity element of G. If $\sum_{i=1}^n h_i\psi_i \in S$, $h_i \in F[H]$, then either*

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(i) $h_i \in \rho$, *for all* i, *or*

(ii) $\{\bar{e}, \bar{\psi}_1, \ldots, \bar{\psi}_n\}$ *are linearly dependent over* Δ.

Proof. This is an easy consequence of the Density Theorem. If (ii) does not hold, the Density Theorem tells us that for any $x_i \in R, 1 \leqq i \leqq n$, there exists an $r \in R$ such that

(1) $er - 0 \in \rho$,

(2) $\psi_i r - x_i \in \rho, 1 \leqq i \leqq n$.

From (2) it follows that

(3) $\sum h_i \psi_i r - \sum h_i x_i \in \rho$.

But since $\sum h_i \psi_i \in S$, (1) and (3) allow us to conclude that

$$\sum h_i x_i \in \rho.$$

Appropriate choices for x_i now give us (i). This completes the proof.

It is a long standing conjecture, and a widely believed one, too, that the Jacobson radical of $F[G]$ is trivial for all G. This in turn would imply that we can always express $F[G]$ as a subdirect sum of primitive rings. This makes the following conjecture which the author proposes a somewhat surprising one.

Conjecture. $F[G]$ is never primitive if G is nontrivial.

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