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UNIFORMLY PERFECT SETS AND DISTORTION OF HOLOMORPHIC FUNCTIONS

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Abstract. We investigate the uniform perfectness on a boundary point of a hyperbolic open set and distortion of a holomorphic function from the unit disk Δ into a hyperbolic domain with a uniformly perfect boundary point, especially of a universal covering map of such a domain from Δ, and we obtain similar results to celebrated Koebe's Theorems on univalent functions.

§1. Uniformly perfect points

We begin by recalling the basic knowledge of the hyperbolic metric on a hyperbolic domain Ω in the complex plane $\mathbf{C}$, that is, $\mathbf{C} \setminus \Omega$ contains at least two points. On an arbitrary hyperbolic domain Ω, we have the hyperbolic metric $\lambda_\Omega(z)|dz|$ with Gaussian curvature -4. The hyperbolic metrics of the unit disk Δ and the upper half plane $\mathbf{H} = \{\mathrm{Im} z > 0\}$ are respectively

$$\lambda_\Delta(z)|dz| = \frac{|dz|}{1 - |z|^2} \text{ and } \lambda_{\mathbf{H}}(z)|dz| = \frac{|dz|}{2\mathrm{Im} z}.$$

The density $\lambda_\Omega(w)$ of the hyperbolic metric on a hyperbolic domain Ω is then defined as follows. Let $f(z)$ be a holomorphic universal covering map from Δ onto Ω. Then the density $\lambda_\Omega(w)$ is determined by

$$\lambda_\Omega(f(z))|f'(z)| = \frac{1}{1 - |z|^2}. \tag{1}$$

Noting that $f(z)$ is locally homeomorphic, we can solve $\lambda_\Omega(w)$ from equation (1). The determination of λ_Ω is independent of the choices of holomorphic covering maps of Ω from Δ because of invariance of the hyperbolic metric $|dz|/(1 - |z|^2)$ under Möbius transformation from Δ onto itself. Then the

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hyperbolic metric is conformally invariant. By $\lambda_{0,1}(z)$ we denote the density of the hyperbolic metric on $\mathbf{C}\setminus\{0,1\}$. From [8] and [9], we have

$$\lambda_{0,1}(z)\geq\frac{1}{2|z|(|\log|z||+\kappa)},\tag{2}$$

where $\kappa=\Gamma(1/4)^4/(4\pi^2)$. Next, by $\operatorname{mod}(A)$ we denote the modulus of an annulus A. Let $A=\{z;\ r<|z-a|<R\}$, $0<r<R$. A calculation implies that whenever $|z-a|=\sqrt{rR}$, we have

$$\lambda_A(z)=\frac{\pi}{2\sqrt{rR}\operatorname{mod}(A)}\tag{3}$$

(see [4]).

Throughout, let W be a hyperbolic open set in the complex plane, that is, $\mathbf{C}\setminus W$ is closed and contains at least two points. We can define the hyperbolic metric on W as the hyperbolic metric on each connected component of W. By $\lambda_W(z)$ we denote the density of the hyperbolic metric on W. For $a\notin W\cup\{\infty\}$, put

$$C(a,W):=\inf\{\lambda_W(z)|z-a|;\ z\in W\}.$$

If $C(a,W)>0$, then a is called a uniformly perfect point with respect to W.

For any $z_0\in W$, put $c(z_0,W):=\lambda_W(z_0)\delta_W(z_0)$, where $\delta_W(z_0):=\operatorname{dist}(z_0,\partial W)$ throughout denotes the euclidean distance from z_0 to ∂W. Then

$$\left\{z;\ |z-z_0|<\frac{c(z_0,W)}{\lambda_W(z_0)}\right\}\subset W.$$

Now we introduce a domain constant

$$C_W:=\inf\{c(z,W);\ z\in W\}.$$

In general, $0\leq C_W\leq\frac{1}{2}$ (see [7]). If every component of W is simply connected, from Koebe $\frac{1}{4}$ Theorem, we easily prove $\frac{1}{4}\leq C_W$. And $C_W=\frac{1}{2}$ if and only if every component of W is convex (see [7]). ∂W is called uniformly perfect, provided that $C_W>0$. There exist many mutually equivalent conditions of uniform perfectness of a closed set (see [19] and [7]).

Proposition 1. $C_W=\inf_{a\in\partial W\setminus\{\infty\}}\{C(a,W)\}$.

Proof. Obviously, for any $a \in \partial W \setminus \{\infty\}$, $C(a, W) \geq C_W$. So we only need to prove that

$$C_W \geq \inf_{a \in \partial W \setminus \{\infty\}} \{C(a, W)\}. \tag{4}$$

For any $n > 0$, there exists a $z_n \in W$ such that $C_W + \frac{1}{n} > \lambda_W(z_n)\delta_W(z_n)$ and for z_n we have $a_n \in \partial W \setminus \{\infty\}$ such that $|z_n - a_n| = \delta_W(z_n)$. Therefore,

$$C_W + \frac{1}{n} > \lambda_W(z_n)|z_n - a_n| \geq C(a_n, W) \geq \inf_{a \in \partial W \setminus \{\infty\}} \{C(a, W)\}.$$

From this (4) follows. ∎

Hence when ∂W is uniformly perfect, any finite point on ∂W is a uniformly perfect one with respect to W. An annulus A is said to separate a from ∞ if the bounded component of $\mathbf{C} \setminus A$ contains a. Below we introduce two domain constants and a notation. For $a \notin W \cup \{\infty\}$, define

$$\begin{aligned} \mathrm{Mod}_a^0(W) &:= \sup\{\mathrm{mod}(A); A \text{ is a round annulus in } W \text{ centered at } a\}, \\ \mathrm{Mod}_a(W) &:= \sup\{\mathrm{mod}(A); A \text{ is a (topological) annulus in } W \\ &\qquad\qquad\qquad\qquad \text{and separates } a \text{ from } \infty\}, \end{aligned}$$

where conventionally $\mathrm{Mod}_a^0(W) = 0$ ($\mathrm{Mod}_a(W) = 0$) if W does not contain any round annuli centered at a (any annuli which separate a from ∞), and

$$\beta_W(z; a) := \inf\left\{\left|\log\frac{|z-a|}{|b-a|}\right|;\ b \in \partial W\right\}.$$

Since a round annulus in W centered at a obviously separates a from ∞, we have $\mathrm{Mod}_a(W) \geq \mathrm{Mod}_a^0(W)$. We shall establish an inequality concerning $C(a, W)$ and $\mathrm{Mod}_a^0(W)$. To this end, we first prove the following result.

LEMMA. *For $a \in \partial W \setminus \{\infty\}$, we have*

$$\mathrm{Mod}_a^0(W) = 2 \sup_{z \in W} \beta_W(z; a). \tag{5}$$

Proof. For $z_0 \in W$ with $\beta_W(z_0; a) \neq 0$, it is clear that $\{|z - a| = \delta\} \cap \partial W = \emptyset$, where $\delta = |z_0 - a|$. Then there must exist in W a round annulus $A = \{z;\ r < |z - a| < R\}$ such that $\partial A \cap \partial W \neq \emptyset$ and $\delta = \sqrt{rR}$. For $b \in \partial A \cap \partial W$, then it is easy to see that

$$\beta_W(z; a) = \left|\log\left|\frac{z-a}{b-a}\right|\right| = \frac{1}{2}\log\frac{R}{r} = \frac{1}{2}\mathrm{mod}(A), \tag{6}$$

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whenever $|z-a| = \sqrt{rR}$, especially,

$$2\beta_W(z_0;a) = \mathrm{mod}(A) \le \mathrm{Mod}_a^0(W).$$

This inequality still holds for $z_0 \in W$ with $\beta_W(z_0;a) = 0$. Therefore

$$2\sup_{z\in W} \beta_W(z;a) \le \mathrm{Mod}_a^0(W). \tag{7}$$

To get (5) we need to prove the converse inequality. We may assume that $\mathrm{Mod}_a^0(W) > 0$, then there exists a sequence of round annuli

$$A_n = \{z;\ r_n < |z-a| < R_n\} \subset W$$

such that $\partial A_n \cap \partial W \neq \emptyset$ and

$$\mathrm{mod}(A_n) + \frac{1}{n} > \mathrm{Mod}_a^0(W).$$

Applying (6) to A_n gives $2\beta_W(z;a) = \mathrm{mod}(A_n)$ whenever $|z-a| = \sqrt{r_n R_n}$. Thus

$$2\sup_{z\in W} \beta_W(z;a) + \frac{1}{n} > \mathrm{Mod}_a^0(W). \tag{8}$$

(5) immediately follows by combining (8) with (7). □

We can prove by applying (2) and the method in [4] the following theorem , which is essentially due to Beardon and Pommerenke [4](see [20] and [23]).

Theorem A. *For $a \in \partial W \setminus \{\infty\}$, we have*

$$\frac{1}{2(\beta_W(z;a)+\kappa)} \le \lambda_W(z)|z-a| \le \frac{\pi}{4\beta_W(z;a)},\ \ z \in W. \tag{9}$$

Combining Theorem A with Lemma immediately shows the following result.

Proposition 2. *For $a \in \partial W \setminus \{\infty\}$, we have*

$$\frac{1}{\mathrm{Mod}_a^0(W)+2\kappa} \le C(a,W) \le \frac{\pi}{2\mathrm{Mod}_a^0(W)}. \tag{10}$$

Observe the domain

$$\Omega_0 := \mathbf{C} \setminus \bigcup_{n=1}^{\infty} [r_n, r_n^2],$$

where r_n is chosen to satisfy $r_{n+1} > r_n^3 > 0$ and $r_n \to +\infty$. It is easy to see that $C_{\Omega_0} = 0$ and from Proposition 2 for any $a \in \partial\Omega_0 \setminus \{\infty\}$, $C(a, \Omega_0) = 0$. Hence in order to consider the local structure of ∂W at a boundary point a, we introduce the quantity

$$C(a, W; R) := \inf\{\lambda_W(z)|z - a|;\ z \in W \cap \{|z - a| < R\}\},$$

where R is a positive constant. For a fixed a, $C(a, W; R)$ decreases as R increases, hence we easily prove that

$$C(a, W) = \lim_{R \to +\infty} C(a, W; R).$$

Then for $a \in \partial W \setminus \{\infty\}$, if $\{a\}$ is not a component of ∂W, it is easy to see from Proposition 2 that $C(a, W; R) > 0$. However, this condition is not necessary to $C(a, W; R) > 0$.

Set

$$L_W(\gamma) = \int_\gamma \lambda_W(z)|dz|,\ \gamma \subset W.$$

It is the hyperbolic length of γ on W. For any annulus A, the hyperbolic length of the core curve, denoted by $\mathrm{Core}(A)$, of A is

$$L_A(\mathrm{Core}(A)) = \frac{\pi^2}{\mathrm{mod}(A)}.$$

Let $\Gamma_W(a)$ be the set of all the closed curves winding around $a \in \partial W \setminus \{\infty\}$ in W. Define for $a \in \partial W \setminus \{\infty\}$

$$I(a, W) := \inf\left\{L_W(\gamma);\ \gamma \in \Gamma_W(a)\right\},$$

where conventionally $I(a, W) = \infty$ if $\Gamma_W(a) = \emptyset$, and

$$I_W := \inf\{I(a, W); a \in \partial W \setminus \{\infty\}\}.$$

PROPOSITION 3. *For $a \in \partial W \setminus \{\infty\}$, we have*

$$I(a, W) \le \frac{\pi^2}{\mathrm{Mod}_a(W)} \le I(a, W)\exp(I(a, W)). \tag{11}$$

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Proof. For an annulus A in W which separates a from ∞, we clearly have

$$\frac{\pi^2}{\mathrm{mod}(A)} = L_A(\mathrm{Core}(A)) \geq L_W(\mathrm{Core}(A)) \geq I(a, W),$$

and therefore the left-hand side of (11) follows from arbitrary choice of A.

It remains to show the right-hand side of (11). From the definition of $I(a, W)$, there exists a sequence of closed curves $\{\gamma_n\}$ in $\Gamma_W(a)$ such that

$$L_W(\gamma_n) < I(a, W) + \frac{1}{n}.$$

For each $n > 0$, we have the geodesic α_n homotopic to γ_n in W such that $L_W(\gamma_n) \geq L_W(\alpha_n)$. $\alpha_n \in \Gamma_W(a)$ is obvious. By the collar lemma (see [14]), there exists a collar A_n of width ω_n around the geodesic α_n in W, that is, $A_n = \{z \in W; d_W(z, \alpha_n) < \omega_n/2\}$, where $d_W(z, \alpha_n)$ denotes the hyperbolic distance of z from α_n, such that A_n is homeomorphic to a round annulus and $\sinh \omega_n \sinh L_W(\alpha_n) = 1$. From the proof of Theorem 5.2 and Corollary 5.3 of [19] (see [13]), it follows that

$$\frac{\pi^2}{\mathrm{mod}(A_n)} \leq L_W(\alpha_n) \exp\{L_W(\alpha_n)\}, \tag{12}$$

so that

$$\frac{\pi^2}{\mathrm{Mod}_a(W)} \leq \left(I(a, W) + \frac{1}{n}\right) \exp\left(I(a, W) + \frac{1}{n}\right).$$

This implies the right-hand side of (11). □

Remark. The similar inequalities concerning C_Ω, I_Ω and $\mathrm{Mod}(\Omega) = \sup\{\mathrm{Mod}_a(\Omega);\ a \in \partial\Omega\}$ have been established, see [19], for a hyperbolic domain Ω. From (10) and (11) we immediately have the following result.

THEOREM 1. *For $a \in \partial W \backslash \{\infty\}$, the following statements are mutually equivalent.*

(I) *a is a uniformly perfect point with respect to W;*
(II) $C(a, W) > 0$;
(III) $I(a, W) > 0$;
(IV) $\mathrm{Mod}_a^0(W) < \infty$;
(V) $\mathrm{Mod}_a(W) < \infty$.

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Proof. Obviously, we only need to imply (V) by (IV). Suppose that $\mathrm{Mod}_a(W) = \infty$, then there exists a sequence of annuli $\{A_n\}$ such that each A_n separates a from ∞ and $\mathrm{mod}(A_n) \to \infty$ $(n \to \infty)$, and furthermore we have a sequence of round annuli $\{B_n\}$ centered at a such that $\mathrm{mod}(B_n) = \mathrm{mod}(A_n) + O(1) \to \infty$ $(n \to \infty)$. This implies $\mathrm{Mod}_a^0(W) = \infty$, which contradicts (IV). □

Remark. From Theorem 1, it is easy to see that $C(a, W) = 0$ if and only if there exists a sequence of annuli $\{A_n\}$ in W such that for each n, A_n separates a from ∞ and $\mathrm{mod}(A_n) \to \infty$ as $n \to \infty$. And we can also require either $\sup\{|z - a|; z \in A_n\} \to 0$ or $\mathrm{dist}(a, A_n) \to \infty$ as $n \to \infty$.

Next, we discuss variation of the domain constant C_Ω of a hyperbolic domain Ω produced under a covering map. It is well-known that C_Ω is quasi-invariant under a conformal mapping. It was indeed proved in [12] that if Ω_0 and Ω_1 are conformally equivalent, then

$$\frac{1}{B} C_{\Omega_1} \le C_{\Omega_0} \le B C_{\Omega_1},$$

where $B = |1 + i \coth \frac{\pi}{3}| = 2.4335....$ Define

$$r_\Omega := \sup\{r;\ \text{the hyperbolic disk } \{z;\ d_\Omega(z, q) < r\} \text{ is}$$
$$\text{simply connected for all } q \in \Omega\},$$

where $d_\Omega(z, q)$ throughout denotes the hyperbolic distance from z to q on Ω. Then $I_\Omega = 2r_\Omega$ (see [11]). Let $p(z)$ be a covering map from Ω onto $p(\Omega)$. From the Principle of Hyperbolic Metric (see below Theorem B), we easily deduce $I_\Omega \ge I_{p(\Omega)}$, so that $r_\Omega \ge r_{p(\Omega)}$. Thus the same argument as in [12] can show the following

PROPOSITION 4. *Let Ω be a hyperbolic domain and $p(z)$ be a covering map from Ω onto $p(\Omega)$. Then*

$$C_{p(\Omega)} \le B C_\Omega.$$

It is clear that the inequality $C_\Omega \le B C_{p(\Omega)}$ does not generally hold, since an arbitrary hyperbolic domain must have a universal covering map from Δ.

§2. Distortion theorems

Distortion theorems concerning univalent analytic functions on Δ are well-known and play an important role in study of Complex Analysis. In this section, we mainly discuss distortion of holomorphic functions and universal covering maps from Δ in terms of uniform perfectness of image domains. The following is the Principle of Hyperbolic Metric (see Chapter III.3 of Nevanlinna[16] and also [15], this principle is sometimes called the Schwarz-Pick lemma), which is a start of our discussion in this section.

Theorem B. *Let $f(z)$ be holomorphic in Δ and Ω be a hyperbolic domain such that $f(\Delta) \subseteq \Omega$. Then*

$$\lambda_\Omega(f(z))|f'(z)| \leq \lambda_\Delta(z), \text{ for } z \in \Delta,$$

with equality if and only if f is a covering map of Ω from Δ.

By applying the Principle of Hyperbolic Metric, we first of all establish a distortion theorem about a function holomorphic in Δ.

Theorem 2. *Let $f(z)$ be holomorphic in Δ and Ω be a hyperbolic domain such that $f(\Delta) \subseteq \Omega$. If for some $a \in \partial\Omega \setminus \{\infty\}$, $c = 2C(a, \Omega) > 0$, then for $z \in \Delta$ we have*

$$|f(0) - a| \left(\frac{1 - |z|}{1 + |z|}\right)^{1/c} \leq |f(z) - a| \leq |f(0) - a| \left(\frac{1 + |z|}{1 - |z|}\right)^{1/c} \tag{13}$$

and

$$|f'(z)| \leq \frac{2|f(0) - a|}{c} \frac{(1 + |z|)^{1/c-1}}{(1 - |z|)^{1/c+1}}. \tag{14}$$

If, in addition, $C_\Omega > 0$ and $f'(0) \neq 0$, we have

$$\{w;\ |w - f(0)| < C_\Omega |f'(0)|\} \subset \Omega. \tag{15}$$

Proof. Applying the Principle of Hyperbolic Metric to $f(z)$ gives

$$\lambda_\Omega(f(z))|f'(z)| \leq \frac{1}{1 - |z|^2}, \ z \in \Delta. \tag{16}$$

Then from the definition of $C(a, \Omega)$ we get

$$\frac{c}{2} \frac{|f'(z)|}{|f(z) - a|} \leq \lambda_\Omega(f(z))|f'(z)| \leq \frac{1}{1 - |z|^2}. \tag{17}$$

Integrating the left-hand and right-hand sides of (17) along the segment $[0,z]$ gives

$$c\left|\log\frac{|f(z)-a|}{|f(0)-a|}\right| \le \log\frac{1+|z|}{1-|z|}.$$

From this (13) follows, and by combining (17) with (13), we deduce (14).

Since $0 < C_\Omega \le \lambda_\Omega(f(0))\delta_\Omega(f(0))$, from (16) we obtain

$$C_\Omega|f'(0)| \le \delta_\Omega(f(0)).$$

This immediately implies (15).

Theorem 2 follows. ∎

We remark on Theorem 2. When $f(\Delta)$ is simply connected with $f(0) = 0$ and $f'(0) = 1$, we have

$$\left\{w;\ |w| < \frac{1}{4}\right\} \subset f(\Delta).$$

This result generalizes Koebe $\frac{1}{4}$ Theorem, since we do not assume that $f(z)$ is univalent. When $f(\Delta)$ is convex with $f(0) = 0$ and $f'(0) = 1$, we have $\{w; |w| < \frac{1}{2}\} \subset f(\Delta)$.

THEOREM 3. *Let $f(z)$ be a universal covering map of Ω from Δ. If $d = 2C_\Omega > 0$, then*

$$\frac{d}{2}|f'(0)|\frac{(1-|z|)^{1/d-1}}{(1+|z|)^{1/d+1}} \le |f'(z)| \le \frac{2}{d}|f'(0)|\frac{(1+|z|)^{1/d-1}}{(1-|z|)^{1/d+1}} \tag{18}$$

and

$$|f(z)-f(0)| \le |f'(0)|\left\{\left(\frac{1+|z|}{1-|z|}\right)^{1/d} - 1\right\}. \tag{19}$$

Proof. For any $z \in \Delta$, there exists a point $a_z \in \partial\Omega$ such that $\delta_\Omega(f(z)) = |f(z) - a_z|$. From (15) it is easy to see that

$$|f(0)-a_z| \ge \frac{d}{2}|f'(0)|.$$

Noting $C(a_z,\Omega) \ge C_\Omega$ and using (13), we have

$$|f(z)-a_z| \ge |f(0)-a_z|\left(\frac{1-|z|}{1+|z|}\right)^{1/d}.$$

An application of the Principle of Hyperbolic Metric yields

$$\lambda_\Omega(f(z))|f'(z)| = \frac{1}{1-|z|^2}. \tag{20}$$

It is well-known that

$$\lambda_\Omega(f(z))\delta_\Omega(f(z)) \le 1. \tag{21}$$

Combining the above inequalities shows

$$\begin{aligned} |f'(z)| &\ge \frac{1}{1-|z|^2}\delta_\Omega(f(z)) \\ &= \frac{1}{1-|z|^2}|f(z)-a_z| \\ &\ge \frac{d}{2}|f'(0)|\frac{(1-|z|)^{1/d-1}}{(1+|z|)^{1/d+1}}. \end{aligned}$$

This is the left-hand side of (18). It is clear from (21) and (20) that

$$|f(0)-a_0| = \delta_\Omega(f(0)) \le \frac{1}{\lambda_\Omega(f(0))} = |f'(0)|.$$

Thus from (14) the right-hand side of (18) follows.

In order to prove (19), we note the elementary formula

$$\int_0^t \frac{(1+x)^{\alpha-1}}{(1-x)^{\alpha+1}}dx = \frac{1}{2\alpha}\left(\frac{1+t}{1-t}\right)^\alpha - \frac{1}{2\alpha}, \tag{22}$$

where α is a non-zero real constant. For $z \in \Delta$, using the right-hand side of (18) we have

$$|f(z)-f(0)| = \left|\int_0^z f'(\zeta)d\zeta\right| \le \frac{2}{d}|f'(0)|\int_0^{|z|}\frac{(1+x)^{1/d-1}}{(1-x)^{1/d+1}}dx.$$

Thus applying (22) to the last integration on the above inequality implies (19). □

Remark. (A) In Theorem 3, when Ω is simply connected, we have that $d = 2C_\Omega \ge 1/2$ and f is a conformal mapping, and then it follows from (18) that

$$\frac{1}{4}|f'(0)|\frac{1-|z|}{(1+|z|)^3} \le |f'(z)| \le 4|f'(0)|\frac{1+|z|}{(1-|z|)^3}$$

and from (19) that

$$|f(z) - f(0)| \leq |f'(0)| \frac{4|z|}{(1-|z|)^2}.$$

Comparing them with the corresponding inequalities in Koebe Distortion Theorem for a conformal mapping from Δ onto Ω, then we have reason to ask whether the coefficients $d/2$ and $2/d$ respectively in both the sides of (18) are necessary.

(B) The lower bound corresponding to (19) for $|f(z) - f(0)|$ does not exist unless $f(z)$ is conformal. This is because $f(z)$ can take $f(0)$ at other point in Δ than zero if $f(z)$ is not univalent.

Another distortion theorem on a universal covering map can be established by another way.

THEOREM 4. *Let $f(z)$ be a universal covering map of Ω from Δ. Assume that $d = 2C_\Omega > 0$. Then*

$$|f'(0)| \frac{(1-|z|)^{2/d-1}}{(1+|z|)^{2/d+1}} \leq |f'(z)| \leq |f'(0)| \frac{(1+|z|)^{2/d-1}}{(1-|z|)^{2/d+1}} \tag{23}$$

and

$$|\arg f'(z) - \arg f'(0)| \leq \frac{2}{d} \log \frac{1+|z|}{1-|z|}. \tag{24}$$

Proof. Let $F(z)$ be the universal covering map of Ω from Δ with $F(0) = 0$ and $F'(0) = 1$ (Here we assume $0 \in \Omega$ for the moment). From the Principle of Hyperbolic Metric, we have

$$\lambda_\Omega(F(z))|F'(z)| = \lambda_\Delta(z).$$

Taking the logarithm of the above equality and, then, differentiating it give

$$\frac{\partial}{\partial w}[\log \lambda_\Omega(w)](F(z))F'(z) + \frac{1}{2}\frac{F''(z)}{F'(z)} = \frac{\partial}{\partial z} \log \lambda_\Delta(z) = \frac{\overline{z}}{1-|z|^2}.$$

Thus

$$|F''(0)| = 2\left|\frac{\partial}{\partial w} \log \lambda_\Omega(0)\right| = |\nabla \log \lambda_\Omega(0)|.$$

By Theorem 4 in [17] and by noting $\lambda_\Omega(0) = \lambda_\Delta(0) = 1$, we have

$$|\nabla \log \lambda_\Omega(0)| \leq \frac{2}{\delta_\Omega(0)} \leq \frac{2}{C_\Omega},$$

and therefore

$$|F''(0)| \le \frac{4}{d}. \tag{25}$$

For each $z \in \Delta$ define

$$g(\zeta) := \frac{f\left(\frac{\zeta+z}{1+\overline{z}\zeta}\right) - f(z)}{(1-|z|^2)f'(z)}.$$

It is easy to see that $g(\zeta)$ is a universal covering map from Δ onto $L(\Omega)$, where $L(w) = (w - f(z))/[(1-|z|^2)f'(z)]$ is a linear transformation. Also $g(0) = 0$ and $g'(0) = 1$. A simple calculation reveals

$$g''(0) = (1-|z|^2)\frac{f''(z)}{f'(z)} - 2\overline{z}.$$

Applying (25) to $g(\zeta)$ and noting $d = 2C_\Omega = 2C_{L(\Omega)}$ give

$$|g''(0)| = \left|(1-|z|^2)\frac{f''(z)}{f'(z)} - 2\overline{z}\right| \le \frac{4}{d}.$$

Multiply both the sides of this inequality by $|z|/(1-|z|^2)$ to get

$$\left|\frac{zf''(z)}{f'(z)} - \frac{2|z|^2}{1-|z|^2}\right| \le \frac{4}{d}\frac{|z|}{1-|z|^2}.$$

This implies

$$\frac{2|z|^2 - \frac{4}{d}|z|}{1-|z|^2} \le \mathrm{Re}\frac{zf''(z)}{f'(z)} \le \frac{2|z|^2 + \frac{4}{d}|z|}{1-|z|^2} \tag{26}$$

and

$$\left|\mathrm{Im}\frac{zf''(z)}{f'(z)}\right| \le \frac{4}{d}\frac{|z|}{1-|z|^2}. \tag{27}$$

We note

$$\mathrm{Re}\frac{zf''(z)}{f'(z)} = |z|\frac{\partial}{\partial|z|}\log|f'(z)|$$

and

$$\mathrm{Im}\frac{zf''(z)}{f'(z)} = |z|\frac{\partial}{\partial|z|}\arg f'(z).$$

Thus (26) and (27) respectively yield

$$\frac{2|z| - \frac{4}{d}}{1-|z|^2} \le \frac{\partial}{\partial|z|}\log|f'(z)| \le \frac{2|z| + \frac{4}{d}}{1-|z|^2}$$

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and

$$-\frac{4}{d}\frac{1}{1-|z|^2} \le \frac{\partial}{\partial |z|}\arg f'(z) \le \frac{4}{d}\frac{1}{1-|z|^2}.$$

Integrating both the sides of the above two inequalities along the segment $[0, z]$ respectively implies (23) and (24). □

The following is a consequence of Theorems 3 and 4, which generalizes the celebrated distortion theorem of a univalent analytic function on Δ.

COROLLARY 1. *Assume that K is a compact subset of hyperbolic domain G. Then for every covering map $f : G \to f(G)$ such that $C_{f(G)} \ge k > 0$, we have*

$$\frac{1}{M} \le \frac{|f'(z)|}{|f'(w)|} \le M, \text{ for } z,\ w \in K, \tag{28}$$

where M are a positive constant depending on K and k.

Proof. It suffices to prove the right-hand side of (28). Let h be a universal covering map of G from Δ. Then $g = f(h) : \Delta \to f(G)$ is a covering map. We can find a r, $0 < r < 1$, such that $h(\Delta_r) \supset K$, $\Delta_r = \{|z| < r\}$. For a pair of z and w in K, there exist z_0 and w_0 in Δ_r such that $h(z_0) = z$, $h(w_0) = w$. From Proposition 4 it follows that $s = C_G \ge 0.42 C_{f(G)} \ge 0.42k > 0$. Applying Theorem 4 respectively to h and g gives

$$\frac{|h'(w_0)|}{|h'(z_0)|} \le \frac{(1+r)^{2/s}}{(1-r)^{2/s}}$$

and

$$\frac{|f'(z)h'(z_0)|}{|f'(w)h'(w_0)|} = \frac{|g'(z_0)|}{|g'(w_0)|} \le \frac{(1+r)^{2/k}}{(1-r)^{2/k}}.$$

Combining the above inequalities implies the right-hand side of (28). □

We can also establish the corresponding inequalities to (13) for half plane, angular domain and other special domains.

THEOREM 5. *Let $f(z)$ be holomorphic in $\mathbf{H}$ and $f(\mathbf{H}) \subseteq \Omega$. If for some $a \in \partial\Omega \setminus \{\infty\}$, $c = 2C(a, \Omega) > 0$, then for any $0 < \delta < \frac{\pi}{2}$, we have*

$$|f(z)| \le C_0(1 + |z|^{1/c}),\ |\arg z - \frac{\pi}{2}| < \delta, \tag{29}$$

where C_0 is a positive constant depending on δ, a and a fixed point z_1 in $\mathbf{H}$ and $f(z_1)$.

Proof. It is well-known (see [3]) that for a fixed point z_1 in $\mathbf{H}$, we have

$$\sinh^2 d_{\mathbf{H}}(z,z_1) = \frac{|z-z_1|^2}{4\mathrm{Im}[z]\mathrm{Im}[z_1]} = O(|z|), \tag{30}$$

whenever $|\arg z - \frac{\pi}{2}| < \delta$ and $z \to \infty$.

On the other hand, recalling the definition of hyperbolic distance between two points we obtain

$$\begin{aligned} d_\Omega(\zeta,\zeta_0) &= \inf_\gamma \int_\gamma \lambda_\Omega(\zeta)|d\zeta| \\ &\geq \frac{c}{2}\inf_\gamma \int_\gamma \frac{|d\zeta|}{|\zeta-a|} \\ &\geq \frac{c}{2}\left|\log\left|\frac{\zeta-a}{\zeta_0-a}\right|\right|, \end{aligned}$$

where the infimum is taken over all the curves γ connecting ζ and ζ_0 in Ω. Noting $\sinh^2 x > e^{2x}/4 - 1/2$, this yields

$$\begin{aligned} \sinh^2 d_\Omega(\zeta,\zeta_0) &\geq \sinh^2\left\{\frac{c}{2}\log\left|\frac{\zeta-a}{\zeta_0-a}\right|\right\} \\ &> \frac{1}{4}\left|\frac{\zeta-a}{\zeta_0-a}\right|^c - \frac{1}{2}, \text{ for } \zeta,\zeta_0 \in \Omega. \end{aligned} \tag{31}$$

Then the desired inequality (29) can be derived from $d_\Omega(f(z), f(z_1)) \leq d_{\mathbf{H}}(z,z_1)$ and by combining (30) with (31). ∎

Let $D(z_0,\theta,\delta) := \{z;\ |\arg(z-z_0)-\theta| < \delta\}$ be an angular domain. Transformation

$$w = M(z) = \{e^{-i(\theta-\delta)}(z-z_0)\}^{\frac{\pi}{2\delta}}$$

maps conformally $D(z_0,\theta,\delta)$ onto the upper half plane $\mathbf{H}$. And $w = \exp(\frac{\pi}{R-r}(z-Ri))$ maps conformally the band domain $\{r < \mathrm{Im} z < R\}$ onto the upper half plane $\mathbf{H}$. Then from Theorem 5 the following results immediately follow.

Corollary 2. *Let $f(z)$ be holomorphic in $D(z_0,\theta,\delta)$ and $f(D(z_0,\theta,\delta)) \subseteq \Omega$. If for some $a \in \partial\Omega \setminus \{\infty\}$, $c = 2C(a,\Omega) > 0$, then for any $0 < \delta_0 < \delta$, we have*

$$|f(z)| \leq C_0(1 + |z|^{\frac{\pi}{2c\delta}}), \text{ for } z \in D(z_0,\theta,\delta_0), \tag{32}$$

where C_0 is a positive constant depending on δ_0, δ, a and a fixed point z_1 in $D(z_0,\theta,\delta_0)$ and $f(z_1)$.

COROLLARY 3. *Let $f(z)$ be holomorphic in $E = \{r < \mathrm{Im} z < R\}$ and $f(E) \subseteq \Omega$. If for some $a \in \partial\Omega \setminus \{\infty\}$, $c = 2C(a, \Omega) > 0$, then for any $0 < \delta_0 < (R - r)/2$, we have*

$$(33) \quad |f(z)| \leq C_0 \exp\left(\frac{\pi}{(R-r)c}|z|\right), \text{ for } z \in \{r + \delta_0 < \mathrm{Im} z < R - \delta_0\},$$

where C_0 is a positive constant depending on δ_0, a and a fixed point z_1 in E and $f(z_1)$.

Remark. The inequalities (29), (32) and (33) are sharp. For example, observe function $h(z) = \{e^{-i(\theta-\delta)}(z-z_0)\}^{\frac{\pi}{2\delta}}$. It maps conformally $D(z_0, \theta, \delta)$ onto the upper half plane $\mathbf{H}$. Obviously, $h(z)$ satisfies the condition of Corollary 2 with $\Omega = \mathbf{H}$. In fact it is easy to see that for any $a \in \{\mathrm{Im} z = 0\}$, $c = 2C(a, \mathbf{H}) = 1$. Thus

$$|h(z)| = |z - z_0|^{\frac{\pi}{2\delta}} \sim |z|^{\frac{\pi}{2\delta}\frac{1}{c}},$$

as $z \to \infty$, $z \in D(z_0, \theta, \delta)$.

Corollary 2 has an application in iteration theory of meromorphic functions. Let $f(z)$ be a transcendental meromorphic function in the complex plane. Let $f^n(z)$ denote the n-th iterate of f : $f^1(z) = f(z)$, $f^n(z) = f(f^{n-1}(z)) = f^{n-1}(f(z))$. Then $f^n(z)$ is defined for all $z \in \mathbf{C}$ except for a countable set of the poles of f, f^2, $\ldots$, f^{n-1}. Define Fatou set of $f(z)$ as

$$F(f) := \{z \in \mathbf{C};\ \{f^n\} \text{ is defined and normal in some neighborhood of } z\}.$$

$F(f)$ is open and each $f^n(z)$ is analytic in $F(f)$. It is well-known that $F(f)$ is completely invariant, that is, $z \in F(f)$ if and only if $f(z) \in F(f)$, and thus for any connected component U of $F(f)$, called a stable domain of f, $f^n(U)$ is contained in a component U_n of $F(f)$. If for some n, $U_n = U$, then U is called a periodic domain of f; If for $n \neq m$, $U_n \neq U_m$, then U is called a wandering domain of f. We refer to [5] for more information.

THEOREM 6. *Let f be a meromorphic function and U be a stable domain of f. Assume that there exist an angular domain $D(z_0, \theta, \delta) \subset U$ and an $a \notin U$ such that $C(a, U) > 0$. Then for any positive integer n, we have*

$$(34) \qquad |f^n(z)| \leq C_n(1 + |z|^{\frac{t\pi}{4\delta}}), \text{ for } z \in D(z_0, \theta, \delta_0),$$

where $0 < \delta_0 < \delta$, $t = \max\{4, 1/C(a, U)\}$ and C_n is a positive constant depending on a, δ_0, δ, n and a fixed point z_1 in U and $f^n(z_1)$.

Proof. If $U_n = U$, then f^n satisfies the condition of Corollary 2 with $\Omega = U$; If $U_n \neq U$, then $U_n \cap U = \emptyset$, and $U_n \subset \mathbf{C} \setminus \{\arg(z - z_0) = \theta\}$. Noting the fact that $\mathbf{C} \setminus \{\arg(z - z_0) = \theta\}$ is simply connected and $C_{\mathbf{C}\setminus\{\arg(z-z_0)=\theta\}} = 1/4$, we also have that f^n satisfies the condition of Corollary 2 with $\Omega = \mathbf{C} \setminus \{\arg(z - z_0) = \theta\}$. Thus (34) follows from Corollary 2. □

We remark on Theorem 6. (I) (34) with $t = 4$ holds without the assumption of $C(a, \Omega) > 0$ when U is a wandering domain of f.

(II) When U is simply connected, (34) with $t = 1/C_U \leq 4$ holds, which was established in [6] and [18] by different methods with $t = 4$ for the case when f is an entire function, for an unbounded stable domain of an entire function f is simply connected (see [2]).

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