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ON A DIOPHANTINE EQUATION

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In this paper the equation $x^2 + 3^{2k} = y^n$ where $n \geqslant 3$ is studied. For $n = 3$, it is proved that it has a solution only if $k = 3K+2$ and then there is a unique solution $x = 46 \times 3^{3K}$ and $y = 13 \times 3^{2K}$. For $n > 3$ theorems are proved which determine a large number of values of k and n for which this equation has no solution. It is proved that if this equation has a solution for $n > 3$, then n is odd and $k = 2^{\delta}.k'$ where $\delta \geqslant 1$, $(2,\delta) = 1$, $k' \equiv 15 \pmod{20}$ and all the primes divisors p of n are congruent to 11 (mod 12).

1. INTRODUCTION

Many special cases of the equation $x^2 + C = y^n$ where x and y are positive integers and $n \geqslant 3$ have been considered over the years, but recently Cohn has studied this equation extensively. In [**3**] he has solved this equation completely for most values of C less than 100. For $C = 2^k$, Cohn [**2**] has proved that when k is odd there are three families of solutions and recently Arif and Abu Muriefah [**1**] have studied the same equation when k is even and they have put forward a conjecture and verified it for most values of k less than 200.

In this paper we confine ourselves to the study of the equation $x^2 + C = y^n$ for $C = 3^{2k}$. The first result for general n is due to Lebesgue [**4**] who proved that when $k = 0$ the equation has no solution, so we shall assume that $k \geqslant 1$. We solve the equation completely for n equal to 3 and for n even and greater than or equal to 4. For the other values of n we prove some theorems giving necessary conditions for the solvability of the equation. Our work suggests the following.

CONJECTURE. There are no solutions for the diophantine equation

$$x^2 + 3^{2k} = y^n, \qquad \text{where} \quad n \geqslant 3 \tag{1}$$

unless $k = 3K + 2$ and $n = 3$ and then there is a unique solution $x = 46 \times 3^{3K}$ and $y = 13 \times 3^{2K}$.

We are able to prove this conjecture for a large class of values of k and have verified it for all values of k less than or equal to 100 with eleven exceptions.

Our method of proof is similar to that of Cohn [**3**] and we use some of the results proved in that paper. Without loss of generality we can assume that x is positive and we consider two solutions of (1) different if they have different values of x.

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2. We first deal with the case n even and we will use the following lemma to prove the next theorem.

Lemma 1. (Störmer [7].) *The diophantine equation $x^2+1=2y^n$ has no solution in integers $x>1$, $y\geqslant 1$ and n odd, $n\geqslant 3$.*

Theorem 1. *The diophantine equation $x^2+3^{2k}=y^n$ has no solution if n is even and greater than or equal to 4.*

Proof: If x is odd then $x^2+3^{2k}\equiv 2 \pmod 8$, yielding no solution. So we assume that x is even and y is odd. First assume that $(3,x)=1$. Putting $n=2t$ with $t\geqslant 2$ we obtain $(y^t+x)(y^t-x)=3^{2k}$. Since $(3,x)=1$, we get $y^t+x=3^{2k}$ and $y^t-x=1$. By adding the last two equations we get $2y^t=1+3^{2k}$. If t is even this equation is not true modulo 3 and if t is odd then it follows from Lemma 1 that it has no solution. Now if $3\mid x$ then of course $3\mid y$. Suppose that $x=3^uX$, $y=3^vY$ where $u>0$, $v>0$ and $(3,X)=(3,Y)=1$. Then

$$3^{2u}X^2+3^{2k}=3^{2tv}Y^{2t}. \tag{2}$$

If $u<k$, then by cancelling 3^{2u} in (2) we get $X^2+3^{2(k-u)}=3^{2tv-2u}Y^{2t}$ and considering this equation modulo 3 we deduce that $tv-u=0$, then $X^2+3^{2(k-u)}=Y^{2t}$. But it is proved above that this equation has no solution. If $k\leqslant u$, we get $3^{2(u-k)}X^2+1=3^{2tv-2k}Y^{2t}$ and considering this modulo 3 we get $2tv-2k=0$, so (2) becomes $\left(3^{u-k}X\right)^2+1=Y^{2t}$ and this equation is known to have no solution [4].

3. Now we consider the equation when n is an odd integer and suppose that p is an odd prime that divides n. Then we can write (1) as $x^2+3^{2k}=\left(y^{n/p}\right)^p$. So it is sufficient to consider the equation $x^2+3^{2k}=y^p$.

In fact it is sufficient to consider the case $(3,x)=1$. Because if $3\mid x$ then using the hypotheses in the proof of the last theorem, and by similar argument we get

$$X^2+3^{2(k-u)}=Y^p, \quad \text{where} \quad 2u=pv, \tag{3}$$

with $(3,X)=1$ and the equation is reduced to the same kind of equation (1) with a smaller value of k. So we proceed to consider the equation

$$x^2+3^{2k}=y^p, \quad \text{where} \quad k>0,\ p \text{ an odd prime and } (3,x)=1. \tag{4}$$

Theorem 2. *If the diophantine equation (4) has a solution then either $p=3$ and there is the unique solution $k=2$, $x=46$, $y=13$ or k is even and $p\equiv 11 \pmod{12}$.*

Proof: We factorise equation(4) in the field $Q(i)$, to obtain

$$\left(x+3^ki\right)\left(x-3^ki\right)=y^p,$$

where the factors in the left hand side are coprime. Thus

$$x + 3^k i = (a + bi)^p, \tag{5}$$

where $y = a^2 + b^2$ is odd, so a and b have opposite parity. On equating real and imaginary parts in (5) we get

$$x = a\left\{a^{p-1} - \binom{p}{2}a^{p-2}b^2 + \cdots + (-1)^{(p-1)/2}pb^{p-1}\right\} \tag{6}$$

$$3^k = b\sum_{r=0}^{(p-1)/2}\binom{p}{2r+1}a^{p-2r-1}\left(-b^2\right)^r. \tag{7}$$

From (6) we deduce that $(3, a) = 1$, and from (7) we deduce that a is even and b is odd. If $p = 3$ then from (7) we get $3^k = b\left(3a^2 - b^2\right)$. If $b = \pm 1$, then $\pm 3^k = 3a^2 - 1$, which is impossible modulo 3. Similarly $b = \pm 3^k$ can be easily eliminated. Hence $b = \pm 3^c$, $1 \leqslant c < k$. Then $\pm 3^{k-c-1} = a^2 - 3^{2c-1}$. Since $(3, a) = 1$, we deduce that $k = c + 1$, whence $3^{2c-1} \pm 1 = a^2$. By considering this equation modulo 4, we get $3^{2c-1} + 1 = a^2$. Then $(a-1)(a+1) = 3^{2c-1}$, whence $a - 1 = 1$ or -3, and $k = 2$. Thus from (6) we get $x = 46$, so $y = 13$. Now if $p > 3$, let $b = \pm 1$. Then (7) considered modulo 3 implies

$$0 \equiv \frac{(1+i)^p - (1-i)^p}{2i} \equiv \pm 1 \pmod 3,$$

which is a contradiction, so $b \neq \pm 1$. Hence $b = \pm 3^\lambda$, $1 \leqslant \lambda < k$. If $\lambda \neq k$ then again considering (7) modulo 3, we get a contradiction. Hence $b = \pm 3^k$, and we arrive at

$$\pm 1 = \sum_{r=0}^{(p-1)/2}\binom{p}{2r+1}a^{p-2r-1}\left(-3^{2k}\right)^r.$$

This equation is exactly equation (1) in [**3**] and we can use Lemma 5 of [**3**] to deduce that the upper sign cannot hold, so $b = -3^k$ and

$$-1 = \sum_{r=0}^{(p-1)/2}\binom{p}{2r+1}a^{p-2r-1}\left(-3^{2k}\right)^r. \tag{8}$$

This implies that k is even, $p \equiv 2 \pmod 3$, $p \equiv 3 \pmod 4$. Consequently $p \equiv 11 \pmod{12}$. This completes the proof of Theorem 2. ∎

Corollary 1. *The diophantine equation (4) has no solution if k is odd.*

We use Theorem 2 to solve equation (1) completely when $p = 3$. When $3 \mid x$ we can deduce from equation (3) that (1) reduces to

$$X^2 + 3^{2(k-u)} = Y^3, \quad \text{where} \quad 2u = 3v,$$

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with $(3, X) = 1$, and from Theorem 2 there is a unique solution when $k - u = 2$. But $3 \mid u$, so let $u = 3K$, then $k = 3K + 2$, $x = 46 \times 3^u = 46 \times 3^{3K}$ and $y = 13 \times 3^v = 13 \times 3^{2K}$. Hence we get the following:

COROLLARY 2. *The diophantine equation $x^2 + 3^{2k} = y^3$ has a solution only if $k = 3K + 2$ and the unique solution is given by $x = 46 \times 3^{3K}$ and $y = 13 \times 3^{2K}$.*

Now we consider equation (8) and obtain conditions for the solvability of (4). We need the following two lemmas to prôve Theorem 3.

LEMMA 2. (Nagell [6].) *Suppose that $N = 2^t.v$ where N, t and v are positive integers, v odd. Suppose further that u and u_1 are odd integers $u \neq u_1$. Then the integer $(u^N - u_1^N)/(u^2 - u_1^2)$ is divisible exactly by 2^{t-1}.*

LEMMA 3. *The integer a defined in (8) is divisible by 4.*

PROOF: We know that a is even and $p \equiv 3 \pmod 4$. Let $a = 2a'$ where $(2, a') = 1$ and $p = 4H + 3$. Then (8) implies

$$\begin{aligned}
-1 &\equiv \frac{p(p-1)}{2} a^2 3^{k(p-3)} - 3^{k(p-1)} \pmod{16} \\
&\equiv (4H+3)(2H+1)4a'^2.3^{4kH} - 3^{k(2+4H)} \pmod{16} \\
&\equiv \pm 4 - 3^{2k} \pmod{16},
\end{aligned}$$

which is not true. This concludes the proof. □

THEOREM 3. *In equation (8), if $2^S \| a$ then $S \geqslant 2$ and $2^{2S-3} \| k$.*

PROOF: Since $4 \mid a$, let $a = 2^S.a'$, where $S \geqslant 2$ and $(2, a') = 1$. Also k even implies that $k = 2^\delta.k'$, $(2, k') = 1$ and $\delta \geqslant 0$. By rewriting (8) we obtain

$$3^{k(p-1)} - 1 = \sum_{r=0}^{(p-3)/2} \binom{p}{2r+1} a^{p-2r-1} \left(-3^{2k}\right)^r.$$

The right hand side of this equation is exactly divisible by 2^{2S}. For the left hand side, we know that $p \equiv 3 \pmod 4$, so using Lemma 2, where

$$N = k(p-1) = 2^{\delta+1}.k'\left(\frac{p-1}{2}\right) = 2^{\delta+1}.v,$$

and v is odd we find that this side is exactly divisible by $2^{\delta+3}$. Consequently $\delta + 3 = 2S$ and hence $2^{2S-3} \| k$. □

COROLLARY 3. *The diophantine equation (4) has no solution if $k = 2^{2m}.k'$, where $(2, k') = 1$ and $m \geqslant 0$.*

EXAMPLES.

1. The diophantine equation $x^2 + 9 = y^n$ has no solution. This was first shown by Ljunggren [**5**].
2. Consider the diophantine equation $x^2 + 3^{40} = y^n$.

Here $k \equiv 2 \pmod 3$, so if $3 \mid n$, then there is a unique solution $x = 46.3^{18}$, $y = 13.3^9$, $n = 3$ and if n even then there is no solution. Finally if $(6, n) = 1$ then there is no solution when $(3, x) = 1$ (Corollary 3), so let $x = 3^u X$, $y = 3^v Y$ where $u > 0$, $v > 0$ and $(3, X) = (3, Y) = 1$. Then we have only the equation

$$X^2 + 3^{2(20-u)} = Y^n,$$

where $nv = 2u$ and $0 < u < 20$. Corollary 3 can solve this equation except when

$$20 - u = 2,\ 6,\ 8,\ 10,\ 14,\ 18.$$

1. If $20 - u = 2$, then $u = 18$, so $nv = 36$ which is impossible since $(6, n) = 1$.
2. If $20 - u = 6$, then $u = 14$, so $nv = 48$, thus $n = 7$, so $X^2 + 3^{12} = Y^7$ with $(3, X) = 1$ which has no solution (Theorem 2).
3. If $20 - u = 8$ then $nv = 24$ which is impossible since $(6, n) = 1$.
4. If $20 - u = 10$, then we get $n = 5$, so $X^2 + 3^{20} = Y^5$ with $(3, X) = 1$ which has no solution (Theorem 2).
5. If $20 - u = 14$, then $nv = 12$ which is impossible since $(6, n) = 1$.
6. If $20 - u = 18$, then $nv = 4$ which is impossible since $(6, n) = 1$.

Corollary 3 does not solve equation (4) if an odd power of 2 exactly divides k, for example, $k = 2, 8, 10, \ldots$. In the following we give a theorem which can solve some of these values of k, that is when an odd power of 2 exactly divides k and when $(5, k) = 1$. From Corollary 1 it is sufficient to consider k even.

THEOREM 4. *The diophantine equation (4) where $p > 3$ has no solution if $(5, k) = 1$.*

PROOF: Since k is even, $3^{2k} \equiv 1 \pmod 5$. Considering equation (8) modulo 5, remembering $p \equiv 3 \pmod 4$, we get

$$-1 \equiv \frac{(a+i)^p - (a-i)^p}{2i} \equiv \frac{(a+i)^3 - (1-i)^3}{2i} = 3a^2 - 1 \pmod 5,$$

which implies $5 \mid a$. Then [**3**, Lemma 3] implies that $3^{8k} \equiv 1 \pmod{25}$. But 3 is a primitive root of 25, hence $5 \mid k$, so if $(5, k) = 1$, then equation (4) as no solution. ∎

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EXAMPLES.

1. The diophantine equation $x^2 + 3^4 = y^n$ has the unique solution $x = 46$, $y = 13$, $n = 3$. This result is also in [**3**].
2. The diophantine equation $x^2 + 3^{48} = y^n$ has no solution.

Now if an odd power of 2 exactly divides k and $5 \mid k$, then none of the above theorems solve (4). The following theorem can solve this problem partially.

THEOREM 5. *The diophantine equation (4) where $p > 3$ has no solution if $k = 2^\delta.k'$ where $k' \equiv 1 \pmod 4$, $\delta \geqslant 1$.*

PROOF: Let us define the sequences of rational integers $\{u_m\}$ and $\{v_m\}$ by setting

$$(a - 3^k i)^m = u_m + 3^k v_m i,\ m > 0. \tag{9}$$

Obviously

$$u_1 = a,\ v_1 = -1,\ u_2 = a^2 - 3^{2k},\ v_2 = -2a.$$

From equation (5) where $b = -3^k$ we get

$$x + 3^k i = (a - 3^k i)^p, \tag{10}$$

where $a = 2^S.a'$, $S \geqslant 2$, $(2, a') = 1$ and $p \equiv 3 \pmod 4$. From (9) and (10) we get

$$x + 3^k i = (a - 3^k i)^p = u_p + 3^k v_p i. \tag{11}$$

So a solution of (4) exists if $v_p = 1$, for some p. Now

$$\begin{aligned} u_{p+1} + 3^k v_{p+1} i &= (a - 3^k i)^p (a - 3^k i) \\ &= (u_p + 3^k v_p i)(a - 3^k i). \end{aligned}$$

By equating the imaginary parts in this relation we get

$$v_{p+1} = a v_p - u_p. \tag{12}$$

Also from (11) we get

$$\begin{aligned} u_{p+2} + 3^k v_{p+2} i &= (a - 3^k i)^p (a - 3^k i)^2 \\ &= (u_p + 3^k v_p i)(a^2 - 2a.3^k i - 3^{2k}). \end{aligned}$$

Again by equating the imaginary parts in this relation and using (12) we deduce

$$\begin{aligned} v_{p+2} &= (a^2 - 3^{2k}) v_p - 2 u_p a \\ &= (a^2 - 3^{2k}) v_p + 2a(v_{p+1} - a v_p) \\ &= 2a v_{p+1} - (a^2 + 3^{2k}) v_p. \end{aligned} \tag{13}$$

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Since $4 \mid a$, then from (13) we can deduce that

$$\begin{aligned} v_3 &= 2a\,v_2 - \left(a^2 + 3^{2k}\right)v_1 \\ &= -2^{2S+2}a'^2 + a^2 + 3^{2k} \\ &\equiv 2^{2S}a'^2 + 3^{2k} \pmod{2^{2S+2}} \\ &\equiv 2^{2S} + 3^{2k} \pmod{2^{2S+2}} \end{aligned}$$

and

$$\begin{aligned} v_{p+2} + \left(a^2 - 3^{2k}\right)v_p &= 2av_{p+1} \\ &= 2a[2av_p - \left(a^2 + 3^{2k}\right)v_{p-1}] \\ &\equiv -2a\left(2^{2S} + 3^{2k}\right)v_{p-1} \pmod{2^{2S+2}}. \end{aligned}$$

Continuing, we get

$$v_{p+2} + \left(a^2 - 3^{2k}\right)v_p \equiv \pm 2a\left(2^{2S} + 3^{2k}\right)v_2 \equiv 0 \pmod{2^{2S+2}}.$$

This implies that

$$v_{p+2} \equiv -\left(2^{2S} + 3^{2k}\right)v_p \pmod{2^{2S+2}}.$$

So we deduce that when $p \equiv 3 \pmod 4$

$$\begin{aligned} v_p &\equiv -\left(a^2 + 3^{2k}\right)v_{p-2} \pmod{2^{2S+2}}, \\ &\equiv \left(a^2 + 3^{2k}\right)^2 v_{p-4} \pmod{2^{2S+2}}, \\ &\equiv \left(2^{2S} + 3^{2k}\right)^{(p-3)/2} v_3 \pmod{2^{2S+2}}, \\ &\equiv \left(2^{2S} + 3^{2k}\right)^{(p-1)/2} \pmod{2^{2S+2}}. \end{aligned}$$

By using the Binomial Theorem we get

(14) $$v_p \equiv \left\{\left(\frac{p-1}{2}\right)2^{2S}.3^{k(p-3)} + 3^{k(p-1)}\right\} \pmod{2^{2S+2}}.$$

From Theorem 3 we have $k = 2^{2S-3}k'$. Suppose $k' = 4r + 1$. Since $p \equiv 3 \pmod 4$ we have two cases:

Case 1. $p = 8H + 7$. Then from (14) we get, since $3^{2S} \equiv 1 \pmod{2^{2S+2}}$

$$v_p \equiv \left\{(4H+3)2^{2S}.3^{2^{2S-3}(8H+4)(4r+1)} + 3^{2^{2S-3}(8H+6)(4r+1)}\right\} \pmod{2^{2S+2}}$$

(15) $$\equiv \left\{2^{2S}.3.3^{2^{2S-1}} + 3^{3.2^{2S-3}}\right\} \pmod{2^{2S+2}}.$$

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But $3^{2^{2S-2}} \equiv 1 + 2^{2S} \pmod{2^{2S+2}}$ and $3^{2^{2S-1}} \equiv 1 \pmod 8$. On substituting in (13) we get

$$\begin{aligned} v_p &\equiv \left\{3.2^{2S} + \left(1 + 2^{2S}\right)^3\right\} \pmod{2^{2S+2}} \\ &\equiv 1 + 2^{2S+1} \pmod{2^{2S+2}}. \end{aligned}$$

CASE 2. $p = 8H + 3$. Then from (14) we get

$$\begin{aligned} v_p &\equiv \left\{(4H+1)2^{2S}.3^{2^{2S-3}.8H(4r+1)} + 3^{2^{2S-3}(8H+2)(4r+1)}\right\} \pmod{2^{2S+2}} \\ &\equiv \left\{2^{2S} + 3^{2^{2S-2}}\right\} \pmod{2^{2S+2}} \\ &\equiv 1 + 2^{2S} + 2^{2S} \pmod{2^{2S+2}} \\ &\equiv 1 + 2^{2S+1} \pmod{2^{2S+2}}. \end{aligned}$$

In both cases $v_p \neq 1$, hence the diophantine equation (4) has no solution. □

EXAMPLES. The diophantine equation $x^2 + 3^{20} = y^n$ and $x^2 + 3^{80} = y^n$ have no solutions.

THEOREM 6. *If $3 \mid k$ and $(7, k) = 1$, then the diophantine equation (4) where $p > 3$ may have a solution only if $p \equiv 11 \pmod{24}$.*

PROOF: Since $3 \mid k$, therefore $3^{2k} \equiv 1 \pmod 7$. From (8) we get

$$-1 \equiv \frac{(a+i)^p - (a-i)^p}{2i} \pmod 7 \tag{16}$$

From Theorem 1 we have only the following two cases for p:

CASE 1: $p = 8H + 3$. Since $(a \pm i)^8 \equiv a^2 + 1 \pmod 7$ therefore (16) becomes

$$-1 \equiv \left(a^2 + 1\right)^H \left(3a^2 - 1\right) \pmod 7.$$

We consider the different values of a:

1. $a^2 \equiv 0 \pmod 7$, then from Lemma 3 of [3] we get $3^{12k} \equiv 1 \pmod{49}$. But the order of 3 modulo 49 equals to 7, hence $7 \mid k$, which is not true.
2. $a^2 \equiv 1 \pmod 7$, then $2^{H+1} \equiv -1 \pmod 7$, which is not true.
3. $a^2 \equiv 2 \pmod 7$, then $5.3^H \equiv -1 \pmod 7$, so $H \equiv 4 \pmod 6$ and $p \equiv 2 \pmod 3$.
4. $a^2 \equiv 4 \pmod 7$, then $4.5^H \equiv -1 \pmod 7$, so $H \equiv 1 \pmod 6$ and $p \equiv 2 \pmod 3$. Thus when $p \equiv 3 \pmod 8$, we deduce $p \equiv 2 \pmod 3$, that is $p \equiv 11 \pmod{24}$.

CASE 2: $p = 8H + 7$. From (16) we get

$$1 \equiv \left(a^2 + 1\right)^H \pmod 7.$$

We consider the different values of a:

1. $a^2 \equiv 0 \pmod 7$, as above, this is not possible.
2. $a^2 \equiv 1 \pmod 7$, then $2^H \equiv 1 \pmod 7$ so $H \equiv 0 \pmod 3$ and $p \equiv 1 \pmod 3$.
3. $a^2 \equiv 2 \pmod 7$, then $3^H \equiv 1 \pmod 7$, so $H \equiv 0 \pmod 6$ and $p \equiv 1 \pmod 3$.
4. $a^2 \equiv 4 \pmod 7$, then $5^H \equiv 1 \pmod 7$, so $H \equiv 0 \pmod 6$ and $p \equiv 1 \pmod 3$.

Thus when $p \equiv 7 \pmod 8$, we deduce $p \equiv 1 \pmod 3$, which is not true (Theorem 1). ∎

From Theorems 2, 3, 4, and 5 we are able to solve the equation $x^2 + 3^{2k} = y^n$, where $(3, x) = 1$ for all $k \leqslant 100$ except when $k = 30, 70$. And if $3 \mid x$, then we examine the following equation

$$X^2 + 3^{2(k-u)} = Y^n,$$

where $(3, X) = 1$, $nv = 2u$, $0 < u < k$ for a given $k \leqslant 100$. The problem arises when $k - u = 30$ or 70 and n has a prime divisor $p \equiv 11 \pmod{12}$ but Theorem 6 can solve this problem for some values of k. As an example we take $k = 53$.

EXAMPLE. Consider the diophantine equation $x^2 + 3^{106} = y^n$. As before it is sufficient to consider the equation

$$X^2 + 3^{2(53-u)} = Y^n,$$

with $(6, n) = 1, (3, X) = 1$, $nv = 2u$ and $0 < u < 53$. For all values of u, this equation has no solution except when $53 - u = 30$, then $nv = 46$, that is $n = 23$, hence

$$X^2 + 3^{60} = Y^{23}.$$

From Theorem 6 this equation has no solution. So the given equation has no solution.

By using the above method we are able to verify the conjecture for $k \leqslant 100$ except possibly for the values $k = 30, 41, 52, 63, 70, 81, 85, 89, 92, 93, 96$.

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