ON AN INVERSION OPERATOR

FOR THE FOURIER TRANSFORMATION

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(received September 9, 1959)

1. Introduction. In an earlier paper [1] we presented a representation theory for the Fourier transformation defined by

$$(1.1) \qquad F(x) = \frac{d}{dx}(2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} ((e^{ixy}-1)/iy)\, f(y)dy,$$

for functions f in certain function spaces. This theory made use of an operator

$$(1.2) \qquad F_{k,t}[F] = (-ik/t)^{k+1}(2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} (x-ik/t)^{-(k+1)}F(x)dx,$$

where k = 1, 2, ..., and it was stated without proof that this operator is an inversion operator for the Fourier transformation; that is, that under certain conditions

$$(1.3) \qquad \lim_{k\to\infty} F_{k,t}[F] = f(t).$$

Here we propose to investigate the circumstances under which (1.3) is true, the limit in (1.3) being taken in various senses of convergence.

The derivation of the inversion operator (1.2) is outlined in [2], where it is shown that a certain operator applied to the Fourier transformation changes it into the Laplace transformation. Then an application of the Widder-Post inversion operator for the Laplace transformation yields (1.2).

As $k \to \infty$, the operator (1.2) tends formally to the classical inversion operator for the Fourier transform,

$$(1.4) \qquad f(t) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-itx}F(x)dx,$$

since

Can. Math. Bull. vol. 3, no. 2, May 1960

$$\lim_{k \to \infty} (-ik/t)^{k+1}(x-ik/t)^{-(k+1)} = e^{-itx} ;$$

but actually (1.3) has somewhat greater utility than (1.4) since the integral in (1.4) fails to exist, in the Lebesgue sense, unless $F \in L_1(-\infty, \infty)$, while the integral in (1.2) exists, for example, if $F \in L_p(-\infty, \infty)$ for any p, $1 \leq p \leq \infty$.

It is known that the Fourier transform F of f, defined by (1.1), exists if $f \in L_p(-\infty, \infty)$ where $1 \leq p \leq 2$, or if $|x|^{1-2/q} f(x) \in L_q(-\infty, \infty)$, where $q \geq 2$. When $f \in L_1(-\infty, \infty)$, the differentiation indicated in (1.1) can be carried under the integral sign. We shall show in section 2 that under any of these circumstances just mentioned, (1.3) holds in the sense of pointwise convergence almost everywhere, and in section 4 we shall show that, under the same hypotheses, (1.3) holds in the sense of mean convergence. Section 3 is given over to two lemmas needed in section 4.

A word about notation. Throughout this paper, p will stand for a fixed but arbitrary number satisfying $1 \leq p \leq 2$, while q will stand for a fixed but arbitrary number satisfying $q \geq 2$. Also, if a number $r > 1$ is given, we shall understand by r' the number defined by

$$1/r + 1/r' = 1.$$

2. Pointwise convergence. The following theorem shows the pointwise convergence.

THEOREM 1. If F is defined by (1.1), where either

(a) $f \in L_p(-\infty, \infty)$,

or

(b) $|x|^{1-2/q} f(x) \in L_q(-\infty, \infty)$,

then at every point $t \neq 0$ in the Lebesgue set of f

$$\lim_{k \to \infty} F_{k,t}[F] = f(t) .$$

Proof. Let $t \neq 0$ and let k be an arbitrary positive integer. Then, as a function of x,

$$(x-ik/t)^{-(k+1)} \in L_p(-\infty, \infty) .$$

Also, an easy application of the residue calculus shows that

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$$(2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} (x-ik/t)^{-(k+1)} e^{-ixy} dx$$

$$= \begin{cases} (2\pi)^{\frac{1}{2}}(-i)^{k+1} y^k e^{ky/t}/k!, & y>0,\ t<0, \\ -(2\pi)^{\frac{1}{2}}(-i)^{k+1} y^k e^{ky/t}/k!, & y<0,\ t>0, \\ 0, & yt>0. \end{cases}$$

Hence by [4, theorem 35 if $f \in L_1(-\infty, \infty)$, theorem 75 if $f \in L_p(-\infty, \infty)$, $1 < p < 2$, theorem 49 if $f \in L_2(-\infty, \infty)$, and theorem 81 if $|x|^{1-2/q} f(x) \in L_q(-\infty, \infty)$, $q > 2$],

(2.1) $$F_{k,t}[F] = (-ik/t)^{k+1}(2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} (x-ik/t)^{-(k+1)} F(x)dx$$

$$= \begin{cases} (k/t)^{k+1}(k!)^{-1} \int_0^{\infty} e^{-ky/t} y^k f(y)dy, & t > 0, \\ -(k/t)^{k+1}(k!)^{-1} \int_{-\infty}^{0} e^{-ky/t} y^k f(y)dy, & t < 0. \end{cases}$$

For $s > 0$, let

$$g+(s) = \int_0^{\infty} e^{-sy} f(y)dy,$$

and

$$g-(s) = \int_0^{\infty} e^{-sy} f(-y)dy.$$

These integrals are clearly finite. Also let

$$L_{k,t}[g] = (-1)^k (k/t)^{k+1} g^{(k)}(k/t) / k!, \quad k = 1, 2, \ldots .$$

Then by [5, chapter 2, § 5, theorem 5a], for $t > 0$

$$L_{k,t}[g+] = (k/t)^{k+1}(k!)^{-1} \int_0^{\infty} e^{-ky/t} y^k f(y)dy = F_{k,t}[F],$$

and for $t < 0$,

$$L_{k,-t}[g-] = (-k/t)^{k+1}(k!)^{-1} \int_0^{\infty} e^{ky/t} y^k f(-y)dy$$

$$= -(k/t)^{k+1}(k!)^{-1} \int_{-\infty}^{0} e^{-ky/t} y^k f(y)dy = F_{k,t}[F].$$

Hence by [5, chapter 7, § 6, theorem 6a], if t is in the Lebesgue set of f and $t \neq 0$,

$$\lim_{k\to\infty} F_{k,t}[F] = f(t).$$

3. Preliminary lemmas. Lemma 1 is preliminary to theorem 2 and lemma 2 to theorem 3.

LEMMA 1. If $f \in L_r(0, \infty)$, where $1 \leq r < \infty$, then

$$\lim_{y\to 1} \int_0^\infty |f(yt)-f(t)|^r \, dt = 0 .$$

Proof. From [3, p. 397, ex. 19], if $g \in L_r(-\infty, \infty)$,

$$\lim_{h\to 0} \int_{-\infty}^\infty |g(x+h)-g(x)|^r \, dx = 0 .$$

Let $g(x) = e^{x/r} f(e^x)$. Then $g \in L_r(-\infty, \infty)$, and hence

$$\int_{-\infty}^\infty | e^{(x+h)/r} f(e^{x+h}) - e^{x/r} f(e^x) |^r \, dx = o(1)$$

as $h \to 0$.

But then, setting $e^x = t$, $e^h = y$, we obtain

$$\int_0^\infty | y^{1/r} f(yt)-f(t)|^r \, dt = o(1)$$

as $y \to 1$. But then, as $y \to 1$,

$$\int_0^\infty |f(yt)-f(t)|^r \, dt$$

$$\leq 2^r \left\{ \int_0^\infty |y^{1/r} f(yt)-f(t)|^r \, dt + |y^{1/r}-1|^r \int_0^\infty |f(yt)|^r \, dt \right\}$$

$$= 2^r \left\{ \int_0^\infty |y^{1/r} f(yt)-f(t)|^r \, dt + |1-y^{-1/r}|^r \int_0^\infty |f(t)|^r \, dt \right\}$$

$$= o(1)$$

LEMMA 2. If $t^{1-2/r} f(t) \in L_r(0, \infty)$, for some r, $1 < r < \infty$, then

$$\lim_{y\to 1} \int_0^\infty t^{r-2} |f(yt)-f(t)|^r \, dt = 0 .$$

Proof. An application of the previous lemma to $t^{1-2/r} f(t)$ gives us

$$\lim_{y\to 1} \int_0^\infty t^{r-2} | y^{1-2/r} f(yt) - f(t)|^r dt = 0 .$$

But then, as $y \to 1$

$$\int_0^\infty t^{r-2} | f(yt) - f(t) |^r dt$$

$$\le 2^r \left\{ \int_0^\infty t^{r-2} |y^{1-2/r} f(yt) - f(t)|^r dt + |y^{1-2/r} - 1|^r \int_0^\infty t^{r-2} |f(yt)|^r dt \right\}$$

$$= 2^r \left\{ \int_0^\infty t^{r-2} |y^{1-2/r} f(yt) - f(t)|^r dt + |y^{1-2/r} - 1|^r / y^{r-1} \int_0^\infty t^{r-2} |f(t)|^r dt \right\}$$

$= o(1)$.

4. Mean convergence. Theorem 2 deals with the case $f \in L_p(-\infty, \infty)$, and theorem 3 with the case $|x|^{1-2/q} f(x) \in L_q(-\infty, \infty)$.

THEOREM 2. If $f \in L_p(-\infty, \infty)$, and F is defined by (1.1), then

$$\lim_{k\to\infty} \int_{-\infty}^\infty |F_{k,t}[F] - f(t)|^p dt = 0 .$$

Proof. Suppose first that $t > 0$. Then using (2.1) and the fact that

$$(k/t)^{k+1} (k!)^{-1} \int_0^\infty e^{-ky/t} y^k dy = 1 ,$$

we deduce

$$(4.1) \quad |F_{k,t}[F] - f(t)| \le (k/t)^{k+1} (k!)^{-1} \int_0^\infty e^{-ky/t} y^k |f(y) - f(t)| dy.$$

Then an application of Hölder's inequality to (4.1), if $p > 1$, yields

$$(4.2) \quad |F_{k,t}[F] - f(t)|$$

$$\le (k/t)^{k+1} (k!)^{-1} \left\{ \int_0^\infty e^{-ky/t} y^k |f(y) - f(t)|^p dy \right\}^{1/p}$$

$$\cdot \left\{ \int_0^\infty e^{-ky/t} y^k dy \right\}^{1/p'}$$

$$= \left\{ (k/t)^{k+1} (k!)^{-1} \int_0^\infty e^{-ky/t} y^k |f(y) - f(t)|^p dy \right\}^{1/p} ,$$

and this inequality remains true if $p = 1$, since then it reduces to (4.1).

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Similarly, if $t < 0$,

$$(4.3) \quad |F_{k,t}[F]-f(t)| \leq \{-(k/t)^{k+1} \int_{-\infty}^{0} e^{-ky/t} y^k |f(y)-f(t)|^p dy\}^{1/p}.$$

Now

$$\int_{-\infty}^{\infty} |F_{k,t}[F]-f(t)|^p dt = \left(\int_0^{\infty} + \int_{-\infty}^{0}\right) |F_{k,t}[F]-f(t)|^p dt = I_1 + I_2 .$$

Consider first I_1. By lemma 1, given a positive number ε, there is a positive number δ so that if $|y-1| \leq \delta$,

$$\int_0^{\infty} |f(yt)-f(t)|^p dt < \varepsilon.$$

Clearly we may assume $\delta < 1$. Then from (4.2),

$$I_1 = \int_0^{\infty} |F_{k,t}[F] - f(t)|^p dt$$

$$\leq (k^{k+1}/k!) \int_0^{\infty} t^{-(k+1)} dt \int_0^{\infty} e^{-ky/t} y^k |f(y)-f(t)|^p dy$$

$$= (k^{k+1}/k!) \int_0^{\infty} dt \int_0^{\infty} e^{-ky} y^k |f(ty)-f(t)|^p dy$$

$$= (k^{k+1}/k!) \int_0^{\infty} e^{-ky} y^k dy \int_0^{\infty} |f(yt)-f(t)|^p dt$$

$$= (k^{k+1}/k!) \left(\int_0^{1-\delta} + \int_{1-\delta}^{1+\delta} + \int_{1+\delta}^{\infty}\right) e^{-ky} y^k dy \int_0^{\infty} |f(yt)-f(t)|^p dt$$

$$= J_1 + J_2 + J_3 .$$

But

$$J_1 = (k^{k+1}/k!) \int_0^{1-\delta} e^{-ky} y^k dy \int_0^{\infty} |f(yt)-f(t)|^p dt$$

$$\leq 2^p (k^{k+1}/k!) \int_0^{1-\delta} e^{-ky} y^k dy \int_0^{\infty} (|f(yt)|^p + |f(t)|^p) dt$$

$$= 2^p (k^{k+1}/k!) \int_0^{1-\delta} e^{-ky} (y^{k-1} + y^k) dy \int_0^{\infty} |f(t)|^p dt$$

$$\leq M (k^{k+1}/k!) \int_0^{1-\delta} e^{-ky} y^{k-1} dy ,$$

where

$$M = 2^{p+1} \int_0^{\infty} |f(t)|^p dt.$$

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But if $k > 1$, $e^{-(k-1)y}y^{k-1}$ steadily increases on the interval $0 \leq y \leq 1$, so that

$$J_1 \leq M\,(k^{k+1}/k!)e^{-(k-1)(1-\delta)}(1-\delta)^{k-1}\int_0^{1-\delta} e^{-y}dy$$

$$\leq M\,e^{1-\delta}\,(k^{k+\frac{1}{2}}e^{-k}/k!)k^{\frac{1}{2}}(e^{\delta}(1-\delta))^k \rightarrow 0$$

as $k \rightarrow \infty$ on using Stirling's formula and the fact that $e^{\delta}(1-\delta) < 1$.

Similarly $J_3 \rightarrow 0$ as $k \rightarrow \infty$, and also

$$J_2 = (k^{k+1}/k!)\int_{1-\delta}^{1+\delta} e^{-ky}y^k dy \int_0^{\infty} |f(yt)-f(t)|^p\,dt$$

$$\leq \varepsilon\,(k^{k+1}/k!)\int_{1-\delta}^{1+\delta} e^{-ky}y^k dy$$

$$\leq \varepsilon\,(k^{k+1}/k!)\int_0^{\infty} e^{-ky}y^k dy = \varepsilon\,.$$

Hence,

$$\overline{\lim}_{k\rightarrow\infty} I_1 \leq \varepsilon\,,$$

and thus since ε is arbitrary,

$$\lim_{k\rightarrow\infty} I_1 = 0\,.$$

Similarly,

$$\lim_{k\rightarrow\infty} I_2 = 0\,,$$

and thus

$$\lim_{k\rightarrow\infty}\int_{-\infty}^{\infty} |F_{k,t}[F] - f(t)|^p\,dt = 0.$$

THEOREM 3. If $|x|^{1-2/q} f(x) \in L_q(-\infty, \infty)$, and F is defined by (1.1), then

$$\lim_{k\rightarrow\infty}\int_{-\infty}^{\infty} |t|^{q-2}\,|F_{k,t}[F] - f(t)|^q\,dt = 0\,.$$

Proof. As in the proof of theorem 2, but with p replaced by q, we have, if $k > q - 2$,

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$$|F_{k,t}[F] - f(t)|$$

$$\leq \begin{cases} \{(k/t)^{k+1}(k!)^{-1} \int_0^\infty e^{-ky/t} y^k |f(y)-f(t)|^q \, dy\}^{1/q}, & t > 0, \\ \{-(k/t)^{k+1}(k!)^{-1} \int_{-\infty}^0 e^{-ky/t} y^k |f(y)-f(t)|^q \, dy\}^{1/q}, & t < 0. \end{cases}$$

Now

$$\int_{-\infty}^{\infty} |t|^{q-2} |F_{k,t}[F] - f(t)|^q \, dt = \left(\int_0^\infty + \int_{-\infty}^0\right) |F_{k,t}[F] - f(t)|^q \, dt$$

$$= I_3 + I_4.$$

Consider first I_3. By lemma 2, given a positive number ε, there is a positive number δ so that if $|y-1| \leq \delta$,

$$\int_0^\infty t^{q-2} |f(yt)-f(t)|^q \, dt < \varepsilon .$$

Clearly we may assume $\delta < 1$. But then,

$$I_3 = \int_0^\infty t^{q-2} |F_{k,t}[F] - f(t)|^q \, dt$$

$$\leq (k^{k+1}/k!) \int_0^\infty t^{q-k-3} dt \int_0^\infty e^{-ky/t} y^k |f(y)-f(t)|^q \, dy$$

$$= (k^{k+1}/k!) \int_0^\infty t^{q-2} dt \int_0^\infty e^{-ky} y^k |f(ty)-f(t)|^q \, dy$$

$$= (k^{k+1}/k!) \int_0^\infty e^{-ky} y^k dy \int_0^\infty t^{q-2} |f(yt)-f(t)|^q \, dt$$

$$= (k^{k+1}/k!) \left(\int_0^{1-\delta} + \int_{1-\delta}^{1+\delta} + \int_{1+\delta}^\infty \right) e^{-ky} y^k dy \int_0^\infty t^{q-2} |f(yt)-f(t)|^q \, dt$$

$$= J_4 + J_5 + J_6 .$$

J_4 and J_6 tend to zero as $k \to \infty$, in much the same manner as J_1 and J_3 of theorem 2, and as in that theorem, $J_5 \leq \varepsilon$. Hence

$$\overline{\lim}_{k \to \infty} I_3 \leq \varepsilon ,$$

so that since ε is arbitrary,

$$\lim_{k \to \infty} I_3 = 0 .$$

Similarly

$$\lim_{k \to \infty} I_4 = 0 ,$$

and thus

$$\lim_{k \to \infty} \int_{-\infty}^{\infty} |t|^{q-2} |F_{k,t}[F] - f(t)|^q dt = 0 .$$

REFERENCES

1. P.G. Rooney, On the representation of functions as Fourier transforms, Canad. J. Math. 11 (1959), 168-174.

2. P.G. Rooney, On the inversion of general transformations, Canad. Math. Bull. 2 (1959), 19-24.

3. E.C. Titchmarsh, The Theory of Functions, 2nd edition (Oxford, 1939).

4. E.C. Titchmarsh, The Theory of Fourier Integrals, 2nd edition (Oxford, 1948).

5. D.V. Widder, The Laplace Transform (Princeton, 1941).

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