Can. J. Math., Vol. XXVI, No. 4, 1974, pp. 931–947

OSCILLATION CRITERIA FOR QUASILINEAR EQUATIONS

W. ALLEGRETTO

1. Introduction. Several authors have recently considered the problem of establishing sufficient criteria to guarantee the oscillation or non-oscillation of all solutions of a second order elliptic equation or system. We mention in particular the papers of C. A. Swanson, [**15**; **16**], K. Kreith [**9**], Kreith and Travis [**10**], Noussair and Swanson [**13**], Allegretto and Swanson [**3**], Allegretto and Erbe [**2**] and the references therein.

However, all of the criteria obtained in the above mentioned papers, appear to stem from a *linear* comparison theorem, and therefore can only be applied to nonlinear operators which can be "majorized" by linear oscillatory operators. Consequently, the type of nonlinear operator which can be considered is severely restricted.

The main purpose of this paper is to obtain a description of the oscillatory behaviour of the solutions of the scalar equation $\mathscr{L}u = f$ in a domain Ω for a much wider class of quasilinear operators $\mathscr{L}$ than that previously considered. In particular, we shall show that subject to certain conditions on the coefficients of $\mathscr{L}$, on Ω and on f, either u oscillates or its minimum decays in a specified fashion. Since in the cases previously considered it was always assumed that f was either zero or of fixed sign, related to that of u, some of our results are new even in the linear case. As another departure from most previous considerations, we will establish results which will not depend on a suitable comparison theorem or on another extension of the Swanson-Picone identity, although these concepts are used to obtain additional results. The primary tools that we shall use are some simple considerations involving the maximum principle and some results from ordinary differential equations.

Finally, we remark that simplicity of presentation and relation to known results were considered in the formulations of our theorems. Consequently the criteria obtained are not, by and large, the most general possible with the methods used. For example, by merely complicating the presentation, analogous theorems can be obtained for solutions in various Sobolev spaces, for operators whose coefficients satisfy weaker conditions than those we shall state, for special systems of first order equations, etc. We also note, on the same argument, that the introduction of Green's functions, or of eigenvalues or of any other quantity whose calculation could present as much (or more) difficulty as the original problem were deliberately avoided.

Received February 26, 1973 and in revised form, July 24, 1973.

2. Some preliminary considerations. Let Ω denote a domain of n-dimensional Euclidean space $\mathbf{R}^n$, and let $C^2(\Omega)$ denote the space of real valued, twice continuously differentiable functions defined in Ω. As is usual, we denote by $x = (x_1, \ldots x_n)$ the points of $\mathbf{R}^n$ and differentiation with respect to x_i by D_i for $i = 1, \ldots n$. We consider the elliptic operator $\mathscr{L}$ defined in $C^2(\Omega)$ by:

$$(1) \qquad \mathscr{L}u(x) = Lu(x) + d(x, T(u)(x))$$

where L is a quasilinear operator given by:

$$(2) \qquad Lu(x) = \sum_{i,j=1}^{n} a_{ij}(x) D_i D_j u(x) + 2 \sum_{j=1}^{n} b_j(x) D_j u(x) - e(x, u(x)).$$

The coefficients a_{ij}, b_j are assumed real, continuous and the matrix $(a_{ij}(x))$ symmetric uniformly positive definite in Ω. We shall assume that T is an operator from $C^2(\Omega)$ to $C(\Omega)$ and that d, e denote continuous maps from $\Omega \times \mathbf{R}$ to $\mathbf{R}$ with (at least) the following properties:

(i) $d(x, \xi)$ is non-decreasing in ξ for every $x \in \Omega$;

(ii) $d(x, T(-f)(x)) = -d(x, T(f)(x))$ and $e(x, -\xi) = -e(x, \xi)$ for all $f \in C^2(\Omega)$ and $(x, \xi) \in \Omega \times \mathbf{R}$;

(iii) there is a non-negative continuous function $p(x)$ and a number $\delta \geqq 0$ such that if $\xi \geqq 0$ then $e(x, \xi) \leqq p(x)[\delta + \xi]$ for all $x \in \Omega$.

Note that the above conditions are satisfied by the functions: $e \equiv 0$, $T =$ identity, $d(x, \xi) = \sum_{i=1}^{k} q_i(x) \xi^{\gamma_i}$ with $q_i(x) \geqq 0$ and γ_i odd integers for $i = 1, \ldots, k$. Consequently, the operator $\mathscr{L}$ includes as special cases those operators whose expression involves a polynomial in the function, if, for example, the polynomial coefficients are non-negative and only odd powers of the function are involved.

Let $\bar{\Omega}$ denote the closure of Ω in the topology induced by the standard one point compactification of $\mathbf{R}^n$. Given a subset Γ of $\bar{\Omega}$ and a function $u \in C^2(\Omega)$, we define u to be *oscillatory at* Γ if and only if u has at least one zero in $N \cap \Omega$, for any neighbourhood N of Γ. If Ω is unbounded and we set $\Gamma = \{\infty\}$, then the above definition reduces to the usual definition of oscillation (at ∞).

Even if we restrict ourselves to the set of solutions of the equation $\mathscr{L}u = 0$ in Ω, it is clear that some solutions may oscillate at $\partial\Omega$ while others do not. For example, the equation:

$$\mathscr{L}u = e^{-2y} \frac{\partial^2 u}{\partial x^2} + \frac{\partial}{\partial y}\left(e^{-2y} \frac{\partial u}{\partial y}\right) + e^{-2y} u = 0$$

has, in particular, solutions $u = \sin x$, $v = e^y$, where we take $\Omega = \mathbf{R}^2$, $\Gamma = \{\infty\}$. We shall obtain criteria which will ensure that this cannot happen.

The following notation will be used throughout in the sequel: given any function h mapping $\mathbf{R}^n$ to $\mathbf{R}^+$ we shall denote by $S_{\alpha,h}$ the surface $\{x | h(x) = \alpha\}$; by $\Omega_{\alpha,h}$ the set $\Omega \cap S_{\alpha,h}$ and by $\Omega_{\alpha,\beta,h}$ the set $\Omega \cap \{x | \alpha < h(x) < \beta\}$. If f is any function defined in Ω then we will set $m_h(f, \alpha) = \inf\{f(x) | x \in \Omega_{\alpha,h}\}$. If the function h which is used is obvious from the context, we shall drop the subscript h and write S_α for $S_{\alpha,h}$, Ω_α for $\Omega_{\alpha,h}$, etc.

The following lemma and related corollary will be useful in what follows. Since these are standard results (similar ones were established, for example, by Agmon in [**1**, p. 16] for the case $k \leqq 0$) we merely sketch the proofs:

LEMMA 1. *The nonlinear boundary value problem:*

$$(3)\qquad \begin{aligned} &y'' - a|y'| = k \qquad \text{for } \alpha \leqq r \leqq \beta \\ &y(\alpha) = m(\alpha) \geqq 0 \\ &y(\beta) = 0 \end{aligned}$$

where a, k are continuous and a is non-negative, has a unique solution y and

$$(3')\qquad y(r) = m(\alpha)\,\frac{\int_r^\beta \exp\left[\int_\alpha^t \tilde{a}\right]}{\int_\alpha^\beta \exp\left[\int_\alpha^t \tilde{a}\right]} + G[-k]$$

for some function $\tilde{a}$, dependent on y, such that $|\tilde{a}| = a$, where G is the Green's Operator associated with the problem $w'' - \tilde{a}w' = k$ and zero boundary conditions.

Proof. The equation in (3) satisfies a Nagumo condition [**7**, p. 143] and suitable lower and upper solutions for (3) can easily be constructed. Consequently, it is known [**7**, p. 144] that there is a C^2 solution y to the boundary value problem (3). That this solution is unique follows from a small variation of the maximum principle. Finally, we set $\tilde{a} = a\,\mathrm{sign}(y')$ to obtain (3'). (Note that $\tilde{a}$ is, at least, Lebesque integrable.)

The usefulness of Lemma 1 comes from the fact that the kernel P of G can be obtained explicitly. It is:

$$P(l,\xi) = \begin{cases} \dfrac{o(l)g(\xi)}{W(\xi)}, & \text{for } \alpha \leqq l \leqq \xi \leqq \beta \\[2mm] \dfrac{o(\xi)g(l)}{W(\xi)}, & \text{for } \alpha \leqq \xi \leqq l \leqq \beta \end{cases}$$

where

$$o(l) = \int_\alpha^l \exp\left[\int_\alpha^t \tilde{a}\right]dt$$

$$g(l) = \int_l^\beta \exp\left[\int_\alpha^t \tilde{a}\right]dt$$

$$W(l) = \exp\left[\int_\alpha^l \tilde{a}\right]\int_\alpha^\beta \exp\left[\int_\alpha^t \tilde{a}\right]dt.$$

https://doi.org/10.4153/CJM-1974-088-0 Published online by Cambridge University Press

COROLLARY 1. *Let y be the solution of Problem* (3) *and further assume that k is non-negative and that*

$$m(\alpha) \geqq \int_\alpha^{r_0} \int_t^{r_0} \exp\left[\int_t^\xi a\right] k(\xi)d\xi dt \quad \textit{for all } r_0 \in (\alpha, \beta).$$

Then $y' \leqq 0$ in (α, β) and therefore $\tilde{a} = -a$.

Proof. It is obvious that in this case $y'(\alpha) \leqq 0$ and that y' can change sign at most once in (α, β). If y' is positive somewhere in (α, β) then there is a point $r_0 \in (\alpha, \beta)$ such that $y'(r_0) = 0$, and $y' > 0$ in (r_0, β). Clearly y has a minimum at r_0 and $\tilde{a} = -a$ to the left of r_0. Consequently,

$$y'(\alpha) = -\int_\alpha^{r_0} k(t) \exp\left[\int_\alpha^t a\right]dt$$

and therefore,

$$y(\beta) > y(r_0) = -\int_\alpha^{r_0} \int_t^{r_0} \exp\left[\int_t^\xi a\right] k(\xi)d\xi dt + m(\alpha) \geqq 0.$$

The contradiction establishes the corollary.

Oscillation criteria. Assume now that there is a C^2 function h mapping a subdomain of $\mathbf{R}^n$ onto $\mathbf{R}^+$ and with the following properties:

(i) grad(h) is never zero;

(ii) if $\alpha', \alpha'' \in \mathbf{R}^+$ and $\alpha' < \alpha''$ then the set

$$\bigcup_{\alpha'<\alpha<\alpha''} S_\alpha$$

is a bounded domain with boundary $S_{\alpha'} \cup S_{\alpha''}$;

(iii) Ω admits the representation:

$$\Omega = \bigcup_{\alpha>\alpha_0} S_\alpha$$

for some $\alpha_0 \in \mathbf{R}^+$, and $a_{ij}, b_j, p \in C(\bar{\Omega}_{\alpha_0,\beta})$ for any $\beta \geqq \alpha_0$, and $i, j = 1, \ldots, n$.

Obviously, for any given Ω there may be many such functions, and each one will give rise to oscillation criteria.

We shall obtain a description of the oscillatory behaviour of u at $\Gamma = \partial\Omega - S_{\alpha_0}$ in terms of the behaviour of u on S_α for α large. Note that if Ω is unbounded then $\{\infty\} \in \Gamma$. It is useful to introduce the following notation:

$$a(r) = \sup_{h(x)=r} \Bigg\{ \Bigg| \sum_{i,j=1}^n a_{ij}(x)D_iD_jh(x) + 2\sum_{j=1}^n b_j(x)D_jh(x)$$
$$+ 2\sum_{i,j=1}^n a_{ij}(x)D_i\rho(x)\rho^{-1}(x) \cdot D_jh(x)\Bigg| \cdot \left(\sum_{i,j=1}^n a_{ij}(x)D_ih(x)D_jh(x)\right)^{-1}\Bigg\}$$

where ρ denotes any positive function such that

$$\sum_{i,j=1}^{n} a_{ij} D_i D_j \rho + 2 \sum_{j=1}^{n} b_j D_j \rho \geqq \rho p.$$

in Ω.

For definiteness and convenience we choose the function ρ given by: $\rho(x) = w(h(x))$ where;

$$w(r) = \exp\left[\int_{\alpha_0}^{r} \int_{\alpha_0}^{t} \hat{p}(\xi) \exp\left[\int_{\xi}^{t} a_1\right] d\xi dt\right]$$

$$\hat{p}(\xi) = \sup_{h(x)=\xi} \left[\frac{p(x)}{\sum_{i,j=1}^{n} a_{ij}(x) D_i h(x) D_j h(x)}\right]$$

$$a_1(\xi) = \sup_{h(x)=\xi} \left\{\left|\sum_{i,j=1}^{n} a_{ij}(x) D_i D_j h(x) + 2 \sum_{j=1}^{n} b_j(x) D_j h(x)\right| \times \left(\sum_{i,j=1}^{n} a_{ij}(x) D_i h(x) D_j h(x)\right)^{-1}\right\}.$$

We remark that if $p \equiv 0$, then $w \equiv 1$.

LEMMA 2. *Let $Lu \leqq f$ in $\Omega_{\alpha,\beta}$ and assume u is non-negative in $\Omega_{\alpha,\beta}$* and *$f \in C(\bar{\Omega}_{\alpha,\beta})$. Then, for all $x \in \Omega_{\alpha,\beta}$, u satisfies the inequality:*

$$u(x) \geqq \left\{m(u,\alpha) w^{-1}(\alpha) \frac{\int_{h(x)}^{\beta} \exp\left(\int_{\alpha}^{t} \tilde{a}\right) dt}{\int_{\alpha}^{\beta} \exp\left(\int_{\alpha}^{t} \tilde{a}\right) dt} + G(-f^*)(h(x))\right\}_+ \rho(x) \tag{4}$$

where (i) *$\tilde{a}$ is a function dependent on f, α, β and the coefficients of L such that* $|\tilde{a}(r)| = a(r)$,

(ii) $$f^*(r) = \sup_{h(x)=r} \left\{\frac{|f(x)| + \delta p(x)}{\sum_{i,j=1}^{n} a_{ij}(x) D_i h(x) D_j h(x)}\right\} w^{-1}(r),$$

(iii) *G is the inverse operator associated with the problem*

$$y'' - \tilde{a} y' = f^* \qquad \alpha < r < \beta$$
$$y(\alpha) = y(\beta) = 0.$$

Proof. Let $Lu \leqq f$ in $\Omega_{\alpha,\beta}$ and $u \geqq 0$ in $\Omega_{\alpha,\beta}$. By assumption, this implies that $e(x, u(x)) \leqq p(x)[\delta + u(x)]$ in $\Omega_{\alpha,\beta}$. Let A denote the operator defined in $C^2(\Omega)$ by the expression

$$A(\phi) = \sum_{i,j=1}^{n} a_{ij} D_i D_j \phi + 2 \sum_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij} D_j \rho \cdot \rho^{-1} + b_i\right) D_i \phi. \tag{5}$$

https://doi.org/10.4153/CJM-1974-088-0 Published online by Cambridge University Press

Following Protter and Weinberger [**14**, p. 90], we introduce the function v defined by: $v(x) = u(x)\rho^{-1}(x)$. A trivial calculation shows that v satisfies in $\Omega_{\alpha,\beta}$ the inequality

$$A(v) \leqq (|f| + p)\rho^{-1}$$

and the boundary conditions:

$$m(v, \alpha) = m(u, \alpha)w^{-1}(\alpha); \quad m(v, \beta) \geqq 0.$$

A lower bound in $\Omega_{\alpha,\beta}$ can be obtained for v by classical procedures (see, for example, Protter and Weinberger [**14**, p. 134] for analogous arguments): let y denote the solution of the problem

$$\text{(6)} \qquad \begin{aligned} &y'' - a|y'| = f^*, \qquad \alpha < r < \beta \\ &y(\alpha) = m(u, \alpha)w^{-1}(\alpha); \quad y(\beta) = 0 \end{aligned}$$

whose existence is guaranteed by Lemma 1. The function $y(h(x))$ satisfies for $x \in \Omega_{\alpha,\beta}$ (and consequently $\alpha < h(x) < \beta$) the inequality:

$$\begin{aligned} A(y(h(\cdot)))(x) &\geqq \sum_{i,j=1}^{n} a_{ij}D_ih(x)D_jh(x)\{y''(h(x)) - a|y'(h(x))|\} \\ &\geqq \rho^{-1}(x)\{|f|(x) + \delta p(x)\}. \end{aligned}$$

Since $\partial\Omega_{\alpha,\beta}$ is given by $S_\alpha \cup S_\beta$, if $x \in \partial\Omega_{\alpha,\beta}$ then $y(h(x)) \leqq v(x)$. By the Hopf maximum principle it follows that $v(x) \geqq y(h(x))$ and consequently, $u(x) \geqq \rho(x)y(h(x))_+$ in $\Omega_{\alpha,\beta}$. Since by Lemma 1 the solution of (6) is given by the expression in the large brackets on the right hand side of inequality (4), the conclusion of Lemma 2 follows.

COROLLARY 2. *Let the conditions of Lemma 2 hold. Then for* $x \in \Omega_{\alpha,\beta}$,

$$u(x) \geqq \left\{m(u, \alpha)w^{-1}(\alpha) - \int_\alpha^\beta \left(\int_\alpha^t \exp\left(\int_t^\xi \tilde{a}\right)\right) f^*(t)dt\right\} \times \rho(x) \frac{\int_{h(x)}^\beta \exp\left[\int_\alpha^t \tilde{a}\right]dt}{\int_\alpha^\beta \exp\left[\int_\alpha^t \tilde{a}\right]dt}.$$

Proof. We note that, employing the previously introduced notation,

$$\begin{aligned} G[-f^*](r) &= -\int_\alpha^r \frac{o(\xi)g(r)}{W(\xi)} f^*(\xi)d\xi - \int_r^\beta \frac{o(r)g(\xi)}{W(\xi)} f^*(\xi)d\xi \\ &\geqq -g(r)\left[\int_\alpha^\beta \frac{o(\xi)f^*(\xi)}{W(\xi)}d\xi\right] \end{aligned}$$

and the estimate is immediate.

We can now state our first Theorem:

THEOREM 1. *Let $\mathscr{L}u = f$ in Ω, and define*

$$I = \int_\alpha^\infty \exp\left[-\int_\alpha^t a\right] dt.$$

Further assume that $u(x_0) \geqq v(x_0)$ implies $(Tu)(x_0) \geqq (Tv)(x_0)$. Finally, assume that there is a positive constant c such that:

$$\text{(i)} \qquad \int_\alpha^\infty \exp\left[-\int_\alpha^t a\right] z(t)dt = +\infty$$

$$\text{(ii)} \qquad \int_\alpha^\infty \int_\alpha^\xi \exp\left[\int_\xi^t a\right] z(t)dtd\xi = +\infty$$

for any positive constant $\alpha \geqq \alpha_0$, where

$$H(r) = w(r)\int_r^\infty \exp\left[-\int_{\alpha_0}^t a\right]dt, \quad \text{if } I < \infty$$
$$= w(r), \qquad \text{if } I = +\infty$$

and

$$z(r) = m\left\{\frac{d(\cdot, T(cH(h)))}{\sum a_{ij}D_ihD_jh}, r\right\} w^{-1}(r).$$

Then either u oscillates at Γ, or for all α sufficiently large,

$$\begin{aligned} m(|u|, \alpha) &\leqq w(\alpha) \int_\alpha^\infty \left[\int_\alpha^t \exp\left(\int_\xi^t a\right) d\xi\right] f^*(t)dt \\ &\qquad + w(\alpha)c \int_\alpha^\infty \exp\left[\int_{\alpha_0}^t -a\right]dt, \quad \text{if } I < \infty \end{aligned} \tag{7}$$

$$m(|u|, \alpha) \leqq w(\alpha) \int_\alpha^\infty \left[\int_\xi^t \exp\left(\int_\xi^t a\right) d\xi\right] f^*(t)dt + cw(\alpha), \quad \text{if } I = \infty.$$

where f^* is as defined in Lemma 2.

Proof. If u does not oscillate at Γ and if (7) is not valid for all α sufficiently large, then we can choose a value α_1 for which (7) fails and we may assume, without loss of generality, that $u(x)$ is positive if $h(x) \geqq \alpha_1$, and satisfies $Lu \leqq |f|$ for such x. Next, we note that, for all $\beta > \alpha_1$,

$$\int_{\alpha_1}^\beta \int_t^\beta \exp\left[\int_t^\xi a\right] f^*(\xi)d\xi dt = \int_{\alpha_1}^\beta \int_{\alpha_1}^t \exp\left[\int_\xi^t a\right] f^*(t)d\xi dt.$$

Therefore, since α_1 was chosen so that (7) is not valid, the condition of Corollary 1 is satisfied and we can set $\tilde{a} = -a$ in Lemmas 1, 2 and Corollary 2. Conse-

https://doi.org/10.4153/CJM-1974-088-0 Published online by Cambridge University Press

quently, for any $\beta > \alpha_1$ we find that in $\Omega_{\alpha_1,\beta}$:

$$u(x) \geqq \left\{m(u, \alpha_1)w^{-1}(\alpha_1) - \int_{\alpha_1}^{\beta}\left(\int_{\alpha_1}^{t} \exp \int_{\xi}^{t} a\right)f^*(t)dt\right\} \times \rho(x)\frac{\int_{h(x)}^{\beta} \exp\left[\int_{\alpha_1}^{t} -a\right]dt}{\int_{\alpha_1}^{\beta} \exp\left[\int_{\alpha_1}^{t} -a\right]dt}$$

Letting β approach $+\infty$ yields:

$$\begin{aligned} u(x) &\geqq c\rho(x) \int_{h(x)}^{\infty} \exp\left[\int_{\alpha 0}^{t} -a\right]dt, && \text{if } I < \infty \\ &\geqq c \cdot \rho(x), && \text{if } I = +\infty \end{aligned}$$

for all x such that $h(x) \geqq \alpha_1$. From the monotonicity properties of d and T, it follows that, for $x \in \{x|h(x) \geqq \alpha_1\}$, $u(x)$ satisfies the inequality:

$$Lu \leqq |f| - d(x, T(cH)(x)).$$

Again setting $v(x) = u(x)\rho^{-1}(x)$ we find that $v(x)$ satisfies in $\{x|h(x) \geqq \alpha_1\}$, the inequality:

$$A(v) \leqq \{(|f| + \delta p) - d(\cdot, T(cH)\cdot)\}\rho^{-1}$$

where A is the operator defined by (5). Let β again denote any scalar larger than α_1. Repeating the argument of Lemma 2, we conclude that in $\Omega_{\alpha_1,\beta}$ u satisfies the inequality

$$(8) \qquad u(x) \geqq \rho(x)\left[f_1(h(x)) + \pi\frac{\int_{h(x)}^{\beta} \exp\left[\int_{\alpha 0}^{t} -a\right]}{\int_{\alpha_1}^{\beta} \exp\left[\int_{\alpha 0}^{t} -a\right]}\right] \geqq \rho(x)f_1(h(x))$$

where π is nonnegative and f_1 is the solution of the problem:

$$(9) \qquad \begin{aligned} &f_1'' - a|f_1'| = -z(r), \quad \alpha < r < \beta \\ &f_1(\alpha_1) = f_1(\beta) = 0 \end{aligned}$$

(and, consequently, $f_1(h(x))$ satisfies $A(f_1(h(\cdot))) \geqq -d(\cdot, T(cH)\cdot)\rho^{-1}$ in $\Omega_{\alpha_1,\beta}$).

We observe that the right hand side of (9) is nonpositive and conclude that $f_1'(\alpha_1) \geqq 0$ and f_1' changes sign precisely once in (α_1, β). Let $r_0(\beta)$ denote a point such that $f_1'(r) \geqq 0$ for $r \in (\alpha_1, r_0(\beta))$ and $f_1'(r) < 0$ for $r \in (r_0(\beta), \beta)$. A trivial calculation shows that:

$$f_1(r) = \begin{cases} \int_{\alpha_1}^{r} \int_{t}^{r_0(\beta)} \exp\left[-\int_{t}^{\xi} a\right]z(\xi)d\xi dt, & \text{for } \alpha_1 \leqq r \leqq r_0(\beta) \\ \int_{r}^{\beta} \int_{r_0(\beta)}^{t} \exp\left[-\int_{\xi}^{t} a\right]z(\xi)d\xi dt, & \text{for } r_0(\beta) \leqq r \leqq \beta. \end{cases}$$

Consequently,

$$(10)\quad \int_{\alpha_1}^{r_0(\beta)} \int_t^{r_0(\beta)} \exp\left[-\int_t^{\xi} a\right] z(\xi)d\xi dt = \int_{r_0(\beta)}^{\beta} \int_{r_0(\beta)}^{t} \exp\left[-\int_{\xi}^{t} a\right] z(\xi)d\xi dt.$$

This implies that $\lim_{\beta\to\infty} r_0(\beta) = +\infty$, since otherwise there would exist a sequence $\{\beta_n\}$ and a constant θ such that $\lim_{n\to\infty}\{r_0(\beta_n)\} = \theta$ and $\lim_{n\to\infty}\{\beta_n\} = +\infty$. In such a case, substituting β_n for β in equation (10) would lead to a contradiction, since the left side of (10) would remain bounded, whereas the right side would diverge (by condition (ii)) as $n \to +\infty$. Letting β approach $+\infty$, and choosing a fixed $r > \alpha_1$ we note that:

$$\begin{aligned} f_1(r) &\geqq \int_{\alpha_1}^{r} \int_r^{r_0(\beta)} \exp\left[-\int_t^{\xi} a\right] z(\xi)d\xi dt \\ &= \left\{\int_{\alpha_1}^{r} \exp\left[\int_{\alpha_1}^{t} a\right]dt\right\}\left\{\int_r^{r_0(\beta)} \exp\left[-\int_{\alpha_1}^{\xi} a\right] z(\xi)d\xi\right\}. \end{aligned}$$

Since the second integral is divergent by condition (i), we conclude that $f_1(r)$ can be made arbitrarily large by choosing sufficiently large values of β. This contradicts inequality (8) and establishes the theorem.

Despite their similarity, conditions (i) and (ii) are independent of each other. However condition (i) will imply condition (ii) under very general conditions on the decay of

$$\exp\left[-\int_{\alpha_0}^{t} a\right]$$

and the growth of z.

It is clear from the proof of Theorem 1 that if the integrals in condition (i), (ii) diverge for *all* positive constants c, then we can replace inequality (7) in Theorem 1 by:

$$(7')\quad m(|u|, \alpha) \leqq w(\alpha) \int_{\alpha}^{\infty}\left[\int_{\alpha}^{t} \exp\left(\int_{\xi}^{t} a\right)d\xi\right] f^*(t)dt$$

both for $I < \infty$ and $I = +\infty$.

We remark that conditions (i), (ii) are *not* sufficient for oscillation of u, so that we *cannot* change the conclusion of Theorem 1 to read "u oscillates" (unless, of course, $f^* \equiv 0$). We illustrate this remark by considering the example:

$$\Delta u + r^4 u = e^{-r}\left(1 - \frac{2}{r} + r^4\right)$$

in the outside of a sphere in $\mathbf{R}^3$. We note that all the conditions of Theorem 1 are satisfied but yet the equation has the non-oscillatory solution $u = e^{-r}$. However we can conclude the following:

COROLLARY 3. *Let the conditions of Theorem* 1 *hold, except assume that the integrals in* (i), (ii) *diverge for all positive constants c, and that the right hand side of* (7′) *converges to zero as* $\alpha \to +\infty$. *Further assume that u has a continuous extension to* $\bar{\Omega}$. *Then u has a zero in* Γ.

Several other corollaries can now be stated by specializing d, T, e. As an example, let T = identity, $e(x, \xi) = p(x)\xi^{\gamma_0}$ and

$$d(x, \xi) = \sum_{i=1}^{k} q_i(x)\xi^{\gamma_i}$$

where $p(x), q_i(x)$ are non-negative, γ_i are the ratio of two odd positive integers for $i = 0, \ldots, n$ and $0 < \gamma_0 \leqq 1$. Note that if ξ is non-negative then

$$e(x, \xi) \leqq p(x)[\gamma_0^{(\gamma_0-1)\gamma_0} - \gamma_0^{(\gamma_0-1)} + \xi].$$

Theorem 1 is thus clearly applicable, and we have:

COROLLARY 4. *Let*

$$\mathscr{L}u = \sum_{i,j=1}^{n} a_{ij}D_iD_ju + 2\sum_{j=1}^{n} b_jD_ju - pu^{\gamma_0} + \sum_{i=1}^{k} q_iu^{\gamma_i} = f$$

where:

$$\text{(i)} \qquad \int_\alpha^\infty \sum_{i=1}^{k} (H)^{\gamma_i}(t)m\left(\frac{q_i}{\sum a_{ij}D_ihD_jh}, t\right) w^{-1}(t) \times \exp\left[-\int_\alpha^t a\right]dt = +\infty,$$

$$\text{(ii)} \qquad \int_\alpha^\infty \int_\alpha^t \exp\left[-\int_\xi^t a\right] \times \sum_{i=1}^{k} m\left(\frac{q_i}{\sum a_{ij}D_ihD_jh}, \xi\right) w^{-1}(\xi)H^{\gamma_i}(\xi)d\xi dt = +\infty,$$

H, w are as given in Theorem 1, *and* $\delta = \gamma_0^{(\gamma_0-1)\gamma_0} - \gamma_0^{(\gamma_0-1)}$. *Then either u oscillates at* Γ *or, for all* α *sufficiently large, satisfies* (7′).

COROLLARY 5. *Let the conditions of Corollary* 4 *hold, but further assume that* $f \equiv 0$ *and* $\delta p \equiv 0$. *Then all solutions oscillate.*

Similar theorems can be stated for the case where T is a shift operator by making a small change in the proof of Theorem 1.

COROLLARY 6. *Let* $Tu(x) = u(\tau(x))$ *where* τ *is a regular function mapping* Ω *to* Ω *and taking the set* $\{x|h(x) \geqq \alpha\}$ *to the set* $\{x|h(x) \geqq \lambda(\alpha)\}$ *for all* α *suffi-*

ciently large, where $\lim_{\alpha\to+\infty}\lambda(\alpha) = +\infty$. *Let*

$$\mathscr{L}u(x) = \sum_{i,j=1}^{n} a_{ij}(x)D_iD_ju(x) + 2\sum_{j=1}^{n} b_j(x)D_ju(x) + q_0(x)u(\tau(x)) = f(x)$$

where $q_0 \geqq 0$ *and*:

(i) $$\int_\alpha^\infty m\left(\frac{q_0}{\sum a_{ij}D_ihD_jh}, t\right)\exp\left[-\int_\alpha^t a\right]m[H(h(\tau\cdot)), t] = +\infty$$

(ii) $$\int_\alpha^\infty \int_\alpha^t \exp\left[-\int_\xi^t a\right] \times m\left(\frac{q_0}{\sum a_{ij}D_ihD_jh}, \xi\right) m[H(h(\tau\cdot)), \xi]d\xi dt = +\infty$$

Then either u *oscillates at* Γ *or satisfies* (7′) *for all* α *sufficiently large.*

Proof. If we assume that u does not oscillate at Γ and does not satisfy (7′) for all α sufficiently large, it follows that there is a constant α_1 such that: (7′) fails at α_1, and if $h(x) > \alpha_1$ then both $u(\tau(x))$ and $u(x)$ are positive. By the same procedure as in Theorem 1 we can conclude that there is a constant c such that $u \geqq cH$ for $h(x) > \alpha_1$, and consequently that there is a value α_2 such that (7′) is not valid for $\alpha = \alpha_2$ and for $h(x) \geqq \alpha_2$, u satisfies the inequality:

$$\sum_{i,j=1} a_{ij}D_iD_ju + 2\sum_{j=1}^{n} b_jD_ju \leqq |f| - cq_0H(h(\tau\cdot)).$$

The divergence of the integrals in (i), (ii) then gives a contradiction.

We note in passing that the growth estimate $e(x, \xi) \leqq [\delta + \xi]p(x)$, can be considerably weakened if we assume that u is bounded. Analogous results hold in this case if we assume, for example, that $|\xi| < N$ and $x \in \Omega$ imply $e(x, \xi) \leqq K(N)p(x)$.

We illustrate the above results by considering the very special equation:

(11) $\mathscr{L}u = \Delta u + c(x)u^\gamma = f$

where Ω is now assumed to be the complement of a sphere, $\Gamma = \{\infty\}$, $h(x) = \{\sum_{i=1}^n x_i^2\}^{1/2}$, $c(x) \geqq 0$ and continuous for all $x \in \Omega$, and γ is the ratio of two odd positive integers. In this case, $a = (n-1)/r$ and consequently $I = +\infty$ for $n = 2$ and $I < \infty$ for $n \geqq 3$. It is clear that for this operator condition (ii) of Theorem 1 is implied by condition (i) and we can conclude, therefore, that if

$$\int_\alpha^\infty m(c, r)r^{(2-n)\gamma+1-n}dr = +\infty$$

then either u oscillates at $\{\infty\}$, or, for all α sufficiently large, satisfies the inequality:

$$(12)\quad m(|u|,\alpha) \leqq \frac{1}{\ln(\alpha)} \int_\alpha^\infty t \ln t \left\{ \sup_{|x|=t} |f(x)| \right\} dt \quad n = 2$$

$$m(|u|,\alpha) \leqq \int_\alpha^\infty \frac{1}{(n-2)} \left\{ \sup_{|x|=t} |f(x)| \right\} t \left[\left(\frac{t}{\alpha} \right)^{n-2} - 1 \right] dt \quad (n \geqq 3).$$

It does not seem possible to obtain these results from any of the previously known theorems.

If the elliptic operator considered is of a more special type, for example if we consider

$$\begin{aligned} \mathscr{L}_1 u &= + \sum_{i,j=1}^n D_i(A_{ij} D_j u) - 2 \sum B_j D_j u + C u^\gamma - D u^\epsilon \\ &= L_1 u + C u^\gamma - D u^\epsilon \end{aligned}$$

with $0 < \epsilon \leqq 1 \leqq \gamma$ and γ, ϵ the ratio of two odd positive integers, then the Swanson-Picone identity can be combined with the above methods to obtain results which do not depend on the *minimum* of the coefficients on various surfaces. We remark that the coefficients of $\mathscr{L}_1$ are assumed to satisfy analogous conditions to those satisfied by the coefficients of $\mathscr{L}$, and in particular, C and D are assumed non-negative. Finally, to relate our results to some of the previous work we shall restrict ourselves to the case where Ω contains the complement of a sphere, $h(x) = \{\sum_{i=1}^n x_i^2\}^{1/2}$, and $\Gamma = \{\infty\}$.

Let the symbols H, a, w denote the analogous functions for $\mathscr{L}_1$ to those defined for $\mathscr{L}$, and let (r, ω) denote hyperspherical coordinates in $\mathbf{R}^n$.

THEOREM 2. *Let $\mathscr{L}_1 u = f$ in Ω and assume that the ordinary differential equation*

$$(13)\quad +(A(r) r^{n-1} y')' - (r^{n-1} C(r, \eta)) y = 0$$

has oscillatory solutions at ∞ for all positive constants η, where:

$$(13')\quad A(r) = r^{-2} \int_U \sum_{i,j=1}^n A_{ij} x_i x_j d\omega,$$

$$(13'')\quad C(r,\eta) = \int_U \left\{ \frac{|f|}{\eta H} - C(H\eta)^{\gamma-1} + D(H\eta)^{\epsilon-1} + \sum_{i,j=1}^n A^{ij} B_i B_j - \sum_{i=1}^n D_i(B_i) \right\} d\omega,$$

$(A_{ij}) = (A^{ij})^{-1}$, U denotes the full range of the angular variables ω, $d\omega$ denotes the angular component of the volume element $d\Omega$, and the integrands in (13′), (13″) are viewed as functions of (r, ω). Then either u oscillates at $\{\infty\}$ or satisfies (7′) for all α sufficiently large. In particular if $f^*(t) \equiv 0$ then u oscillates.

Proof. By exactly the same procedure as the one adopted for Theorem 1, we may assume that if u is not oscillatory and if (7′) is not satisfied for all α sufficiently large, then there are positive constants α_1, η_0 such that if x satisfies $|x| > \alpha_1$ then:

(14) $u(x) \geqq \eta_0 H(x)$.

For any $\beta > \alpha_1$, the well-known Swanson-Picone identity, [**15**; **16**; **17**], gives, for any function $\phi \in C_0^\infty(\Omega_{\alpha_1}, \beta)$, the identity:

$$(15)\quad \begin{aligned} &\int_{\Omega\alpha_1,\beta}\left[\sum_{i,j=1}^n A_{ij}u^2 D_i\left(\frac{\phi}{u}\right)D_j\left(\frac{\phi}{u}\right) + 2\sum_{j=1}^n B_j\phi u D_j\left(\frac{\phi}{u}\right) + \sum_{j=1}^n A^{ij}B_iB_j\phi^2\right]d\Omega \\ &= \int_{\Omega\alpha_1,\beta}\left\{-\phi L_1(\phi) + \sum_{i,j=1}^n A^{ij}B_iB_j\phi^2\right\}d\Omega + \int_{\Omega\alpha_1,\beta}\frac{\phi^2}{u}L_1(u)d\Omega. \end{aligned}$$

As is customary, the term $\sum_{i,j=1}^n A^{ij}B_iB_j\phi^2$ is added to both sides of (15) to ensure that the integrand on the left side is a non-negative definite form. Consequently,

$$0 \leqq \int_{\Omega\alpha_1,\beta}\left\{-\phi L_1(\phi) + \sum_{i,j=1}^n A^{ij}B_iB_j\phi^2\right\}d\Omega + \int_{\Omega\alpha_1,\beta}\left\{\frac{|f|}{u} - Cu^{\gamma-1} + Du^{\epsilon-1}\right\}\phi^2 d\Omega.$$

By means of estimate (14), integration by parts, and the assumption that $0 < \epsilon \leqq 1 \leqq \gamma$ we conclude that the inequality:

$$(16)\quad \begin{aligned} 0 \leqq \int_{\Omega\alpha_1,\beta}\Bigg[\sum_{i,j=1}^n A_{ij}D_i\phi D_j\phi + \Bigg\{\sum_{i,j=1}^n A^{ij}B_iB_j + \frac{|f|}{H\eta_0} - C(\eta_0 H)^{\gamma-1} \\ + D(H\eta_0)^{\epsilon-1} - \sum_{i=1}^n D_i(B_i)\Bigg\}\phi^2\Bigg]d\Omega \end{aligned}$$

is valid for any function $\phi \in C_0^\infty(\Omega_{\alpha_1,\beta})$. To establish a contradiction we need to find a function ϕ which makes the right hand side of (16) negative. By the averaging procedures introduced by Kreith and Travis [**10**], and Noussair [**12**], a sufficient condition for the existence of such a ϕ is that (13) be oscillatory for all η. Since, if this is the case, then by the standard theory of eigenvalues there exist a scalar $\beta_1 > \alpha_1$, and a function $\psi \in C_0^\infty((\alpha_1,\beta_1))$ such that:

$$(16')\quad \int_{\alpha_1}^{\beta_1}\left[A(r)(\psi'(r))^2 + C(r,\eta_0)\psi^2(r)\right]r^{n-1}dr < 0.$$

But the integral in (16′) is precisely the integral on the right hand side of

https://doi.org/10.4153/CJM-1974-088-0 Published online by Cambridge University Press

(16) expressed in spherical coordinates and with the choices $\phi(x) = \psi(|x|)$, $\beta = \beta_1$. This gives the desired contradiction.

We remark that Theorem 2 extends the *linear* result of Kreith and Travis [**10**] and Noussair [**12**] which is obtained by setting $f \equiv 0$, and $\epsilon = \gamma = 1$. Theorem 2 can also be thought of as a Sturm comparison theorem between solutions of $\mathscr{L}_1 u = f$ and the operator defined by the right hand side of (16), of which equation (13) is, for some η, a majorant.

We also note that inherent in the proof of Theorem 2 is the assumption that the coefficients $A_{ij}(x)$ and $B_j(x)$ $(i, j = 1, \ldots, n)$ are differentiable. This condition can be weakened by technical arguments based on the same ideas as those used in the proof of Theorem 2 (see, for example [**2**]).

Several corollaries can now be stated giving oscillation criteria for $\mathscr{L}_1$ by employing any of the well know oscillation criteria for ordinary differential equations. We merely state two of the Leighton-Wintner type and one of the Kneser type.

Corollary 7. *Let u be a solution of $\mathscr{L}_1 u = f$ in Ω and further assume that the coefficients of equation* (13) *satisfy the following conditions:*

(1) $$\int_\alpha^\infty \frac{1}{A(r)}\, dr = +\infty$$

(2) $$\int_\alpha^\infty \left(C(r, \eta) + \frac{A'(r)(n-1)}{2r} + \frac{A(r)(n-1)^2}{4r^2} - \frac{A(r)(n-1)}{2r^2} \right) dr = -\infty$$

for all positive constants η. Then either u oscillates at $\{\infty\}$ or satisfies (7′) *for all α sufficiently large.*

Proof. A simple change of variable shows that (13) is oscillatory if the equation:

$$+(A(r)y')' - \left(C(r, \eta) + \frac{A'(r)(n-1)}{2r} + \frac{A(r)(n-1)^2}{4r^2} - \frac{A(r)(n-1)}{2r^2} \right) y = 0$$

is oscillatory. A sufficient condition for this to happen is given by conditions (1), (2) by the standard Leighton-Wintner test [**11**].

Corollary 8. *Let u be a solution of $\mathscr{L}_1(u) = f$ in Ω, and, further, let the coefficients of equation* (13) *satisfy:*

(1) $A(r) < K$ *for some constant* K,

(2) $$\int_\alpha^\infty \left(rC(r, \eta) + \frac{A(r)(n-2)^2}{4r} + \frac{A'(r)}{2}(n-2) \right) dr = -\infty$$

for all positive constants η. Then either u oscillates or satisfies (7′) *for all α sufficiently large.*

Proof. It is easy to see that (13) is oscillatory if

$$+(rA(r)y')' - \left(rC(r,\eta) + \frac{A'(r)}{2}(n-2) + \frac{A(r)}{4r}(n-2)^2\right)y = 0$$

is oscillatory. Corollary 8 then follows by again using the Leighton-Wintner test [**11**].

COROLLARY 9. *If u satisfies $\mathscr{L}_1(u) = f$ in Ω and the coefficients of equation* (13) *satisfy:*

$$\limsup_{r\to\infty}\left\{r^2\left[\frac{C(r,\eta)}{A(r)} + \frac{A'(r)}{A(r)}\frac{(n-1)}{2r} + \frac{(n-1)(n-3)}{4r^2} + \frac{1}{2}\frac{A''(r)}{A(r)} - \frac{1}{4}\left(\frac{A'(r)}{A(r)}\right)^2\right]\right\} < -\frac{1}{4}$$

for all $\eta > 0$, then either u oscillates or satisfies (7′) *for all α sufficiently large.*

Proof. It is easy to see that (13) oscillates if

$$+y'' - \left(\frac{C(r,\eta)}{A(r)} + \frac{A'(r)}{A(r)}\frac{(n-1)}{2r} + \frac{(n-1)(n-3)}{4r^2} + \frac{1}{2}\frac{A''(r)}{A(r)} - \frac{1}{4}\left(\frac{A'(r)}{A(r)}\right)^2\right)y = 0$$

is oscillatory. We then merely apply Kneser's classical result [**8**].

To illustrate the above results, we once again consider the special example:

(11) $\Delta u + c(x)u^\gamma = f$

on the outside of a sphere in $\mathbf{R}^n$, where *we now further assume that* $\gamma \geqq 1$. *A*s a consequence of Corollary 8 we conclude that if:

$$\text{(17)}\quad \lim_{R\to\infty}\left[\int_{\Omega\alpha,R}\left\{\eta c(x)(r^{2-n})^\gamma - \frac{|f|}{\eta}\right\}d\Omega - \mu(U)\frac{(n-2)^2}{4}\ln R\right] = +\infty$$

for all positive constants η, α, and u satisfies equation (11) then either u oscillates at $\{\infty\}$ or satisfies conditions (12) (here $\mu(U)$ denotes the measure of the surface of the unit sphere). Consequently if $f \equiv 0$ and condition (17) is satisfied, then u must oscillate. It is interesting to note that for $n = 2$ and $f \equiv 0$, condition (17) reduces to:

$$\text{(18)}\quad \lim_{R\to\infty}\left[\int_{\Omega\alpha,R} c(x)d\Omega\right] = +\infty.$$

It is obvious that we can consider by exactly the same methods the more general operator:

$$\mathscr{L}_1 u = L_1 u - \sum_{i=1}^{m} D_i u^{\epsilon_i} + \sum_{i=1}^{p} C_i u^{\gamma_i}$$

with $0 \leqq \epsilon_i \leqq 1 \leqq \gamma_i$ and ϵ_i, γ_i the ratio of two odd positive integers. However the problem of using the Swanson-Picone identity for the *sublinear* case $0 < \gamma_i < 1$, which was allowed in Theorem 1, remains open. We also note that the conditions on ϵ_i can be weakened if u is assumed bounded.

Clearly, analogous results also apply to the nonlinear equation

$$(19) \quad \mathscr{L}_2 u = \sum_{i,j=1}^{n} a_{ij} D_i D_j u + b(x, u, D_1 u, \dots, D_n u) = f$$

if we assume that, for $u > 0$, b satisfies relations which imply that, as a consequence of (19), $\mathscr{L} u \leqq |f|$ where $\mathscr{L}$ is an operator of the same type as (1).

We conclude by examining the one-dimensional criteria which follow from the above theorems. Unlike the case for partial differential equations, oscillation criteria for nonlinear ordinary differential equations have been obtained by numerous authors. The survey article by Wong, [**18**], gives an extensive bibliography up to 1968. More recent references can be found in the articles of Coffman and Wong [**4**], and Erbe [**5**]. It seems unreasonable to expect that for this special case relevant new results can be obtained by the above methods. However, several classical results can be obtained immediately. We illustrate these remarks by considering the equation

$$(20) \quad y'' + d(x, y) = 0$$

in the interval $x > 0$. Here $d(x, \xi)$ is assumed to be continuous, nondecreasing in ξ for each fixed x, and such that $d(x, -\xi) = -d(x, \xi)$ for all $(x, \xi) \in \mathbf{R}^2$. Theorem 1 is immediately applicable and, noting that for this case $a \equiv 0$, $I = +\infty$, $h(x) = x$, we conclude that if

$$(21) \quad \int_{\alpha}^{\infty} d(x, \eta) = +\infty$$

for all constants $\eta > 0$ then any solution of (20) oscillates. Similarily, by a trival variation of Theorem 2 and a classical result of Zlamal [**19**], we conclude that if

$$(22) \quad \int_{\alpha}^{\infty} t^{\lambda} \inf_{\eta \leqq y < \infty} \left(\frac{d(t, y)}{y} \right) = +\infty$$

for some $\lambda < 1$ and all $\eta > 0$ then all solutions of (20) oscillate. Criteria (21) and (22) are well-known; they are contained, for example, in results of Wong [**18**], and Izyumova [**6**].

References

1. S. Agmon, *Unicité et convexité dans les problèmes differentiels* (University of Montreal Press, Montreal, 1966).

2. W. Allegretto and L. Erbe, *Oscillation criteria for matrix differential inequalities*, Can. Math. Bull. *16* (1973), 5–10.

3. W. Allegretto and C. A. Swanson, *Oscillation criteria for elliptic systems*, Proc. Amer. Math. Soc. *27* (1971), 325–330.
4. C. V. Coffman and J. S. Wong, *Oscillation and non-oscillation of solutions of generalized Emden-Fowler equations*, Trans. Amer. Math. Soc. *167* (1972), 399–434.
5. L. Erbe, *Oscillation criteria for nonlinear delay equations*, Can. Math. Bull. *16* (1973), 49–56.
6. D. V. Izyumova, *On the conditions for the oscillation and non-oscillation of nonlinear second-order differential equations*, Differencial'nye Uravnenija *2* (1966), 1572–1586.
7. L. K. Jackson, *Subfunctions and second-order ordinary differential inequalities*, "Lectures on Ordinary Differential Equations" (Academic Press, New York, 1970).
8. A. Kneser, *Untersuchungen über die reelen Nullstellen der Integrale linearen Differentialgleichungen*, Math. Ann. *42* (1893), 409–435.
9. K. Kreith, *Oscillation theorems for elliptic equations*, Proc. Amer. Math. Soc. *15* (1964), 341–344.
10. K. Kreith and C. Travis, *Oscillation criteria for self-adjoint elliptic differential equations*, Pacific J. Math. *41* (1972), 743–753.
11. W. Leighton, *The detection of the oscillation of solutions of a second order linear differential equation*, Duke Math. J. *17* (1950), 57–61.
12. E. Noussair, *Elliptic equations of order 2m*, J. Differential Equations *10* (1971), 100–111.
13. E. Noussair and C. A. Swanson, *Oscillation criteria for differential systems*, J. Math. Anal. Appl. *36* (1971), 575–580.
14. M. Protter and H. Weinberger, *Maximum principles in differential equations* (Prentice-Hall, New York, 1967).
15. C. A. Swanson, *Oscillation criteria for nonlinear matrix differential inequalities*, Proc. Amer. Math. Soc. *24* (1970), 824–827.
16. ——— *Comparison theorems for quasilinear elliptic differential inequalities*, J. Differential Equations 7 (1970), 243–249.
17. ——— *A comparison theorem for eigenfunctions*, Proc. Amer. Math. Soc. *37* (1973), 537–540.
18. J. S. Wong, *On second order nonlinear oscillation*, Funkcial. Ekvac. *11* (1968), 207–234.
19. M. Zlámal, *Oscillation criterions*, Časopis Pěst. Mat. *75* (1950), 213–217.

University of Alberta,
Edmonton, Alberta