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Low-degree Hurwitz stacks in the Grothendieck ring

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With an appendix by Aaron Landesman and Federico Scavia

Abstract

For $2 \leq d \leq 5$, we show that the class of the Hurwitz space of smooth degree d, genus g covers of $\mathbb{P}^1$ stabilizes in the Grothendieck ring of stacks as $g \to \infty$, and we give a formula for the limit. We also verify this stabilization when one imposes ramification conditions on the covers, and obtain a particularly simple answer for this limit when one restricts to simply branched covers.

1. Introduction

The main results of this paper are Grothendieck ring analogs of classical theorems on the density of discriminants of number fields of degree at most 5 (see [DH69, Bha05, Bha10]). Let $\mathrm{Hur}_{d,g,k}$ be the moduli stack of degree d covers of $\mathbb{P}^1$ with Galois group S_d by smooth geometrically connected genus g curves over a field k, see Definition 5.3. Let $\mathrm{Hur}^s_{d,g,k}$ be the open substack of $\mathrm{Hur}_{d,g,k}$ corresponding to simply branched covers, i.e. the open subset where the map to $\mathbb{P}^1$ has geometric fibers with at least $d-1$ points. The main results of this paper are that for each $d \leq 5$, the classes of these moduli spaces converge in the Grothendieck ring as $g \to \infty$, to particularly nice limits. More precisely, we work in a suitably defined Grothendieck ring of stacks $\widehat{\widetilde{K_0}}(\mathrm{Stacks}_k)$, see Definition 2.6, where as usual $\mathbb{L} := \{\mathbb{A}^1\}$ is the class of the affine line.

Theorem 1.1 (Theorem A). *Suppose $2 \leq d \leq 5$ and k is a field of characteristic not dividing $d!$. In $\widehat{\widetilde{K_0}}(\mathrm{Stacks}_k)$,*

$$\lim_{g\to\infty} \frac{\{\mathrm{Hur}^s_{d,g,k}\}}{\mathbb{L}^{\dim \mathrm{Hur}_{d,g,k}}} = 1 - \mathbb{L}^{-2}.$$

Theorem 1.2 (Theorem B). *Suppose $2 \leq d \leq 5$ and k is a field of characteristic not dividing $d!$. In $\widehat{\widetilde{K_0}}(\mathrm{Stacks}_k)$,*

$$\lim_{g\to\infty} \frac{\{\mathrm{Hur}_{d,g,k}\}}{\mathbb{L}^{\dim \mathrm{Hur}_{d,g,k}}} = \begin{cases} 1-\mathbb{L}^{-2} & \text{if } d=2,\\ (1+\mathbb{L}^{-1})(1-\mathbb{L}^{-3}) & \text{if } d=3,\\ \dfrac{1}{(1-\mathbb{L}^{-1})}\displaystyle\prod_{x\in\mathbb{P}^1_k}(1+\mathbb{L}^{-2}-\mathbb{L}^{-3}-\mathbb{L}^{-4}) & \text{if } d=4,\\ \dfrac{1}{(1-\mathbb{L}^{-1})}\displaystyle\prod_{x\in\mathbb{P}^1_k}(1+\mathbb{L}^{-2}-\mathbb{L}^{-4}-\mathbb{L}^{-5}) & \text{if } d=5.\end{cases}$$

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The products on the right in the cases $d = 4$ and $d = 5$ are motivic Euler products in the sense of Bilu [Bil17, BH21]. Alternatively, these can be viewed as powers in the sense of power structures, as introduced by Gusein-Zade, Luengo, and Melle-Hernàndez [GLM04], see § 2.10.

Theorem 1.1 is a special case of Corollary 10.6 whereas Theorem 1.2 is a special case of Corollary 10.7. Both are consequences of Theorem 10.5, describing the limits of branched covers with specified ramification, along with rates of convergence. These results lead to conjectures in higher degree, see § 1.5.

Remark 1.3. The results Theorems 1.1 and 1.2 of this paper are stated above with the restriction that the Galois group of the cover is S_d. These results continue to hold when one removes this restriction, except that when $d = 4$, covers with Galois group D_4 must be removed. One can deduce these claims from Lemma 9.6.

1.4 Motivation

Motivations for Theorems 1.1 and 1.2 come from number theory, topology, and algebraic geometry.

1.4.1 *Arithmetic motivation.* One can also view results relating to counting number fields of bounded discriminant as 'point counting analogs' of the stabilization of Hurwitz spaces. To spell this out, our main results on stabilization of the classes of Hurwitz spaces suggest the number of $\mathbb{F}_q$ points of these Hurwitz spaces also stabilize in g. (This is not actually implied by our results, because we work in the dimension filtration of the Grothendieck ring, and so it is possible that high codimension substacks of these Hurwitz spaces contain many $\mathbb{F}_q$ points which could potentially alter the $g \to \infty$ limiting behavior of the $\mathbb{F}_q$ point counts.) In the degree 3 case, stabilization of the number of $\mathbb{F}_q$ points was shown by Datskovsky and Wright in [DW88]. Their results actually count S_3 covers of any global field using Shintani zeta functions. However, a more geometric proof counting S_3 covers of any curve over a finite field has also been given by J. Gunther ('Counting cubic curve covers over finite fields', private communication). These results have also been generalized to work in degrees 4 and 5 by Bhargava, Shankar, and Wang in [BSW15]. Analogs over $\mathbb{Q}$ were known much earlier than these results over global function fields. That is, instead of counting $\mathbb{F}_q$ points of Hurwitz spaces, corresponding to S_d covers of $\mathbb{P}^1_{\mathbb{F}_q}$, the arithmetic analog is to count S_d extensions of $\mathbb{Q}$. When $d = 3$, these counts were carried out by Davenport and Heilbronn [DH69, DH71]. When $d = 4$ and $d = 5$, the number field counting was done by Bhargava in [Bha05, Bha10, Bha14]. Our theorems can thus be viewed as Grothendieck ring analogs of these number field counting results. Indeed, the 'Euler products' occurring in Theorem 1.2 with $\mathbb{L}$ replaced by p are exactly those that occur in the densities of discriminants of S_d-number fields of degree $d \leq 5$ (see [DH69, Bha05, Bha10]), which demonstrates, in particular, the great success of the notion of motivic Euler products. Similarly to our methods, the methods behind the number field counting results only apply when $d \leq 5$ because they rely on specific parametrizations [DF64, Bha04, Bha08] of low-degree covers of $\operatorname{Spec} \mathbb{Z}$.

1.4.2 *Topological motivation.* We now describe topological results demonstrating stabilization of Hurwitz spaces. One striking result is due to Ellenberg, Venkatesh, and Westerland [EVW16], which has deep applications to number theory. Their result [EVW16, Theorem p. 732] implies that the dimension of the ith homology $h_i(\mathrm{Hur}^s_{3,g,\mathbb{C}}, \mathbb{Q})$ stabilizes as $g \to \infty$. Unfortunately, although their methods apply in the case of degree 3 covers, they already fail to apply when $d = 4$, see the remarks in [EVW16, p. 732].

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If, instead of working with covers of $\mathbb{P}^1$, one works with the full moduli stack of curves with marked points, $\mathcal{M}_{g,n}$, then these stacks satisfy certain homological stabilities, due to Harer, Madsen-Weiss, and others. See, for example, [MW07] and the survey article [Hat11].

1.4.3 *Algebrogeometric motivation.* Finally, from an algebraic geometric viewpoint, there are some further related unirationality results on objects of low degree and genus. For degrees $d \leq 5$ a simple parametrization of degree d covers was originally given in [Mir85, Theorem 1.1], [CE96, Theorem 4.4], and [Cas96, Theorem 3.8], see also Theorems 3.13, 3.14, and 3.16 (as well as [Poo08, Proposition 5.1] and [Woo11, Theorem 1.1]).

There have also been results proving stabilization of algebraic data relating to $\mathrm{Hur}_{d,g,k}$. The rational Picard groups stabilize when $d \leq 5$, due to Deopurkar and Patel [DP15, Theorem A]. Also, the rational Chow groups were fully determined for $d = 3$, and the rational Chow groups were shown to stabilize for $d = 4$ and $d = 5$ (removing D_4 covers when $d = 4$), in [CL22, Theorem 1.1].

There have also been some related stabilization results working in the Grothendieck ring. For example, the class of smooth hypersurfaces of degree d in $\mathbb{P}^n$ stabilizes as $d \to \infty$ in the Grothendieck ring. This, and various related results are shown by the second and third authors in [VW15]. Building on this, Bilu and Howe prove more general stabilization results for sections of vector bundles in the Grothendieck ring [BH21, Theorem A]. The use of these results will be crucial in the present paper.

1.5 Conjectures and questions motivated by Theorems A and B

The most natural question following Theorem 1.1 is whether the pattern continues for higher d. The continuation of analogies of this pattern have been conjectured in several different domains.

1.6 Arithmetic conjectures

In the context of counting degree d number fields whose Galois closure has Galois group S_d, Bhargava [Bha07, Conjecture 1.2] has conjectured that an analog of Theorem 1.2 holds for all d (which, as mentioned above, is known for $d \leq 5$). Bhargava has given a specific conjectural expression for the Euler factors. It is natural to ask whether Theorem 1.2 holds for $d \geq 6$ using the analogous Euler factors. That is, one may ask whether Theorem 10.5 holds for $d \geq 6$ when all types of ramification are allowed. Further, the heuristics of [Bha07] also predict the analog of Theorem 1.1 in the number field counting setting for all d (which again is a theorem for $d \leq 5$; see [Bha14, Theorem 1.1]). Bhargava's heuristics more generally apply to give a conjecture for counting S_d degree d fields with various ramification restrictions, and the analogy in the Grothendieck ring setting would be to conjecture that Theorem 10.5 holds for $d \geq 6$.

The heuristics above are based on a mass formula proven by Bhargava [Bha07, Theorem 1.1]. We prove an analogous mass formula in the Grothendieck ring in Theorem 8.3, which we now state a consequence of. To make a precise statement, let $\mathscr{X}_d$ denote the stack over k whose T points are finite locally free degree d Gorenstein covers Z of $T \times_{\operatorname{Spec} k} \operatorname{Spec} k[\varepsilon]/(\varepsilon^2)$ so that for each geometric point $\operatorname{Spec} \kappa \to T$, $Z \times_{T \times \operatorname{Spec} k[\varepsilon]/(\varepsilon^2)} (\operatorname{Spec} \kappa \times \operatorname{Spec} k[\varepsilon]/(\varepsilon^2))$ has one-dimensional Zariski tangent space at each point. We write $R \vdash d$ to mean that R is a partition of d. Given $R \vdash d$ comprised of t_i copies of r_i for $i = 1, \ldots n$, we define $r(R) := \sum_{i=1}^{n} (r_i - 1) t_i$ to be its ramification order. We can then deduce the following corollary of Theorem 8.3, also see Remark 8.8, by summing over partitions of d in the same way that [Bha07, Proposition 2.3] was deduced from [Bha07, Proposition 2.2].

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Corollary 1.7. *For $d \geq 1$ and k a field of characteristic not dividing $d!$, in $\widehat{\widetilde{K_0}}(\mathrm{Stacks}_k)$,*

$$\{\mathscr{X}_d\} = \sum_{R \vdash d} \mathbb{L}^{-r(R)} = \sum_{j=0}^{d-1} q(d, d-j)\mathbb{L}^{-j},$$

where $q(d, d-j)$ is the number of partitions of d into at exactly $d-j$ parts.

The above heuristics can be expanded to make predictions when other finite groups replace S_d. These expanded heuristics are often called the Malle–Bhargava principle (see [Woo16]), though in complete generality the predictions are not always correct. This principle, as long as one is imposing only geometric local conditions (i.e. only local conditions on ramification) naturally extends to the Grothendieck ring setting. Then, one can ask in what generality the predictions of the principle hold. Moreover, in the field counting setting, one naturally counts extensions of global fields other than $\mathbb{Q}$ or $\mathbb{F}_q(t)$, and the analog here would be replacing $\mathbb{P}^1$ with another fixed curve, which is another interesting direction to try to understand.

In addition to the above conjectures on S_d extensions, there are also many open questions about Grothendieck ring versions of other extension counting problems. One particularly accessible problem may be that of counting D_4 extensions. In [CDO02, Corollary 1.4], the number of D_4 extensions of $\mathbb{Q}$ was computed when counted by discriminant, though the answer does not appear to have a simple closed form, and was expressed in terms of a sum over quadratic extensions of $\mathbb{Q}$. However, in [ASVW21, Theorem 1] these extensions were counted by conductor, and there was a closed-form answer, expressed in terms of an Euler product.

Question 1.8. What is the asymptotic class of the locus of D_4 covers of $\mathbb{P}^1$ in the Grothendieck ring $\widehat{\widetilde{K_0}}(\mathrm{Stacks}_k)$ when counted by discriminant or conductor?

Similarly, it would be interesting to compute the class of abelian covers of $\mathbb{P}^1$.

Question 1.9. Fix an abelian group G. What is the asymptotic class of the locus of G covers of $\mathbb{P}^1$ in the Grothendieck ring $\widehat{\widetilde{K_0}}(\mathrm{Stacks}_k)$ when counted by discriminant or conductor?

One way to approach this question could be to use that the moduli spaces of abelian covers of $\mathbb{P}^1$ can be described in terms of certain configuration spaces of (colored) points on $\mathbb{P}^1$. The classes of such configuration spaces can be extracted from [VW15, § 5].

1.10 Error terms and second-order terms

It would be interesting to understand the error terms in Theorem 10.5. More precisely, in Theorem 10.5, we show the equalities of Theorems 1.1 and 1.2 hold not just in the limit, but even hold for any fixed g up to codimension

$$r_{d,g} := \min\left(\frac{g + c_d}{\kappa_d}, \frac{g + d - 1}{d} - 4^{d-3}\right),$$

for $c_3 = 0$, $c_4 = -2$, $c_5 = -23$, $\kappa_3 = 4$, $\kappa_4 = 12$, and $\kappa_5 = 40$. We say two classes of dimension d are equal modulo codimension r in $\widehat{\widetilde{K_0}}(\mathrm{Stacks}_k)$ if their difference lies in filtered part of $\widehat{\widetilde{K_0}}(\mathrm{Stacks}_k)$ of dimension at most $d - r$. Concretely, in degree 3, a special case of Theorem 10.5 may be stated as follows.

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Corollary 1.11. *Suppose k is a field of characteristic not dividing* 6. *Then*

$$\frac{\{\mathrm{Hur}_{3,g,k}\}}{\mathbb{L}^{\dim \mathrm{Hur}_{3,g,k}}} \equiv (1+\mathbb{L}^{-1})(1-\mathbb{L}^{-3})$$

modulo codimension $g/4$ in $\widehat{\widehat{K_0}}(\mathrm{Stacks}_k)$.

Focusing on the degree 3 case, Roberts' conjecture [Rob01] states that the number of degree 3 field extensions of $\mathbb{Q}$ of discriminant at most X is $\alpha X + \beta X^{5/6} + o(X^{5/6})$, for appropriate constants α, β. This was proved in [BST13] and [TT13] independently. Moreover, the error term was further improved to $\alpha X + \beta X^{5/6} + O(X^{2/3+\varepsilon})$ in [BTT21].

In the context of function fields, one might similarly expect $\alpha_q q^{\dim \mathrm{Hur}_{3,g,k}} + \beta_q q^{5/6 \dim \mathrm{Hur}_{3,g,k}} + o(q^{5/6 \dim \mathrm{Hur}_{3,g,k}})$ to count the number of degree 3 extensions of $\mathbb{F}_q(t)$ of genus g, for some constants α_q, β_q. Progress towards this was made in [Zha13]. In the context of the Grothendieck ring, as mentioned above, we were able to compute the class of the Hurwitz stack up to codimension

$$r_{3,g} := \min\left(\frac{g}{4}, \frac{g+2}{3} - 1\right) = \min\left(\frac{g}{4}, \frac{g-1}{3}\right).$$

Hence, once $g \geq 4$, $r_{3,g} = g/4$. Since $\dim \mathrm{Hur}_{3,g,k} = 2g+4$, we find $\frac{5}{6} \dim \mathrm{Hur}_{3,g,k} = \dim \mathrm{Hur}_{3,g,k} - (g+2)/3$, and so a weakened form of Roberts' conjecture is the following.

Conjecture 1.12. Suppose k is a field of characteristic not dividing 6. Then

$$\frac{\{\mathrm{Hur}_{3,g,k}\}}{\mathbb{L}^{\dim \mathrm{Hur}_{3,g,k}}} \equiv (1+\mathbb{L}^{-1})(1-\mathbb{L}^{-3})$$

modulo codimension $(g-1)/3$ in $\widehat{\widehat{K_0}}(\mathrm{Stacks}_k)$.

Remark 1.13. Note that $(g-1)/3$ is, in fact, the second term in the minimum defining $r_{3,g}$. There is only one step in our proof where the error term we introduce has codimension less than $(g-1)/3$, namely when we apply the sieve of [BH21] in Proposition 9.10 and Lemma 9.11. Thus, if the sieving machinery could be improved, it may lead to a proof of Conjecture 1.12.

Remark 1.14. In the degree 3 case, it would be quite interesting to actually find the second-order term, instead of just predicting the codimension of the error. The paper [BTT21] improves the error term in Davenport–Heilbronn to $O(X^{2/3+\epsilon})$, where X is the discriminant of the relevant cubic extension. Since $X^{2/3} = X \cdot X^{-1/3}$, when one translates this to a codimension bound in the Hurwitz stack, it suggests one might hope to determine an asymptotic expression for $\{\mathrm{Hur}_{3,g,k}\}$ up to codimension $\frac{1}{3} \dim \mathrm{Hur}_{3,g,k}$. Such a computation would be extremely interesting to us, and we expect it would require tools far beyond those of the current paper.

In addition, it would be interesting, though likely more difficult, to determine the codimension of the error and the second-order terms in degrees 4 and 5.

1.15 Topological conjectures

If Conf_n denotes the configuration space of points on $\mathbb{P}^1$, i.e. the open subscheme of $\mathrm{Sym}^n_{\mathbb{P}^1}$ parameterizing reduced degree n subschemes of $\mathbb{P}^1$, then, for $n \geq 3$, we have $\{\mathrm{Conf}_n\}/\mathbb{L}^{\dim \mathrm{Conf}_n} = 1 - \mathbb{L}^{-2}$ in the Grothendieck ring of varieties. This follows from [VW15, Lemma 5.9(a)] as we explain further toward the end of § 11.3. There is a map $\mathrm{Hur}^s_{d,g,k} \to \mathrm{Conf}_{2g-2+2d}$ sending a curve to its branch locus, see [FP02]. Using this, Theorem 1.1 and the explicit formula for $\{\mathrm{Conf}_{2g-2+2d}\}$ implies that the source and target of this map have classes in $\widehat{\widehat{K_0}}(\mathrm{Stacks}_k)$, defined in Definition 2.6, which differ only by a class of high codimension.

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COROLLARY 1.16. *For $2 \le d \le 5$ and k a field of characteristic not dividing $d!$,*

$$\lim_{g\to\infty} \frac{\{\mathrm{Hur}^s_{d,g,k}\}}{\mathbb{L}^{\dim \mathrm{Hur}^s_{d,g,k}}} = \frac{\{\mathrm{Conf}_{2g-2+2d}\}}{\mathbb{L}^{\dim \mathrm{Hur}^s_{d,g,k}}}$$

in $\widehat{\widetilde{K_0}}(\mathrm{Stacks}_k)$.

It was also conjectured in [EVW16, Conjecture 1.5] that this map $\mathrm{Hur}^s_{d,g,\mathbb{C}} \to \mathrm{Conf}_{2g-2+2d}$ induces an isomorphism on ith homology for fixed d and sufficiently large g. (Technically a slight variant was conjectured in [EVW16, Conjecture 1.5], with $\mathbb{A}^1$ base in place of $\mathbb{P}^1$.) This is, in fact, open for $d \ge 3$, though recent work of Zheng [Zhe24, Theorem 1.2] proves a closely related result in the $d = 3$ case, by finding the stable cohomology of $\mathrm{Hur}_{3,g,\mathbb{C}}$. Theorem 1.1 could be seen as an additional motivation for this conjecture, especially for $d \le 5$.

1.17 Spelling out some questions

Despite the numerous parametrizations mentioned above, the question of whether there exist simple parametrizations of covers of degree 6, or even whether the Hurwitz stack of genus-g degree 6 covers (for large g) is unirational, remains wide open.

Returning to the simply branched case for simplicity, we have now seen several ways in which we could ask whether the spaces $\mathrm{Hur}^s_{d,g,k}$ and $\mathrm{Conf}_{2g-2+2d}$ are similar as $g \to \infty$. The following questions have been raised.

(1) Do they have the same points counts (asymptotically) over $\mathbb{F}_q$?
(2) Do they have the same cohomology, in some stable limit?
(3) Do they have the same normalized limit in the Grothendieck ring?

We also include the following.

(4) Are they piecewise isomorphic up to pieces of codimension going to ∞?

Even though it is not technically about these spaces, in this sequence of questions one should also include the following.

(1′) Are the asymptotic counts of S_d number fields as predicted by Bhargava in [Bha07]?

For $d \ge 6$, it seems progressively harder to believe the questions (1) and (1′), (2), (3), and (4) could have positive answers, though for $d \le 5$ the same parametrizations lead to positive answers to questions (1), (1′), (3), and (4) (and nearly to question (2) for $d = 3$).

1.18 Idea of the proof

The idea of the proof of Theorems 1.1 and 1.2 is simplest to understand in the degree 3 case, so we describe this first. Miranda [Mir85] gave a parametrization of degree 3 covers of a base scheme, and we explain here how we can apply it for degree 3 covers of $\mathbb{P}^1$. Any degree 3 cover of $\mathbb{P}^1$ has a canonical embedding into a $\mathbb{P}^1$-bundle $\mathbb{P}\mathscr{E}$ over $\mathbb{P}^1$. We can write $\mathscr{E} \simeq \mathscr{O}(a) \oplus \mathscr{O}(b)$ where $a + b = g + 2$ and $a \le b$. We can therefore stratify the Hurwitz space by the isomorphism type of the bundle $\mathscr{E}$. The degree 3 curves lie in a particular linear series on $\mathbb{P}\mathscr{E}$. The idea is now to compute the locus of smooth curves in this linear system with particular ramification conditions, and then sum over all splitting types of bundles $\mathscr{E}$. The condition for a degree 3 cover of $\mathbb{P}^1$ to be smooth in a fiber over p can be checked over the preimage of the second-order neighborhood of p in $\mathbb{P}\mathscr{E}$. We directly compute the classes of such curves in such an infinitesimal neighborhood. Using the notion of motivic Euler products, we can 'multiply' these local classes to obtain the global class of smooth curves in $\mathbb{P}\mathscr{E}$ in the relevant linear system, at least up

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to high codimension. We then sum these resulting classes over allowed splitting types of $\mathscr{E}$. It turns out that we must have $\mathscr{E} \simeq \mathscr{O}(a) \oplus \mathscr{O}(b)$ with $a \leq b, 2a \geq b$, and a general member of the relevant linear system on any such bundle gives a smooth trigonal curve. Miraculously, in the simply branched case, this motivic Euler product exactly cancels out with the sum over splitting types of $\mathbb{P}\mathscr{E}$, weighted by their automorphisms. This follows from a motivic Tamagawa number formula for SL_2.

To generalize this idea to the cases of degrees 4 and 5 requires substantial additional work. First, it is no longer the case that curves of degrees 4 and 5 are elements of linear systems on a surface. Rather, there are parametrizations due to Casnati and Ekedahl [CE96, Cas96] describing covers of degree d in terms of pairs of vector bundles $\mathscr{E}$ and $\mathscr{F}$, where $\mathscr{E}$ has rank $d-1$ and $\mathscr{F} \subset \mathrm{Sym}^2 \mathscr{E}$ corresponds to a certain family of quadrics determined by the curve. In the $d = 4$ case, $\mathscr{F}$ has rank 2, corresponding to 4 points in $\mathbb{P}^2$ being a complete intersection of two quadratics, whereas in the case $d = 5$, $\mathscr{F}$ has rank 5, corresponding to 5 points in $\mathbb{P}^3$ being the vanishing locus of the five 4×4 Pfaffians of a certain 5×5 matrix of linear forms. As in the degree 3 case, we can then stratify the Hurwitz stack in terms of the splitting types of $\mathscr{E}$ and $\mathscr{F}$, and compute the classes yielding curves of degree d as sections of a certain vector bundle $\mathscr{H}(\mathscr{E}, \mathscr{F})$ on $\mathbb{P}^1$, depending on $\mathscr{E}$ and $\mathscr{F}$. It is significantly more difficult to calculate the relevant local classes giving the smoothness conditions in fibers in degrees 4 and 5 than it is in degree 3. Nevertheless, we are able to do so by reformulating the question in terms of computing classes of certain classifying stacks for positive dimensional disconnected algebraic groups, and applying a number of results of Ekedahl. The result is Theorem 8.3, which can be viewed as a motivic analog of Bhargava's mass formula [Bha07] counting extensions of local fields in arbitrary degree. The specific splitting types of $\mathscr{E}$ and $\mathscr{F}$ which appear are not nearly so simple as in the degree 3 case, but it turns out that the expressions work out modulo high codimension. For this it is important not to count D_4 covers, i.e. degree 4 covers which factor through a hyperelliptic curve. As in the degree 3 case, it turns out that, at least in the simply branched case, the sum over splitting types of $\mathscr{E}$ and $\mathscr{F}$ cancel out with the local conditions we impose, again by the Tamagawa number formula.

1.19 Outline of the paper

The structure of the remainder of the paper is as follows. In § 2, we give background on the Grothendieck ring of stacks, set up the precise variant we will work in, and recall the notion of motivic Euler products. Then, in § 3 we prove generalizations of parametrizations due to Miranda, Casnati–Ekedahl, and Casnati regarding Gorenstein covers of degree $d \leq 5$. In degrees 3 and 4, generalizations to arbitrary covers of an arbitrary base have been previous shown by Poonen [Poo08, Proposition 5.1] and the third author [Woo11, Theorem 1.1], but in degree 5 we require new arguments, and here we present a (mostly) uniform argument for degrees 3, 4, and 5. In § 4 we upgrade the above-mentioned parametrizations for $d \leq 5$ to describe simple presentations of the stack of degree d Gorenstein covers as a global quotient stack. Having settled the above preliminaries, we define the Hurwitz stacks we will work with in § 5 and prove they are algebraic. We then describe natural stratifications of these Hurwitz stacks that arise from the structure of the parametrizations in § 6. Using these parametrizations, we give descriptions of these strata as quotient stacks in § 7. We next begin our proof of the main theorem by computing the local conditions in the Grothendieck ring corresponding to a cover being smooth with specified ramification conditions in § 8. In § 9 we establish bounds on the codimension of the contributions to the Hurwitz stack from various strata, which will enable us to prove our main result in § 10. The proof for the case of degree 2 is slightly different from that in degrees $3 \leq d \leq 5$, and we complete this in § 11.

1.20 Notation

Let X_Z denote the fibered product $X \times_Y Z$ of schemes, when Y is clear from context. Similarly define $X_R := X \times_Y \operatorname{Spec} R$. For X an integral variety, we use $K(X)$ to denote its function field.

Recall that for G a group, the wreath product $G \wr S_n$ is the semidirect product $G^n \rtimes S_n$ where S_n acts on G^n by the permutation action on the n copies of G. More generally, for $\mathscr{E}$ a category, let $\mathscr{E} \wr BS_j$ denote the corresponding wreath product of categories (see [Eke09b, p. 5]) so that, in particular, $BG \wr BS_j = B(G \wr S_j)$.

For $\mathscr{X}$ a stack, and G a group scheme acting on $\mathscr{X}$, we use $[\mathscr{X}/G]$ to denote the quotient stack. To avoid confusion with this notation, for $\mathscr{X}$ a stack, we use $\{\mathscr{X}\}$ to denote its class in the Grothendieck ring of stacks, see Definition 2.6.

We call an algebraic group G over a field k *special* if every G-torsor over a k-scheme X is trivial Zariski locally on X.

When working in $\widehat{\widetilde{K_0}}(\mathrm{Spaces}_k)$, defined in Definition 2.6, we say two classes $A, B \in \widehat{\widetilde{K_0}}(\mathrm{Spaces}_k)$ of dimension d are *equal modulo codimension* n to mean $A - B$ lies in the dimension $d - n$ filtered part of $\widehat{\widetilde{K_0}}(\mathrm{Spaces}_k)$.

Let $D := \operatorname{Spec} k[\varepsilon]/(\varepsilon^2)$ be the dual numbers. For X a projective scheme over Y, let $\mathrm{Hilb}^d_{X/Y}$ denote the Hilbert scheme parameterizing degree d dimension 0 subschemes of X over Y.

For $X \to Y$ a finite locally free map, and Z an X-scheme, let $\mathrm{Res}_{X/Y}(Z) \to Y$ denote the Weil restriction. Recall (e.g. [BLR90, § 7.6]) that the Weil restriction is the functor defined on T points by $\mathrm{Res}_{X/Y}(Z)(T) = Z(T \times_Y X)$. For Z quasi-projective over X, $\mathrm{Res}_{X/Y}(Z)$ is representable [BLR90, § 7.6, Theorem 4].

2. Background: the Grothendieck ring of stacks and motivic Euler products

In this section, we begin by defining useful variants of the Grothendieck ring. Ultimately, we will compute the classes of Hurwitz stacks in a ring we call $\widehat{\widetilde{K_0}}(\mathrm{Spaces}_k)$, obtained from the usual Grothendieck ring of varieties by quotienting by universally bijective (i.e. bijective on topological spaces after any base change or, equivalently, radicial surjective) morphisms, inverting $\mathbb{L} = \{\mathbb{A}^1\}$, and then completing with respect to the dimension filtration. Following this, we recall basic definitions associated to motivic Euler products, following [Bil17] and [BH21]. We also prove these Euler products satisfy a multiplicativity property (Lemma 2.14).

2.1 Variations of the Grothendieck ring

Recall that we are working over a fixed field k. We begin by introducing the Grothendieck ring of algebraic spaces.

Definition 2.2. Let $K_0(\mathrm{Spaces}_k)$ denote the *Grothendieck ring of algebraic spaces over k*. This is the ring generated by classes $\{X\}$ of algebraic spaces X of finite type over k with relations given by $\{X\} = \{Y\}$ if there is an isomorphism $X \simeq Y$ over k and $\{X\} = \{Z\} + \{X - Z\}$ for any closed sub-algebraic space $Z \subset X$. Let X^{red} denote the reduction of X. Applying this in the case $Z = X^{\mathrm{red}}$, we have $\{X\} = \{X^{\mathrm{red}}\}$. Multiplication is given by $\{X\} \cdot \{Y\} = \{X \times_k Y\}$.

More generally, if S is a finite-type algebraic space over k we can define $K_0(\mathrm{Spaces}_k /S)$ as the free abelian group generated by classes of morphisms $X \to S$ with relations $\{X/S\} = \{Z/S\} + \{X - Z/S\}$ for any closed sub-algebraic space $Z \subset X$, where the implicit maps $f|_Z : Z \to S, f|_{X-Z} : X - Z \to S$ are obtained by restricting the map $f : X \to S$. Multiplication is given by $\{X/S\} \cdot \{Y/S\} = \{X \times_S Y\}$.

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We use a *k-variety* to mean a reduced, separated, finite-type k-scheme. We let Var_k denote the category of k-varieties. One can similarly define $K_0(\mathrm{Var}_k)$, see [BH21, § 2]. Similarly, for S a k-variety, one can analogously define $K_0(\mathrm{Var}_k/S)$, see [BH21, § 2].

Proposition 2.3. *Let S be k-variety. The natural map $\phi : K_0(\mathrm{Var}_k/S) \to K_0(\mathrm{Spaces}_k/S)$, sending $\{X/S\}$ viewed as a k-variety to the same $\{X/S\}$ viewed as a finite type k-space, is an isomorphism.*

Proof. We first show that for any finite-type k-space X, there is a finite collection $X_1, \dots, X_n$ of locally closed k-subspaces isomorphic to schemes, forming a stratification of X. Here, a stratification means that a $\overline{k}$ point of X factors through some X_i. The key input we will need is that finite-type spaces are quasi-separated, and so they contain a dense open isomorphic to a scheme [Ols16, Theorem 6.4.1]. This, together with the facts that $\{X/S\} = \{X^{\mathrm{red}}/S\}$ and that any finite-type k-scheme has a stratification by separated finite-type k-schemes shows that any finite type k-space X has a stratification by locally closed subschemes.

Next, let us show ϕ is surjective. For this, if $\{X/S\}$ is any finite-type algebraic space, we can use the above stratification to write $\{X/S\} = \sum_{i=1}^n \{X_i/S\}$, for X_i varieties over S, implying ϕ is surjective.

We conclude the proof by showing that ϕ is injective. In order to show this, it is enough to show that any single relation in $K_0(\mathrm{Spaces}_k/S)$ is expressible in terms of relations from $K_0(\mathrm{Var}_k/S)$. More precisely, if $\{X/S\} \in K_0(\mathrm{Spaces}_k/S)$ and Z is a closed subspace $Z \subset X$, it suffices to show that the relation $\{X/S\} = \{Z/S\} + \{X - Z/S\}$ can be expressed as the image under ϕ of a sum of relations from $K_0(\mathrm{Var}_k)$. To see this is the case, write $\{X/S\} = \sum_{i=1}^n \{X_i/S\}$ and where $X_1, \dots, X_n$ are k-varieties. Then, let Z_i be the reduction of $X_i \times_X Z$. Note that Z_i is a scheme from the definition of algebraic space because X_i and Z are both schemes. In addition, Z_i is separated since it is a closed subscheme of the separated scheme X_i. Hence, Z_i is a variety. We can also write $\{Z/S\} + \{X - Z/S\} = \sum_{i=1}^n \{Z_i/S\} + \sum_{i=1}^n \{X_i - Z_i/S\}$. Therefore, it suffices to verify that

$$\sum_{i=1}^n \{X_i/S\} = \sum_{i=1}^n \{Z_i/S\} + \sum_{i=1}^n \{X_i - Z_i/S\}$$

is the image under ϕ of a sum of relations in the Grothendieck ring of varieties. Indeed, it is the sum over i of the relations $\{X_i/S\} = \{Z_i/S\} + \{X_i - Z_i/S\}$. ☐

We next introduce the Grothendieck ring of algebraic stacks.

Definition 2.4. The *Grothendieck ring of algebraic stacks* (over k) is the ring $K_0(\mathrm{Stacks}_k)$ generated by classes of algebraic stacks $\{\mathscr{X}\}$ *of finite type over k with affine diagonal*, with the three relations:

(1) $\{\mathscr{X}\} = \{\mathscr{Y}\}$ if there is an isomorphism $\mathscr{X} \simeq \mathscr{Y}$ over k;
(2) $\{\mathscr{X}\} = \{\mathscr{Z}\} + \{\mathscr{U}\}$ for any closed substack $\mathscr{Z} \subset \mathscr{X}$ with open complement $\mathscr{U} \subset \mathscr{X}$;
(3) $\{\mathrm{Spec}_{\mathscr{X}}(\mathrm{Sym}^\bullet_{\mathscr{X}} \mathscr{E})\} = \{\mathscr{X} \times_k \mathbb{A}^n\}$ for $\mathscr{E}$ any locally free sheaf on $\mathscr{X}$ of rank n.

Multiplication in this ring is given by $\{\mathscr{X}\} \cdot \{\mathscr{Y}\} = \{\mathscr{X} \times_k \mathscr{Y}\}$.

Note that condition (3) above follows from the first two in the case of schemes, because vector bundles on schemes are Zariski locally trivial. However, vector bundles over stacks may fail to be Zariski locally trivial, as is the case for nontrivial vector bundles on BG.

Remark 2.5. Let $\mathbb{L} := \{\mathbb{A}^1_k\}$ denote the class of the affine line. The natural map $K_0(\mathrm{Spaces}_k) \to K_0(\mathrm{Stacks}_k)$ induces an isomorphism

$$K_0(\mathrm{Spaces}_k)[\mathbb{L}^{-1}, (\mathbb{L}^n - 1)^{-1}_{n\geq 1}] \xrightarrow{\sim} K_0(\mathrm{Stacks}_k)$$

[Eke09a, Theorem 1.2]. Here $K_0(\mathrm{Spaces}_k)[\mathbb{L}^{-1}, (\mathbb{L}^n - 1)^{-1}]$ denotes the ring obtained from $K_0(\mathrm{Spaces}_k)$ by inverting $\mathbb{L}$, as well as $\mathbb{L}^n - 1$ for all positive integers n. This isomorphism is motivated by Definition 2.4(3) and the fact that inverting the classes of $\mathbb{L}$ and $\mathbb{L}^n - 1$ for all n is equivalent to inverting the classes of GL_n for all n.

In order to apply the results of [BH21] to sieve out smooth covers from all covers, we will need to work in a slight modification of the Grothendieck ring of stacks where we invert universally bijective (i.e. radicial surjective) morphisms and then complete along the dimension filtration.

Definition 2.6. Let k be a field and let $K_0(\mathrm{Spaces}_k)$ denote the Grothendieck ring of algebraic spaces over k from Definition 2.2. From $K_0(\mathrm{Spaces}_k)$, we will construct another ring, $\widehat{\widetilde{K}_0}(\mathrm{Spaces}_k)$, in three steps.

(1) For any universally bijective map $f : X \to Y$ of finite-type algebraic spaces over k, we impose the additional relation that $\{X\} = \{Y\}$. Call the result (only for the next paragraph) $K_0(\mathrm{Spaces}_k)_{\mathrm{RS}}$.
(2) Define $\widetilde{K}_0(\mathrm{Spaces}_k) := K_0(\mathrm{Spaces}_k)_{\mathrm{RS}}[\mathbb{L}^{-1}]$. Like $K_0(\mathrm{Stacks}_k)$, the ring $\widetilde{K}_0(\mathrm{Spaces}_k)$ has a filtration given by dimension with the ith filtered part $F^i\widetilde{K}_0(\mathrm{Spaces}_k) \subset \widetilde{K}_0(\mathrm{Spaces}_k)$ denoting the subset of $\widetilde{K}_0(\mathrm{Spaces}_k)$ spanned by classes of dimension at most $-i$.
(3) Finally, let

$$\widehat{\widetilde{K}_0}(\mathrm{Spaces}_k) := \varprojlim_{i\geq 0} \widetilde{K}_0(\mathrm{Spaces}_k)/F^i\widetilde{K}_0(\mathrm{Spaces}_k)$$

be the completion along the dimension filtration.

Similarly, for $K_0(\mathrm{Stacks}_k)$ the Grothendieck ring of algebraic stacks over k of Definition 2.4, we analogously define $\widehat{\widetilde{K}_0}(\mathrm{Stacks}_k)$ in the same three steps, replacing the word 'spaces' above by 'stacks'.

(1) We first impose the relation $\{X\} = \{Y\}$ for every universally bijective map of algebraic stacks $f : X \to Y$ of finite type with affine diagonal, and denote the result $K_0(\mathrm{Stacks}_k)_{\mathrm{RS}}$.
(2) Define $\widetilde{K}_0(\mathrm{Stacks}_k) := K_0(\mathrm{Stacks}_k)_{\mathrm{RS}}[\mathbb{L}^{-1}]$. Like $K_0(\mathrm{Stacks}_k)$, the ring $\widetilde{K}_0(\mathrm{Stacks}_k)$ has a filtration given by dimension with the ith filtered part $F^i\widetilde{K}_0(\mathrm{Stacks}_k) \subset \widetilde{K}_0(\mathrm{Stacks}_k)$ denoting the subset of $\widetilde{K}_0(\mathrm{Stacks}_k)$ spanned by classes of dimension at most $-i$.
(3) Finally, let

$$\widehat{\widetilde{K}_0}(\mathrm{Stacks}_k) := \varprojlim_{i\geq 0} \widetilde{K}_0(\mathrm{Stacks}_k)/F^i\widetilde{K}_0(\mathrm{Stacks}_k)$$

be the completion along the dimension filtration.

Remark 2.7. In characteristic 0, identifying classes along universally bijective morphisms does not alter the Grothendieck ring. See [BH21, Remarks 2.0.2 and 7.3.2] for some justification of why we are inverting universally bijective morphisms.

But we do not know if inverting universally bijective morphisms alters the Grothendieck ring of spaces or stacks in positive characteristic.

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Since Hurwitz stacks are not, in general, algebraic spaces, but the results of [BH21] apply to the completed Grothendieck ring of algebraic spaces $\widehat{\widetilde{K_0}}(\mathrm{Spaces}_k)$, it will be useful to know that one can also obtain $\widehat{\widetilde{K_0}}(\mathrm{Spaces}_k)$ from $K_0(\mathrm{Stacks}_k)$ by inverting universally bijective maps and completing along the dimension filtration, as we next verify.

Lemma 2.8. *The natural map* $\widehat{\widetilde{K_0}}(\mathrm{Spaces}_k) \to \widehat{\widetilde{K_0}}(\mathrm{Stacks}_k)$ *is an isomorphism.*

Proof. First note that although we constructed $\widehat{\widetilde{K_0}}(\mathrm{Spaces}_k)$ from $K_0(\mathrm{Spaces}_k)$ by first quotienting by universally bijective morphisms and then inverting $\mathbb{L}$, we could have equally well first inverted $\mathbb{L}$ and then inverted universally bijective morphisms. Since localization commutes with taking quotients, we obtain the same result by doing these steps in either order.

Since we can localize and take quotients in any order, using Remark 2.5, we can equivalently obtain $\widehat{\widetilde{K_0}}(\mathrm{Stacks}_k)$ by identifying universally bijective morphisms of spaces and then inverting $\mathbb{L}, \mathbb{L}^n - 1$ and completing along the dimension filtration. To show $\widehat{\widetilde{K_0}}(\mathrm{Spaces}_k) \to \widehat{\widetilde{K_0}}(\mathrm{Stacks}_k)$ is an isomorphism, we wish to show that beginning with $\widetilde{K_0}(\mathrm{Spaces}_k)$ and completing along the dimension filtration is equivalent to first inverting $\mathbb{L}^n - 1$ for all $n \geq 1$ and then completing along the dimension filtration. Indeed, one may define a map $\widehat{\widetilde{K_0}}(\mathrm{Stacks}_k) \to \widehat{\widetilde{K_0}}(\mathrm{Spaces}_k)$ induced by the map $\widetilde{K_0}(\mathrm{Spaces}_k)[(\mathbb{L}^n - 1)^{-1}_{n \geq 0}] \to \widehat{\widetilde{K_0}}(\mathrm{Spaces}_k)$ extended by sending the class of $(\mathbb{L}^n - 1)^{-1} \mapsto \sum_{i \geq 1} \mathbb{L}^{-in}$. Upon completing along the dimension filtration this defines the desired isomorphism $\widehat{\widetilde{K_0}}(\mathrm{Stacks}_k) \to \widehat{\widetilde{K_0}}(\mathrm{Spaces}_k)$ inverse to the natural map $\widehat{\widetilde{K_0}}(\mathrm{Spaces}_k) \to \widehat{\widetilde{K_0}}(\mathrm{Stacks}_k)$ given above. □

Remark 2.9. Due to the equivalence of Lemma 2.8, in what follows, we will work in $\widehat{\widetilde{K_0}}(\mathrm{Spaces}_k)$. In particular, it makes sense to speak of classes of stacks with affine diagonal in $\widehat{\widetilde{K_0}}(\mathrm{Spaces}_k)$ by Lemma 2.8.

2.10 Motivic Euler products

We recall the notion of motivic Euler products in the Grothendieck ring, which is crucial in our proof. See [Bil17] for an introduction to motivic Euler products, and [BH21, § 6] for more details.

We begin by introducing notation to give the definition of motivic Euler products in the setting we will need. For a finite multiset μ, with underlying set I, we write $\mu = (m_i)_{i \in I}$, where m_i is the number of copies of i in μ. Let X be a reduced, quasi-projective scheme over a field k. For any finite multiset $\mu = (m_i)_{i \in I}$, there is a finite surjective map $p : \prod_{i \in I} X^{m_i} \to \prod_{i \in I} \mathrm{Sym}^{m_i} X$. Let U denote the open subscheme of $\prod_{i \in I} X^{m_i}$ where no two coordinates agree and let $C^\mu(X)$ denote the open subscheme $p(U) \subset \prod_{i \in I} \mathrm{Sym}^{m_i} X$. Informally speaking, $C^\mu(X)$ parameterizes configurations of μ-labeled points on X.

More generally, for $\mathcal{X} = (X_i)_{i \in I}$ a collection of reduced, quasi-projective schemes X_i with morphisms to X, and $\mu = (m_i)_{i \in I}$ a multiset, define $C^\mu_X(\mathcal{X})$ as the preimage of $C^\mu(X) \subset \prod_{i \in I} \mathrm{Sym}^{m_i} X$ under the projection $\prod_{i \in I} \mathrm{Sym}^{m_i} X_i \to \prod_{i \in I} \mathrm{Sym}^{m_i}(X)$. As in [BH21, Definition 6.1.8], one can extend this definition to make sense of $C^\mu_X(\mathcal{A})$ as an element of $K_0(\mathrm{Spaces}_k)$ where $\mathcal{A} = (a_i)_{i \in I}$ with a_i in $K_0(\mathrm{Spaces}_k / X)$.

Let $\mathbb{N}$ denote the positive integers. Let $\mathcal{P}$ be the set of non-empty finite multisets of positive integers, and for such a multiset $\mu = (m_i)_{i \in \mathbb{N}}$, let $|\mu| := \sum_i i \cdot m_i$. Following [BDH22, § 2.2.2], for

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$\mathcal{A} = (a_i)_{i\in\mathbb{N}}$ a collection of classes in $K_0(\mathrm{Spaces}_k/X)$, define the *motivic Euler product*

$$\prod_{x\in X}\left(1+\sum_{i=1}^{\infty} a_{i,x}t^i\right) := 1 + \sum_{\mu\in\mathcal{P}} C_X^{\mu}((a_i)_{i\in\mathbb{N}})t^{|\mu|} \in K_0(\mathrm{Spaces}_k)[\![t]\!]. \tag{2.1}$$

Here, $a_{i,x}$ is formal notation to indicate the a_i on which the definition depends. When we write a class $b_i \in K_0(\mathrm{Spaces}_k)$ in place of $a_{i,x}$, it indicates that $a_i = [Y_i \times X] - [Z_i \times X]$, where Y_i, Z_i are algebraic spaces of finite type over k such that $b_i = [Y_i] - [Z_i]$ and $Y_i \times X, Z_i \times X$ have the natural projection map to X.

Let $r \in \mathbb{N}$, and let I be the set of r-tuples of non-negative integers, not all 0. Note that I is a semigroup under coordinate-wise addition. Let $\mathcal{P}_r$ denote the set of non-empty finite multisets of elements of I, and for $\mu \in \mathcal{P}_r$, let $|\mu| \in I$ denote the sum of the elements of μ. More generally, for indeterminates $t_1, \dots, t_r$ one can define, for $\mathcal{A} = (a_{\underline{i}})_{\underline{i}\in I}$, a collection of classes in $K_0(\mathrm{Spaces}_k/X)$,

$$\prod_{x\in X}\left(1+\sum_{\underline{i}\in I} a_{\underline{i},x}\underline{t}^{\underline{i}}\right) := 1 + \sum_{\mu\in\mathcal{P}_r} C_X^{\mu}((a_{\underline{i}})_{\underline{i}\in I})\underline{t}^{|\mu|} \in K_0(\mathrm{Spaces}_k)[\![t_1,\dots,t_r]\!], \tag{2.2}$$

where for $\underline{i} = (i_1, \dots, i_r) \in I$, we write $\underline{t}^{\underline{i}}$ for $t_1^{i_1}\cdots t_r^{i_r}$.

Warning 2.11. The left-hand side of (2.1) is merely (evocative) notation, and has no intrinsic meaning beyond the right-hand side.

In the special cases that we will use them in, motivic Euler products are the same as the power structures of Gusein-Zade, Luengo, and Melle-Hernàndez [GLM04]. We now specialize to the one variable case.

In good circumstances, there is an *evaluation map* at $t = 1$ sending a motivic Euler product, viewed as an element of $K_0(\mathrm{Spaces}_k)[\![t]\!]$ to an element of $\widehat{\widehat{K_0}}(\mathrm{Spaces}_k)$, as in [BH21, Definition 6.4.1 and Notation 6.4.2]. This makes sense whenever the motivic Euler product 'converges at $t = 1$', meaning there are only finitely many terms μ so that $C^{\mu}_{X/S}(a)$ is outside any given piece of the dimension filtration.

Notation 2.12. For a motivic Euler product $\prod_{x\in X}(1 + a_x t)$ which converges at $t = 1$, we use

$$\prod_{x\in X}(1 + a_x t)|_{t=1}$$

to denote the evaluation at $t = 1$ in $\widehat{\widehat{K_0}}(\mathrm{Spaces}_k)$.

We will often also write $\prod_{x\in X}(1 + a_x)$ to also denote the evaluation of the motivic Euler product $\prod_{x\in X}(1 + a_x t)$ at $t = 1$ in $\widehat{\widehat{K_0}}(\mathrm{Spaces}_k)$, in order to shorten notation, but see Warning 2.13.

Warning 2.13. Due to the extreme care with which one must handle motivic Euler products, we acknowledge that Notation 2.12 is not very good notation. It is likely best to think of motivic Euler products as power series in t which are being evaluated at values of t, rather then actual elements in $\widehat{\widehat{K_0}}(\mathrm{Spaces}_k)$, as the manipulations one wants to make have only primarily been established in terms of the power series, and not in terms of their evaluations in $\widehat{\widehat{K_0}}(\mathrm{Spaces}_k)$. We choose to use this convention so as to shorten unwieldy formulas.

In particular, one must be careful that the two expressions $\prod_{x\in X/S}(1 + \sum_{i\in I} a_{i,x}p_i((s_j)_{j\in J}))$ and $\prod_{x\in X/S}(1 + \sum_{i\in I} a_{i,x}t_i)|_{t_i=p_i(\underline{s})}$ do not necessarily agree. However, when these sets indexing

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the variables t_i and s_j are finite, and all $p_i((s_j)_{j\in J})$ are monomials, these two expressions do agree by [BH21, Lemma 6.5.1].

An important lemma will be that these Euler products in $\widehat{\widetilde{K_0}}(\mathrm{Spaces}_k)$ are multiplicative. We now verify this, the key input being multiplicativity of motivic Euler products in $K_0(\mathrm{Spaces}_k)[\![t_1,t_2]\!]$.

Lemma 2.14. *Suppose a and b are two classes in $K_0(\mathrm{Spaces}_k)$ such that the Euler products $\prod_{x\in X}(1+a_xt)$ and $\prod_{x\in X}(1+b_xt)$ converge at $t=1$ in $K_0(\mathrm{Spaces}_k)$. Then,*

$$\prod_{x\in X}(1+a_x)\cdot\prod_{x\in X}(1+b_x)=\prod_{x\in X}((1+a_x)(1+b_x)) \tag{2.3}$$

in $\widehat{\widetilde{K_0}}(\mathrm{Spaces}_k)$.

Proof. We would like to say this follows from multiplicativity of Euler products [Bil17, Proposition 3.9.2.4], but the issue is that when we apply [Bil17, Proposition 3.9.2.4] the left-hand side of (2.3) is equal to

$$\begin{aligned}\left(\prod_{x\in X}(1+a_xt)\cdot\prod_{x\in X}(1+b_xt)\right)\bigg|_{t=1}&=\left(\prod_{x\in X}(1+a_xt)\cdot(1+b_xt)\right)\bigg|_{t=1}\\&=\left(\prod_{x\in X}(1+a_xt+b_xt+a_xb_xt^2)\right)\bigg|_{t=1}.\end{aligned}\tag{2.4}$$

On the other hand, the right-hand side of (2.3) is, by definition,

$$\left(\prod_{x\in X}(1+a_xt+b_xt+a_xb_xt)\right)\bigg|_{t=1}. \tag{2.5}$$

The lemma follows because

$$\prod_{x\in X}(1+a_xt+b_xt+a_xb_xs)|_{s=t=1}=\prod_{x\in X}(1+a_xt+b_xt+a_xb_xt)|_{t=1}$$

and also

$$\prod_{x\in X}(1+a_xt+b_xt+a_xb_xs)|_{s=t=1}=\prod_{x\in X}(1+a_xt+b_xt+a_xb_xt^2)|_{t=1}$$

by [BH21, Lemma 6.5.1]. □

3. Parametrizations of low-degree covers

The key to computing the class of Hurwitz stacks of low-degree covers of $\mathbb{P}^1$ is the parametrization of covers of degree $d\le 5$ of a general base scheme. In the case $d=3$, the first such parametrization was given by Miranda [Mir85, Theorem 1.1], for arbitrary degree 3 covers of an irreducible scheme over an algebraically closed field of characteristic not equal to 2 or 3. Pardini [Par89] later generalized Miranda's result to characteristic 3, and Casnati and Ekedahl [CE96, Theorem 3.4] generalized the result to Gorenstein degree 3 covers of an integral noetherian scheme. Poonen [Poo08, Proposition 5.1] gave a complete parametrization of degree 3 covers of an arbitrary base scheme (see also [Woo11, Theorem 2.1]). When $d=4$, Casnati and Ekedahl [CE96, Theorem 4.4] gave a parametrization of Gorenstein degree 4 covers of an integral noetherian scheme. The third author [Woo11, Theorem 1.1] generalized this to a parametrization of arbitrary degree 4 covers along with the data of a cubic resolvent cover (which is unique in the Gorenstein case) over

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an arbitrary base scheme. When $d=5$, Casnati [Cas96, Theorem 3.8] gave a parametrization of degree 5 covers, satisfying a certainly 'regularity' condition (see Remark 3.7), of an integral noetherian scheme. (We also note that Wright and Yukie [WY92] gave these parametrizations for a covers of a field, and Delone and Faddeev [DF64] and Bhargava [Bha04, Bha08] gave these parametrizations for covers of $\operatorname{Spec}\mathbb{Z}$. Bhargava's parametrizations require additional resolvent data for non-Gorenstein covers. Bhargava, Shankar, and Wang [BSW15, § 3] have refined Wright and Yukie's work for covers of global fields.)

In this section, we will prove similar parametrizations, but suited for our particular application. For our purposes, we would like to parametrize only Gorenstein covers, but over an arbitrary base. For $d=3,4$, such a result could be deduced directly from [Poo08, Proposition 5.1] and [Woo11, Theorem 1.1] by specializing to Gorenstein covers. However, for the case $d=5$ some new arguments are required both to obtain all Gorenstein covers and to generalize to an arbitrary base. For uniformity of exposition, we show how all of the parametrizations of Gorenstein covers can be obtained from the approach of Casnati and Ekedahl.

Casnati and Ekedahl [CE96] prove a structure theorem [CE96, Theorem 2.1] (a reformulation of [CE96, Theorem 1.3]), which describes a minimal resolution of covers of arbitrary degree of an integral scheme. We will need to extend this structure theorem from integral schemes to arbitrary (including non-reduced) bases. Essentially the same proof given in [CE96, Theorem 2.1] applies, suitably replacing Grauert's theorem with cohomology and base change. We thank Gianfranco Casnati for helpful conversations confirming this. We will then apply this structure theorem to obtain our desired parametrizations of covers in degrees 3, 4, and 5, analogously to how it was done by Casnati and Ekedahl in [CE96, Theorems 3.4, 4.4] and [Cas96, Theorem 3.8].

We also upgrade Casnati's result in degree 5 in an additional way to deal with all Gorenstein covers, see Remark 3.7.

3.1 The main structure theorem from Casnati and Ekedahl

We next recall the main structure theorem and give its proof in the more general setting. In essence, it says that degree d Gorenstein covers are classified by linear-algebraic data. It is convenient to describe this as saying that a number of moduli stacks are isomorphic.

We first recall some terminology. We will consider degree d covers which are finite locally free. A finite locally free degree d cover is *Gorenstein* if the scheme-theoretic fiber X_y over $\kappa(y)$ is Gorenstein for every $y \in Y$. For k a field, a subscheme $X \subset \mathbb{P}^n_k$ is *arithmetically Gorenstein* if the affine cone over X, viewed as a subscheme of $\mathbb{A}^{n+1}_k$, is Gorenstein. For $\mathscr{E}$ a rank $d-1$ locally free sheaf of $\mathscr{O}_Y$-modules on Y, let $\pi : \mathbb{P}\mathscr{E} \to Y$ denote the corresponding projective bundle $\mathbb{P}\mathscr{E} := \operatorname{Proj}\operatorname{Sym}^\bullet \mathscr{E}$. We use the term *projective bundle* to describe the projectivization of a vector bundle. For $\mathscr{G}$ a sheaf of $\mathscr{O}_Z$-modules on a scheme or stack Z, we use $\mathscr{G}^\vee := \operatorname{Hom}_{\mathscr{O}_Z}(\mathscr{G}, \mathscr{O}_Z)$ to denote its dual. Finally, for κ a field, a subscheme of $\mathbb{P}^n_\kappa$ is nondegenerate if it is not contained in any hyperplane $H \subset \mathbb{P}^n_\kappa$.

Theorem 3.2 (Generalization of [CE96, Theorem 2.1], see also [CN07, Theorem 2.2]). *Let X and Y be schemes and let $\rho : X \to Y$ be a finite locally free Gorenstein map of degree d, for $d \geq 3$. Fix a vector bundle $\mathscr{E}'$ of rank $d-1$ on Y with corresponding projective bundle $\pi : \mathbb{P} := \mathbb{P}\mathscr{E}' \to Y$, and fix an embedding $i : X \to \mathbb{P}$ such that $\rho = \pi \circ i$. We further require that $\rho^{-1}(y) \subset \pi^{-1}(y) \simeq \mathbb{P}^{d-2}_{\kappa(y)}$ is a nondegenerate subscheme for each point $y \in Y$. A bundle $\mathscr{E}'$ and map i satisfying the above properties exists. Any two such triples $(\mathbb{P}, \pi, i)$ and $(\mathbb{P}_2, \pi_2, i_2)$ are uniquely isomorphic, meaning there is a unique isomorphism $\psi : \mathbb{P} \simeq \mathbb{P}_2$ such that $\pi_2 \circ \psi = \pi$ and $\psi \circ i = i_2$. Moreover, for any such triple $(\mathbb{P}, \pi, i)$ with $\rho = \pi \circ i$, the following properties hold.*

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(i) *Let* $\rho^{\#} : \mathscr{O}_Y \to \rho_* \mathscr{O}_X$ *denote the map defining* $\rho : X \to Y$ *and let* $\mathscr{E} := (\operatorname{coker} \rho^{\#})^\vee$. *Then,* $\mathbb{P} \simeq \mathbb{P}\mathscr{E}$.

(ii) *The composition* $\phi : \rho^* \mathscr{E} \to \rho^* \rho_* \omega_{X/Y} \to \omega_{X/Y}$ *is surjective, and so induces a map* $j : X \to \mathbb{P}\mathscr{E}$, *and* $(\mathbb{P}\mathscr{E}, \sigma : \mathbb{P}\mathscr{E} \to Y, j)$ *is a triple satisfying the properties above. The ramification divisor* $R \subset X$ *of* ρ *satisfies* $\mathscr{O}_X(R) \simeq \omega_{X/Y} \simeq j^* \mathscr{O}_{\mathbb{P}\mathscr{E}}(1)$.

(iii) *There is a sequence* $\mathscr{N}_0, \mathscr{N}_1, \dots, \mathscr{N}_{d-2}$ *of finite locally free* $\mathscr{O}_{\mathbb{P}\mathscr{E}'}$ *sheaves on* $\mathbb{P}\mathscr{E}'$ *with* $\mathscr{N}_0 := \mathscr{O}_{\mathbb{P}\mathscr{E}'}$ *and an exact sequence*

$$0 \longrightarrow \mathscr{N}_{d-2}(-d) \xrightarrow{\alpha_{d-2}} \mathscr{N}_{d-3}(-d+2) \xrightarrow{\alpha_{d-3}} \cdots$$

$$\cdots \xrightarrow{\alpha_2} \mathscr{N}_1(-2) \xrightarrow{\alpha_1} \mathscr{O}_{\mathbb{P}\mathscr{E}'} \longrightarrow \mathscr{O}_X \longrightarrow 0, \qquad (3.1)$$

such that the restriction of (3.1) *to the fiber* $(\mathbb{P}\mathscr{E}')_y := \pi^{-1}(y)$ *over* y *is a minimal free resolution of the structure sheaf of* $X_y := \rho^{-1}(y)$ *for every point* $y \in Y$. *Given* $\rho, \mathscr{E}', i$ *as above, the exact sequence* (3.1) *is unique up to unique isomorphism, such that the isomorphism restricts to the identity map on the final nonzero term* $\mathscr{O}_X$, *among all sequences with the above-listed properties. The locally free sheaves* $\mathscr{F}_i := \pi_* \mathscr{N}_i$ *on* Y *satisfy* $\pi^* \mathscr{F}_i \simeq \mathscr{N}_i$. *Further* $\mathscr{N}_{d-2}$ *is invertible, and, for* $i = 1, \dots, d-3$, *one has*

$$\beta_i := \operatorname{rk} \mathscr{N}_i = \operatorname{rk} \mathscr{F}_i = \frac{i(d-2-i)}{d-1} \binom{d}{i+1}. \qquad (3.2)$$

Moreover, $X_y \subset \mathbb{P}_y$ *is a nondegenerate arithmetically Gorenstein subscheme,* $\pi^* \pi_* \mathscr{N}_i \simeq \mathscr{N}_i$ *for* $0 \le i \le d-2$, *and* $\mathscr{H}om_{\mathscr{O}_{\mathbb{P}\mathscr{E}'}}(\mathscr{N}_\bullet, \mathscr{N}_{d-2}(-d)) \simeq \mathscr{N}_\bullet$. *In addition, the formation of* $\pi_* \mathscr{N}_\bullet$ *commutes with base change on* Y.

(iv) *For* $\mathscr{N}_{d-2}$ *as in* (3.1), *we have* $\mathscr{E}' \simeq \mathscr{E}$ *if and only if* $\mathscr{N}_{d-2} \simeq \pi^* \det \mathscr{E}'$.

(v) *The pushforward of the map* $\alpha_1 : \mathscr{N}_1(-2) \to \mathscr{O}_{\mathbb{P}}$ *along* π *induces an injection* $\mathscr{F}_1 \to \operatorname{Sym}^2 \mathscr{E}$ *and for* $d-3 \ge i \ge 2$, *the pushforward* $\alpha_i : \mathscr{N}_i(-i-1) \to \mathscr{N}_{i-1}(-i)$ *along* π *induces an injection* $\mathscr{F}_i \to \mathscr{F}_{i-1} \otimes \mathscr{E}$.

(vi) *For any point* $y \in Y$, *no subscheme* $X'_y \subset X_y$ *of degree* $d-1$ *is sent under* ρ *to a hyperplane of* $\pi^{-1}(y)$.

Remark 3.3. The statement of Theorem 3.2 differs in several ways from the original statement [CE96, Theorem 2.1].

(1) As pointed out in [CN07, Theorem 2.2], it is necessary to add a nondegenerate hypothesis to the statement (which was an oversight in the original result).
(2) We do not require our base Y to be noetherian.
(3) We do not require our base Y to be integral.
(4) We show that given any two triples $(\mathbb{P}, \pi, i)$ there is a unique isomorphism between them, as in the sense of the statement of Theorem 3.2. In [CE96, Theorem 2.1], it is only shown that the bundle $\mathbb{P}$ is unique.
(5) In property (ii), we additionally show that $(\mathbb{P}\mathscr{E}, \sigma : \mathbb{P}\mathscr{E} \to Y, j)$ is one of the unique above-mentioned triples.
(6) In property (iii), we show the formation of $\pi_* \mathscr{N}_\bullet$ commutes with base change.
(7) In property (iii), we include the requirement that the isomorphism is unique among isomorphisms restricting to the identity on $\mathscr{O}_X$. This assumption was also needed in [CE96], but not explicitly stated there.
(8) We have also added property (v).
(9) We have added property (vi).

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Proof. As a first step, we reduce the proof to the case X and Y are noetherian.

Removing noetherian hypotheses. In view of the asserted uniqueness, by Zariski descent, we may reduce to the case that Y is affine. Because $\rho : X \to Y$ is locally finitely presented as it is finite locally free, we will next show we can spread out all of the above data to a finite-type scheme Y_0. More precisely, as a first step, by [Gro66, Proposition 8.9.1], we can find some finite-type schemes Y_0 and X_0, a map $\rho_0 : X_0 \to Y_0$ and a map $Y \to Y_0$ so that ρ is the base change of ρ_0 along $Y \to Y_0$. By the various spreading out results in [Gro66, § 8] after possibly replacing Y_0 with another finite-type scheme, we may additionally assume ρ_0 is Gorenstein, $\mathscr{E}'$ is the pullback of a vector bundle $\mathscr{E}'_0$ on Y_0, and the triples $(\mathbb{P}, \pi, i)$ and $(\mathbb{P}_2, \pi_2, i_2)$ are base changes of corresponding triples on Y_0. Nearly all parts of the theorem, except the unique isomorphism of two triples $(\mathbb{P}, \pi, i)$ and $(\mathbb{P}_2, \pi_2, i_2)$ and the unique isomorphism in property (iii), follow from the corresponding statement over Y_0. However, if these isomorphisms are not unique, there will be some noetherian scheme to which two different such isomorphisms descend and, hence, this claim can be verified after replacing Y_0 with another noetherian scheme. In particular, it suffices to prove the theorem in the case Y and X are noetherian, and even finite type over $\operatorname{Spec} \mathbb{Z}$. This removes the noetherian hypothesis, addressing Remark 3.3(2).

For the remainder of the proof, we assume X and Y are noetherian. The proof given in [CE96, Theorem 2.1] is broken up into steps A, B, C, and D. Step A has a minor inaccuracy which we next address. The only generalization needed occurs in step B, while steps C and D go through without change.

Addressing step A. We next explain the proof of step A, though we make the additional assumption that the field k is infinite. Before explaining this proof, we remark on an error in the proof of step A from [CE96, Step A, p. 443] when k is finite.

Remark 3.4. Let A be a finite k-algebra A with maximal ideals $\mathfrak{m}_1, \dots, \mathfrak{m}_p$. Let $\eta : A \to k$, be a generalized trace map, i.e. a surjection of k-vector spaces such that the only ideal contained in the kernel is the 0 ideal. It is then claimed that there exists $a \in \ker \eta - \bigcup_{i=1}^p \mathfrak{m}_i$.

This is not always true over finite fields, such as when $k = \operatorname{Spec} \mathbb{F}_2$ and $A = \mathbb{F}_2^5$ and $\eta : A \to k$ is the map given by summing the five coordinates. Indeed, the oversight in [CE96, Step A, p. 443] is that while over infinite fields, $\ker \eta \subset \bigcup_{i=1}^p \mathfrak{m}_i$ implies $\ker \eta \subset \mathfrak{m}_i$ for a single i, this does not always hold over finite fields. It is straightforward to check that this claim holds over an infinite field. Since we cannot have $\ker \eta \subset \mathfrak{m}_i$ by the definition of a generalized trace map, over infinite fields we conclude that $\ker \eta \subsetneq \bigcup_{i=1}^p \mathfrak{m}_i$.

Having explained the error when k is finite, we now conclude our commentary on the proof of step A. As mentioned above, the proof still works correctly in the case k is infinite. We also note that in the statement of [Sch86, Lemma, p. 119] which is cited in [CE96, Step A, p. 443], the subscheme D there should have degree d and lie in $\mathbb{P}^{d-2}$, as opposed to degree $d-2$ in $\mathbb{P}^{d-1}$. Note that in order to apply [Sch86, Lemma, p. 119], it is necessary to use the hypothesis that $X \subset \mathbb{P}\mathscr{E}$ is nondegenerate, a hypothesis which was omitted in [CE96, Theorem 2.1], addressing Remark 3.3(1). At this point, property (vi) follows from [Sch86, Lemma, p. 119], addressing Remark 3.3(8).

Addressing step B. Having established the result when $Y = \operatorname{Spec} \overline{k}$, it remains to carry out the proof for general bases following [CE96, Steps B, C, and D, p. 445–447]. In what follows, we next recapitulate the argument for step B [CE96, p. 445], modifying the application of Grauert's theorem to one of cohomology and base change, which allows us to remove the integrality hypothesis on Y, as in Remark 3.3(3).

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Recall the statement of step B.

Step B: Suppose there is a factorization $\rho = \pi \circ i$, for $\pi : \mathbb{P} \to Y$ a projective $\mathbb{P}^{d-2}$ bundle and $i : X \to \mathbb{P}$ an embedding with X_y a nondegenerate arithmetically Gorenstein subscheme of $\mathbb{P}_y$ for each $y \in Y$. Then, (3.1) exists, is unique up to unique isomorphisms, restricts to a minimal free resolution of $\mathscr{O}_{X_y}$ over each point $y \in Y$, and $\pi^*\pi_*\mathscr{N}_\bullet \simeq \mathscr{N}_\bullet$.

Note that when it is written the resolution is unique up to unique isomorphism in step B, the statement implicitly means such an isomorphism is unique up to those restricting to the identity on $\mathscr{O}_X$, as if such a specification were not given, we could compose with multiplication by a unit. This is the reason for the modification from Remark 3.3(7).

We next observe that it suffices to prove a version of step B where we replace Y with a geometric point over y. To be more precise, in order to verify step B, it suffices to verify step B′, given as follows.

Step B′: Suppose there is a factorization $\rho = \pi \circ i$, for $\pi : \mathbb{P} \to Y$ a projective $\mathbb{P}^{d-2}$ bundle and $i : X \to \mathbb{P}$ an embedding with X_y a nondegenerate arithmetically Gorenstein subscheme of $\mathbb{P}_y$ for each geometric point y of Y. Then, (3.1) exists, is unique up to unique isomorphisms, restricts to a minimal free resolution of $\mathscr{O}_{X_y}$ over each geometric point y of Y and $\pi^*\pi_*\mathscr{N}_\bullet \simeq \mathscr{N}_\bullet$.

We now explain why step B′ implies step B. Indeed, the conditions of X_y being a nondegenerate arithmetically Gorenstein subscheme and for a resolution of $\mathscr{O}_{X_y}$ being a minimal free resolution may be verified after replacing y with a geometric point $\overline{y}$ mapping to y. Therefore, step B′ implies step B.

We next verify step B′. In what follows, we therefore use y to denote a geometric point of Y, as opposed to a point of the underlying topological space whose with scheme structure given as the spectrum of the residue field at that point.

For the remainder of the verification of step B′, we only handle the case $d \geq 4$. The case $d = 3$ is quite analogous to the case $d \geq 4$, though significantly easier as the resolution has length 2.

Define maps j_y, i_y as in the following diagram.

$$\begin{array}{ccccc} X & \xrightarrow{i} & \mathbb{P} & \xleftarrow{j_y} & \mathbb{P}_y \\ & \searrow^{\rho} & \downarrow{\scriptstyle \pi} & & \downarrow \\ & & Y & \xleftarrow{i_y} & y \end{array} \tag{3.3}$$

Letting $\mathscr{I}$ denote the ideal sheaf of X in $\mathbb{P}$, we claim that $j_y^*\mathscr{I}$ is the ideal sheaf of X_y in $\mathbb{P}_y$. To see this, we only need to verify that $j_y^*\mathscr{I} \to j_y^*\mathscr{O}_\mathbb{P} \to j_y^*\mathscr{O}_X$ is exact. Since $\mathscr{O}_X$ is flat over Y, we will verify more generally that for $\mathscr{H}, \mathscr{G}, \mathscr{F}$ three sheaves on X with $\mathscr{F}$ flat over Y, and an exact sequence $0 \to \mathscr{H} \to \mathscr{G} \to \mathscr{F} \to 0$, the pullback sequence $0 \to j_y^*\mathscr{H} \to j_y^*\mathscr{G} \to j_y^*\mathscr{F} \to 0$ is exact. Indeed, this holds because $R^1 j_y^*\mathscr{F} = \mathscr{T}or_1^{\mathscr{O}_\mathbb{P}}(\mathscr{F}, \mathscr{O}_{\mathbb{P}_y}) = \mathscr{T}or_1^{\mathscr{O}_Y}(\mathscr{F}, \kappa(y)) = 0$. Here we are using that $\mathscr{F}$ is flat over Y for the final vanishing and $\mathscr{F} \otimes_{\mathscr{O}_\mathbb{P}} \mathscr{O}_{\mathbb{P}_y} \simeq \mathscr{F} \otimes_{\mathscr{O}_Y} \kappa(y)$ for the equality of $\mathscr{T}or$ sheaves.

Next, [CE96, Step A, p. 443] provides a resolution of $\mathscr{I}_{X_y/\mathbb{P}_y} = j_y^*\mathscr{I}$ of the following form.

$$\begin{aligned} 0 \longrightarrow \mathscr{O}_{\mathbb{P}_y}(-d) \xrightarrow{\alpha_{d-2,y}} \mathscr{O}_{\mathbb{P}_y}(2-d)^{\oplus \beta_{d-3}} \xrightarrow{\alpha_{d-3,y}} \cdots \\ \cdots \xrightarrow{\alpha_{2,y}} \mathscr{O}_{\mathbb{P}_y}(-2) \xrightarrow{\alpha_{1,y}} j_y^*\mathscr{I} \longrightarrow 0. \end{aligned} \tag{3.4}$$

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Note here that we have only verified step A in the case y is the spectrum of an algebraically closed field, but at this point we are assuming that y is a geometric point, as we are verifying step B$'$.

We claim $j_y^*\mathscr{I}$ is 3-regular, in the sense of Castelnuovo–Mumford regularity, i.e. $H^i(\mathbb{P}_y, j_y^*\mathscr{I}(3-i)) = 0$ for $i \geq 1$. To verify this, it follows from the definition of regularity that for an exact sequence $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ of sheaves with $\mathscr{F}'$ $m+1$-regular and $\mathscr{F}$ m-regular, $\mathscr{F}''$ is also m regular. Using this and the fact that $\mathscr{O}_{\mathbb{P}_y}(-k)$ is k-regular (and, hence, it is also $k+1$ regular by [FGI+05, Lemma 5.1(b)]), it follows by induction that $\operatorname{im}\alpha_{d-i,y}$ is $d-i+2$ regular. Therefore, $j_y^*\mathscr{I} = \operatorname{im}\alpha_{1,y}$ is 3-regular. By [FGI+05, Lemma 5.1(b)], we obtain $H^1(\mathbb{P}_y, j_y^*\mathscr{I}(n)) = 0$ for $n \geq 2$. Hence, by cohomology and base change, $R^1\pi_*\mathscr{I}(n) = 0$ for $n \geq 2$. Note that, often, cohomology and base change is only stated in the case y is a point (as opposed to a geometric point) but the case that y is a point follows from the case that y is a geometric point since the vanishing of cohomology groups can be verified after base change to an algebraic closure, using flat base change.

For our next step, we verify that $\pi_*\mathscr{I}(n)$ commutes with base change on Y for $n \geq 2$. For $\mathscr{F}$ a sheaf, let us denote by $\phi_y^i(\mathscr{F}) : R^i\pi_*\mathscr{F} \otimes \kappa(y) \to H^i(X_y, \mathscr{F}|_{X_y})$ the natural base change map. Then we have seen above that, for $n \geq 2$, $\phi_y^1(\mathscr{I}(n))$ is an isomorphism at all y. Further, $R^1\pi_*\mathscr{I}(n)$ is locally free (and, in fact, equal to 0) which implies by cohomology and base change that $\phi_y^0(\mathscr{I}(n))$ is an isomorphism for all $n \geq 2$. In other words, the formation of $\pi_*\mathscr{I}(n)$ then commutes with base change on Y. Further, again by cohomology and base change, $\pi_*\mathscr{I}(n)$ is a locally free sheaf when $n \geq 2$ (since the condition from the theorem on cohomology and base change that ϕ_y^{-1} be an isomorphism is vacuously satisfied).

Set $\mathscr{F}_1 := \pi_*\mathscr{I}(2)$ and $\mathscr{N}_1 := \pi^*\mathscr{F}_1$. Let $\alpha_1 : \mathscr{N}_1(-2) \to \mathscr{I}$ denote the evaluation map coming from the adjunction $\pi^*\pi_*\mathscr{I}(2) \otimes \mathscr{O}_{\mathbb{P}}(-2) \to \mathscr{I}(2) \otimes \mathscr{O}_{\mathbb{P}}(-2) \to \mathscr{I}$. As we have shown above, the formation of $\mathscr{F}_1$, and hence $\mathscr{N}_1$, commutes with base change. Further, naturality of the map α_1, coming from the adjunction, also implies $j_y^*(\alpha_1) = \alpha_{1,y}$. Therefore, α_1 is surjective, as its cokernel has empty support.

We next construct sheaves $\mathscr{F}_i$ and $\mathscr{N}_i$ inductively, with $\mathscr{N}_i = \pi^*\mathscr{F}_i$, for $2 \leq i \leq d-3$. Let $\mathscr{A}_1 := \mathscr{I}$. For $i \geq 2$, assume inductively we have constructed the map α_{i-1} and define $\mathscr{A}_i := \ker\alpha_{i-1}$. Analogously to the above verification that $j_y^*\mathscr{I}$ is 3-regular, it follows that $j_y^*\mathscr{A}_i$ is $i+2$ regular. Therefore, by [FGI+05, Lemma 5.1(b)], $H^1(\mathbb{P}_y, j_y^*\mathscr{A}_i(k)) = 0$ for $k \geq (i+2)-1 = i+1$. Analogously to the above case when $i=1$, it follows from cohomology and base change that $R^1\pi_*\mathscr{A}_i(k) = 0$ for $k \geq i+1$, $\pi_*\mathscr{A}_i(k)$ is locally free for $k \geq i+1$, and the formation of $\pi_*\mathscr{A}_i(k)$ commutes with base change for $k \geq i+1$. Then, set $\mathscr{F}_i := \pi_*\mathscr{A}_i(i+1)$ and $\mathscr{N}_i := \pi^*\mathscr{F}_i$.

We next construct the map $\alpha_i : \mathscr{N}_i \to \mathscr{N}_{i-1}$. Begin with the inclusion $\mathscr{A}_i(i+1) \to \mathscr{N}_{i-1}(1)$ (obtained by twisting the inclusion $\mathscr{A}_i \to \mathscr{N}_{i-1}(-i)$, coming from the definition of $\mathscr{A}_i$, by $i+1$). Apply $\pi^*\pi_*$ to obtain a map $\pi^*\pi_*\mathscr{A}_i(i+1) \to \pi_*\pi^*\mathscr{N}_{i-1}(1)$. Twist by $-i-1$ which yields the composite map

$$\begin{aligned}
\mathscr{N}_i(-i-1) &= (\pi^*\pi_*\mathscr{A}_i(i+1))(-i-1) \\
&\to (\pi^*\pi_*\mathscr{N}_{i-1}(1))(-i-1) \\
&\simeq (\mathscr{N}_{i-1} \otimes \pi^*\pi_*\mathscr{O}(1))(-i-1) \\
&\to \mathscr{N}_{i-1}(-i),
\end{aligned} \tag{3.5}$$

which we call α_i. Since $\mathscr{N}_i$ commutes with base change, and this map is obtained from adjunction, the formation of α_i also commutes with base change. Also, since pushforward is left exact, we obtain condition (v) in the theorem from the above construction of $\mathscr{F}_i$, provided we show the

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above construction is the unique such one as in the statement (which will be done later in the proof). This addresses Remark 3.3(9).

Finally, we similarly construct $\mathscr{F}_{d-2}$, $\mathscr{N}_{d-2}$, and α_{d-2}, assuming we have constructed α_{d-3}. Let $\mathscr{A}_{d-2} := \ker \alpha_{d-3}$. By cohomology and base change, we find $j_y^*\mathscr{A}_{d-2}$ is, in fact, d-regular (as opposed to only $d-1$ regular, as was the case for $\mathscr{A}_i$ with $i < d-2$). Therefore, by cohomology and base change, we find $R^1\pi_*\mathscr{A}_{d-2}(-d) = 0$ and also that $\pi_*\mathscr{A}_{d-2}(-d)$ is locally free and commutes with base change. We set $\mathscr{F}_{d-2} := \pi_*\mathscr{A}_{d-2}(-d)$ and $\mathscr{N}_{d-2} := \pi^*\mathscr{F}_{d-2}$. Analogously to (3.5), there is a canonical map $\alpha_{d-2} : \mathscr{N}_{d-2}(-d) \to \mathscr{N}_{d-3}(-d+2)$ coming from adjunction which commutes with base change. Altogether, we have constructed a complex as in (3.1) which commutes with base change on Y and restricts to the minimal free resolution (3.4) on each fiber $y \in Y$. It follows from Nakayama's lemma that the complex (3.1) is exact, because it is exact when restricted to each fiber over $y \in Y$.

Further, because $\mathscr{N}_i = \pi^*\mathscr{F}_i$, it follows from the projection formula that $\pi_*\mathscr{N}_i \simeq \pi_*(\mathscr{O}_{\mathbb{P}} \otimes \pi^*\mathscr{F}_i) \simeq \pi_*\mathscr{O}_{\mathbb{P}} \otimes \mathscr{F}_i \simeq \mathscr{F}_i$, and so $\pi^*\pi_*\mathscr{N}_i \simeq \mathscr{N}_i$.

We next verify uniqueness of our constructed resolution $\mathscr{N}_\bullet$, up to unique isomorphism, in the sense claimed in property (iii). Suppose $\mathscr{M}_\bullet$ is another such resolution which restricts to a minimal free resolution over each geometric fiber over $y \in Y$. Over any local scheme $\operatorname{Spec} \mathscr{O}_{y,Y} \subset Y$, there is an isomorphism $\phi_U : \mathscr{N}_\bullet|_{\operatorname{Spec} \mathscr{O}_{y,Y}} \simeq \mathscr{M}_\bullet|_{\operatorname{Spec} \mathscr{O}_{y,Y}}$ by a sheafified version of [Eis95, Theorem 20.2]. Such an isomorphism spreads out to an isomorphism over some affine open $U \subset Y$. Further, this isomorphism is unique up to homotopy by a sheafified version of [Eis95, Lemma 20.3]. We claim there are no nonzero homotopies $s : \mathscr{N}_\bullet|_U \to \mathscr{M}_\bullet|_U$. Indeed, such an homotopy would yield a map $s_i : \mathscr{N}_i|_U \to \mathscr{M}_{i+1}|_U$. We wish to show this map is 0. To check it is 0, it suffices to show it is 0 over each $y \in Y$. Over a point $y \in Y$, this corresponds to a map $\mathscr{O}_{\mathbb{P}_y}(a)^{\oplus b} \to \mathscr{O}_{\mathbb{P}_y}(c)^{\oplus d}$ with $c < a$. It follows that there are no nonzero such maps, so the isomorphism ϕ_U is unique. Hence, by this uniqueness, we obtain via Zariski descent an isomorphism $\phi : \mathscr{N}_\bullet \simeq \mathscr{M}_\bullet$. This isomorphism is unique because it is unique when restricted to each member of an open cover.

Addressing steps C and D. We have completed the verification of [CE96, Theorem 2.1, Step B] and now note that steps C and D given in the proof of [CE96, Theorem 2.1] go through without change. Recall that step D states that the factorization $\rho = \pi \circ i$ exists. However, the proof shows more: it shows that the triple $(\mathbb{P}\mathscr{E}, \sigma, j)$ gives such a triple, where $\sigma : \mathbb{P}\mathscr{E} \to Y$ is the structure map. This concludes the verification of part (ii), as mentioned in Remark 3.3(5).

Addressing uniqueness of the triples. At this point, we have proved everything except the uniqueness of the triple $(\mathbb{P}, \pi, i)$. We conclude the proof by verifying this statement, which will complete the verification of the modification noted in Remark 3.3(4). We have shown so far in part (i) that if $(\mathbb{P}_1, \pi_1, i_1)$ and $(\mathbb{P}_2, \pi_2, i_2)$ are two triples as in the statement of Theorem 3.2, then there is an isomorphism $\mu : \mathbb{P}_1 \simeq \mathbb{P}_2$. Since μ is an isomorphism of projective bundles over Y, we have $\pi_1 \circ \mu \simeq \pi_2$. Using this and property (ii), we can reduce to the case that $\mathbb{P}_1 \simeq \mathbb{P}_2 \simeq \mathbb{P}\mathscr{E}$: it suffices to find an automorphism $\psi : \mathbb{P} \to \mathbb{P}$ over Y so that $\psi \circ i = i_2$ and, moreover, show this automorphism ψ is the unique one with this property.

To verify existence and uniqueness of ψ, we first reduce to the case Y is the spectrum of a local ring. We know that both $i_1^*\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1) \simeq \omega_{X/Y}$ and $i_2^*\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1) \simeq \omega_{X/Y}$, by Theorem 3.2(ii). Hence, we obtain that the automorphism ψ is induced by some automorphism ϕ of $\pi_*\omega_{X/Y}$, determined up to unit. The maps i_1 and i_2 induce two surjections $q_1, q_2 : \pi_*\omega_{X/Y} \to \mathscr{O}_Y$ with the maps i_1 and i_2 coming via the linear subsystems $\ker(q_1)$ and $\ker(q_2)$. To show we have an induced map between $\ker(q_1)$ and $\ker(q_2)$, which are both abstractly isomorphic to $\mathscr{E}$, it is enough to show that, up to unit, $q_1 = q_2 \circ \phi$. We may verify this locally and, hence, assume Y is the spectrum of a local ring.

We conclude by verifying existence and uniqueness of ψ in the case Y is the spectrum of a local ring. Using Theorem 3.2(vi), in both of the maps i_1 and i_2, there is no subscheme of degree $d-1$ on the closed fiber contained in a hyperplane, and hence the same holds over the whole local scheme Y. We may rephrase this as the condition that the two relative hyperplane sections of $\mathbb{P}\mathscr{E}$ associated to q_1 and q_2 do not meet $i_1(X)$ and $i_2(X)$. Equivalently, the two hyperplane sections associated to q_1 and q_2 are nowhere vanishing on X and, therefore, related by a unit. By modifying ϕ by this unit, we may assume $q_1 = q_2 \circ \phi$. This verifies that ϕ is unique up to unit and, hence, that ψ is unique. Under the above identifications, the image of $\mathscr{E} \to \pi_* \omega_{X/Y}$ is identified with the kernel of the natural map $\pi_* \omega_{X/Y} \to \mathscr{O}_X$ dual to $\rho^{\#}$. Since this map is also fixed by the resulting automorphism ϕ, the automorphism ϕ of $\pi_* \omega_{X/Y}$ restricts to an automorphism of $\mathscr{E}$ which induces the desired automorphism $\psi : \mathbb{P}\mathscr{E} \to \mathbb{P}\mathscr{E}$. ☐

The following useful corollary tells us that any two 'canonical embeddings' of a Gorenstein cover are related by an automorphism of $\mathbb{P}\mathscr{E}$ coming from $\mathscr{E}$. A special case of this was stated in [CN07, Corollary 2.3], though the proof there seems quite terse, as it omits the verification of uniqueness of the triple $(\mathbb{P}\mathscr{E}', \pi, i)$ which we carry out in Theorem 3.2.

Corollary 3.5. *With notation as in Theorem 3.2, suppose we are given $\rho : X \to Y$ and two embeddings $i_1 : X \to \mathbb{P}\mathscr{E}$ and $i_2 : X \to \mathbb{P}\mathscr{E}$ so that $\rho = \pi \circ i_1 = \pi \circ i_2$ and $\rho^{-1}(y)$ is arithmetically Gorenstein and nondegenerate under both embeddings i_1 and i_2. Then, the unique isomorphism $\psi : \mathbb{P}\mathscr{E} \to \mathbb{P}\mathscr{E}$ taking $i_1(X)$ to $i_2(X)$ is induced by an automorphism of $\mathscr{E}$.*

Proof. This is a direct consequence of the uniqueness property for triples $(\mathbb{P}, \pi, i)$ as stated in Theorem 3.2, applied to two triples $(\mathbb{P}\mathscr{E}, \pi, i_1)$ and $(\mathbb{P}\mathscr{E}, \pi, i_2)$. ☐

3.6 Low-degree parametrizations

We now apply Theorem 3.2, as in the work of Casnati and Ekedahl, to obtain parametrizations of Gorenstein covers of degrees 3, 4, and 5.

Remark 3.7. Our parametrization in degree 5, Theorem 3.16, is stronger than previous work in several ways. The similar result in degree 5 proven in [Cas96, Theorem 3.8] has certain additional restrictions on the covers and sections that Casnati refers to as being 'regular'. This regularity condition amounts to the assumption that the map $\wedge^2 \mathscr{F}^\vee \otimes \det \mathscr{E} \to \mathscr{E}$ associated to a section $\eta \in \mathscr{H}(\mathscr{E}, \mathscr{F})$ is surjective. In addition, [Cas96, Theorem 3.8] does not claim there is a bijection between covers and sections up to automorphisms of $\mathscr{E}$ and $\mathscr{F}$, but only gives constructions of maps in both directions. Further, [Cas96, Theorem 3.8] is stated for degree 5 finite flat surjective maps $X \to Y$ with Y integral and noetherian, whereas ours hold for arbitrary schemes Y.

To introduce notation simultaneously in the cases of degrees 3, 4, and 5, we use the following notation.

Notation 3.8. Let $d \in \{3, 4, 5\}$. Let Y be a scheme. Fix a locally free sheaf $\mathscr{E}$ on Y of rank $d-1$. If $d = 4$, let $\mathscr{F}$ be a locally free sheaf on Y of rank 2 and if $d = 5$, let $\mathscr{F}$ be a locally free sheaf on Y of rank 5. We use the tuple $(\mathscr{E}, \mathscr{F}_\bullet)$ to denote the pair $(\mathscr{E}, \mathscr{F})$ when $d = 4$ or $d = 5$ and to

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denote $\mathscr{E}$ when $d=3$. Define the associated sheaf

$$\mathscr{H}(\mathscr{E},\mathscr{F}_\bullet) := \begin{cases} \operatorname{Sym}^3\mathscr{E}\otimes\det\mathscr{E}^\vee & \text{if } d=3,\\ \mathscr{F}^\vee\otimes\operatorname{Sym}^2\mathscr{E} & \text{if } d=4,\\ \wedge^2\mathscr{F}\otimes\mathscr{E}\otimes\det\mathscr{E}^\vee & \text{if } d=5.\end{cases} \tag{3.6}$$

We will often use $\mathscr{H}$ to denote $\mathscr{H}(\mathscr{E},\mathscr{F}_\bullet)$ when the data $(\mathscr{E},\mathscr{F}_\bullet)$ is clear from context. We will see that sections of the above sheaf $\mathscr{H}$ define subschemes of $\mathbb{P}\mathscr{E}$. When these subschemes have dimension 0 in fibers, we will see they induce degree d locally free covers. The parametrizations for degrees 3, 4, and 5 essentially say that the resulting covers are in bijection with such sections, up to automorphisms of $(\mathscr{E},\mathscr{F}_\bullet)$.

3.9 The resolutions in low degree

In order to state the parametrizations in degrees 3, 4, and 5, we now want a way of associating a subscheme of $\mathbb{P}\mathscr{E}$ to a section. We will give a description of this association separately in the cases that $d=3$, 4, and 5.

Renaming the sheaf $\mathscr{E}'$ appearing in (3.1) as $\mathscr{E}$ and renaming $\mathscr{F}_1$ as $\mathscr{F}$, in the cases $d=3$, 4, and 5, (3.1) becomes respectively

$$0\longrightarrow \pi^*\det\mathscr{E}(-3)\xrightarrow{\sigma}\mathscr{O}_{\mathbb{P}}\longrightarrow\mathscr{O}_X\longrightarrow 0, \tag{3.7}$$

$$0\longrightarrow \pi^*\det\mathscr{E}(-4)\xrightarrow{\sigma}\pi^*\mathscr{F}(-2)\longrightarrow\mathscr{O}_{\mathbb{P}}\longrightarrow\mathscr{O}_X\longrightarrow 0, \tag{3.8}$$

$$\begin{aligned} 0\longrightarrow \pi^*\det\mathscr{E}(-5)\longrightarrow \pi^*\mathscr{F}^\vee\otimes\pi^*\det\mathscr{E}(-3)\xrightarrow{\sigma}\\ \xrightarrow{\sigma}\pi^*\mathscr{F}(-2)\longrightarrow\mathscr{O}_{\mathbb{P}}\longrightarrow\mathscr{O}_X\longrightarrow 0,\end{aligned} \tag{3.9}$$

with the rank of the locally free sheaves $\mathscr{E}$ and $\mathscr{F}$ in the degree 3, degree 4, and degree 5 cases given in Notation 3.8.

3.10 The maps Φ_d in low degree

In the above three cases, corresponding to degrees 3, 4, and 5, respectively, we have isomorphisms

$$\Phi_3 : H^0(Y,\operatorname{Sym}^3\mathscr{E}\otimes\det\mathscr{E}^\vee)\xrightarrow{\sim}H^0(\mathbb{P}\mathscr{E},\pi^*\det\mathscr{E}^\vee(3)), \tag{3.10}$$

$$\Phi_4 : H^0(Y,\operatorname{Sym}^2\mathscr{E}\otimes\mathscr{F}^\vee)\xrightarrow{\sim}H^0(\mathbb{P}\mathscr{E},\pi^*\mathscr{F}^\vee(2)), \tag{3.11}$$

$$\Phi_5 : H^0(Y,\wedge^2\mathscr{F}\otimes\mathscr{E}\otimes\det\mathscr{E}^\vee)\xrightarrow{\sim}H^0(\mathbb{P}\mathscr{E},\wedge^2\pi^*\mathscr{F}\otimes\pi^*\det\mathscr{E}^\vee(1)). \tag{3.12}$$

3.11 The maps Ψ_d in low degree

For $\rho : X\to Y$ a finite locally free surjective Gorenstein map of degree d, we will use $\mathscr{E}^X$ to denote the Tschirnhausen bundle $\operatorname{coker}(\mathscr{O}_Y\to\rho_*\mathscr{O}_X)^\vee$ and $\mathscr{F}^X$ to denote the bundle $\mathscr{F}_1$ in the case we take $\mathscr{E}'$ in Theorem 3.2(iii) to be the Tschirnhausen bundle $\mathscr{E}^X$.

Next, for $3\le d\le 5$, given a section $\eta\in H^0(Y,\mathscr{H}(\mathscr{E},\mathscr{F}_\bullet))$, we define an associated scheme $\Psi_d(\eta)$ over Y.

When $d=3$, we begin with a section $\eta\in H^0(Y,\operatorname{Sym}^3\mathscr{E}\otimes\det\mathscr{E}^\vee)$, which, via Φ_3, can be viewed as an element of $H^0(\mathbb{P}\mathscr{E},\pi^*\det\mathscr{E}^\vee(3))$. Such a section corresponds to a map $\mathscr{O}_{\mathbb{P}\mathscr{E}}\to\pi^*\det\mathscr{E}^\vee(3)$ or, equivalently, a map $\pi^*\det\mathscr{E}(-3)\to\mathscr{O}_{\mathbb{P}\mathscr{E}}$. We let $\Psi_3(\eta)$ denote the support of the cokernel of this map. That is, we define $\Psi_3(\eta)\subset\mathbb{P}\mathscr{E}$ so that on $\mathbb{P}\mathscr{E}$ we have the following exact sequence.

$$\pi^*\det\mathscr{E}(-3)\longrightarrow\mathscr{O}_{\mathbb{P}\mathscr{E}}\longrightarrow\mathscr{O}_{\Psi_3(\eta)}\longrightarrow 0. \tag{3.13}$$

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When $d=4$, given $\eta \in H^0(Y, \mathscr{F}^\vee \otimes \operatorname{Sym}^2 \mathscr{E})$, define $\Psi_4(\eta)$ to be the subscheme of $\mathbb{P}\mathscr{E}$, considered as the support of the cokernel of the map $\pi^*\mathscr{F}(-2) \to \mathscr{O}_{\mathbb{P}(\mathscr{E})}$ corresponding to $\Psi_4(\eta)$.

Finally, when $d=5$, given $\eta \in H^0(Y, \wedge^2 \mathscr{F} \otimes \pi^* \det \mathscr{E}^\vee \otimes \mathscr{E})$, from $\Phi_5(\eta)$ we obtain a corresponding alternating map $\pi^*\mathscr{F}^\vee \otimes \pi^* \det \mathscr{E}(-3) \to \pi^*\mathscr{F}(-2)$. The five 4×4 Pfaffians of this map determine a map of sheaves $\pi^*\mathscr{F}(-2) \to \mathscr{O}_{\mathbb{P}(\mathscr{E})}$, as may be computed locally. Define $\Psi_5(\eta)$ as the support of the cokernel of the map $\pi^*\mathscr{F}(-2) \to \mathscr{O}_{\mathbb{P}(\mathscr{E})}$ in $\mathbb{P}\mathscr{E}$.

Definition 3.12. Let $d \in \{3,4,5\}$, Y be a scheme, and $(\mathscr{E}, \mathscr{F}_\bullet), \mathscr{H}(\mathscr{E}, \mathscr{F}_\bullet)$ be sheaves on Y as in Notation 3.8. We say $\eta \in H^0(Y, \mathscr{H}(\mathscr{E}, \mathscr{F}_\bullet))$ has the *right Hilbert polynomial* at a point $y \in Y$ if the fiber of $\Psi_d(\eta)$ over y has dimension 0 and degree d. We say η has the *right Hilbert polynomial* if it has the right Hilbert polynomial at every $y \in Y$.

Finally, we are ready to state the low-degree parametrizations. The parametrization in degree 3 is as follows.

Theorem 3.13 (Generalization of [CE96, Theorem 3.4], specialization of [Poo08, Proposition 5.1]). *Fix a scheme Y and a rank-2 locally free sheaf $\mathscr{E}$ on Y. The map $\eta \mapsto \Psi_3(\eta)$ induces a bijection between:*

(1) *sections $\eta \in H^0(Y, \operatorname{Sym}^3 \mathscr{E} \otimes \det \mathscr{E}^\vee)$ having the right Hilbert polynomial at every $y \in Y$, up to automorphisms of $\mathscr{E}$; and*
(2) *finite locally free Gorenstein covers $\rho : X \to Y$ of degree 3 such that $\mathscr{E}^\vee \simeq \operatorname{coker} \rho^\#$.*

The following proof extends that given in [CE96, Theorem 3.4]. We note that there the base is assumed to be reduced and noetherian, and the bijection is not stated explicitly. We outline the proof for the reader's convenience.

Proof. We start by constructing the map from (2) to (1). Given such a $\rho : X \to Y$, we obtain from Theorem 3.2, a resolution of $\mathscr{O}_{\mathbb{P}\mathscr{E}}$ as in (3.7), unique up to unique isomorphism. The map σ in (3.7) can be viewed as a section $\sigma \in H^0(\mathbb{P}\mathscr{E}, \pi^* \det \mathscr{E}^\vee(3))$. For Φ_3 as defined in (3.10), we obtain a section $\eta := \Phi_3^{-1}(\sigma) \in H^0(Y, \operatorname{Sym}^3 \mathscr{E} \otimes \det \mathscr{E}^\vee)$. Note that the resulting η has the right Hilbert polynomial at every $y \in Y$ because $X \to Y$ is finite by assumption.

We next show the map $\eta \mapsto \Psi_3(\eta)$ indeed defines a map from (1) to (2). Given η of the right Hilbert polynomial at every $y \in Y$, we obtain a right exact sequence (3.13). The assumption that η has the right Hilbert polynomial yields that the first map in this sequence is injective and, hence, $X \to \mathbb{P}\mathscr{E}$ has a resolution of the form (3.7). This resolution shows X is locally finitely presented over Y. Further, X is finite as it is locally of finite presentation, proper, and quasi-finite [Gro66, 8.11.1]. Flatness of $X \to Y$ may be verified locally, in which case it holds as X is cut out of $\mathbb{P}^1_Y$ by a single equation of degree 3 not vanishing on any fibers. Therefore, X is a finite locally free degree 3 cover of Y. Finally, exactness of (3.7) implies $\mathscr{E}^\vee \simeq \operatorname{coker} \rho^\#$ from Theorem 3.2(iii) and (iv).

It remains to see that these two maps we have defined establish a bijection. For this, we show the compositions of these maps in both orders are equivalent to the identity map. If we begin with a cover $\rho : X \to Y$, (3.7) defines a resolution of $X \to \mathbb{P}\mathscr{E}$ giving X as the vanishing locus $\Psi_3(\eta) \subset \mathbb{P}\mathscr{E}$. To show the other composition is equivalent to the identity, begin with some $\eta \in H^0(Y, \operatorname{Sym}^3 \mathscr{E} \otimes \det \mathscr{E}^\vee)$, and let X denote the associated cover $\Psi_3(\eta)$. The Tschirnhausen bundle $\mathscr{E}^X$ as in § 3.11 associated to X from Theorem 3.2 is then isomorphic to $\mathscr{E}$ using Theorem 3.2(iv), as we may view η as a map $\pi^* \det \mathscr{E}(-3) \to \mathscr{O}_{\mathbb{P}\mathscr{E}}$. Upon choosing such an isomorphism $\mathscr{E} \simeq \mathscr{E}^X$, we obtain a section $\eta^X \in H^0(Y, \operatorname{Sym}^3 \mathscr{E}^X \otimes \det(\mathscr{E}^X)^\vee) \simeq H^0(Y, \operatorname{Sym}^3 \mathscr{E} \otimes \det \mathscr{E}^\vee)$. Using Theorem 3.2(iv), there is an automorphism of $\mathbb{P}\mathscr{E}$ taking $\Psi_3(\eta)$ to $\Psi_3(\eta^X)$.

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From Theorem 3.2(iv) and the fact that the leftmost term of the resolution (3.7) is $\pi^* \det \mathscr{E}(-3)$, we find $\mathscr{E}$ is isomorphic to $\ker(\rho_* \omega_{X/Y} \to \mathscr{O}_Y)$. By Corollary 3.5, this automorphism of $\mathbb{P}\mathscr{E}$ is induced by an automorphism of $\mathscr{E}$. Hence, after composing with the automorphism of $\mathscr{E}$, we can assume η and η^X define isomorphic subschemes of $\mathbb{P}\mathscr{E}$, and so are related via multiplication by a global section $s^{-1} \in \mathscr{O}_Y(Y)$. By composing with an automorphism of $\mathscr{E}$ multiplying by s^{-1}, η and η^X are identified. □

We next verify the parametrization in degree 4.

Theorem 3.14 (Generalization of [CE96, Theorem 4.4], specialization of [Woo11, Theorem 1.1]). *Fix a scheme Y, a rank-3 locally free sheaf $\mathscr{E}$ on Y, and a rank-2 locally free sheaf $\mathscr{F}$ on Y such that there exists an unspecified isomorphism $\det \mathscr{E} \simeq \det \mathscr{F}$. The map $\eta \mapsto \Psi_4(\eta)$ induces a bijection between:*

(1) *sections $\eta \in H^0(Y, \mathscr{F}^\vee \otimes \operatorname{Sym}^2 \mathscr{E})$ having the right Hilbert polynomial at every $y \in Y$, up to automorphisms of $\mathscr{E}$ and $\mathscr{F}$; and*
(2) *finite locally free Gorenstein maps $\rho : X \to Y$ of degree 4 with associated sheaves $\mathscr{E}^X, \mathscr{F}^X$ as in § 3.11 which are isomorphic to $\mathscr{E}$ and $\mathscr{F}$.*

Proof. First we construct the map from (2) to (1). Beginning with a cover $X \to Y$, we obtain a resolution (3.8) and, upon choosing isomorphisms $\mathscr{E}^X \simeq \mathscr{E}$ and $\mathscr{F}^X \simeq \mathscr{F}$, we obtain a section $\eta \in H^0(Y, \mathscr{F}^\vee \otimes \operatorname{Sym}^2 \mathscr{E})$ having the right Hilbert polynomial at every $y \in Y$.

To construct the map from (1) to (2), we must show $\Psi_4(\eta)$ satisfies the properties listed in (2). We first verify $\Psi_4(\eta)$ is a finitely presented Gorenstein cover of Y. On fibers, $\Psi_4(\eta)$ is described as a dimension-0 intersection of two quadrics. Since η has the right Hilbert polynomial at $y \in Y$, it has degree 4 over y. (We parenthetically note that by Bezout's theorem, having the right Hilbert polynomial is equivalent to having dimension 0, which then matches with Casnati and Ekedahl's notion of having 'the right codimension' from [CE96, Definition 4.2].) Gorensteinness follows because $\Psi_4(\eta)$ is a local complete intersection.

We next deduce flatness of $\Psi_4(\eta)$ over Y. We first explain how to reduce to the case that Y is smooth. Let Z denote the moduli space parameterizing pairs of quadrics in $\mathbb{P}^2$ which comes with a universal $\pi : U \to Z$ whose fiber over a pair $[(Q_1, Q_2)]$ is $Q_1 \cap Q_2$. There is an open locus $Z^\circ \subset Z$ where the intersection of these quadrics is zero-dimensional, and hence has constant degree 4 by Bezout's theorem. Let $U^\circ := \pi^{-1}(Z^\circ)$. Since Z is a product of projective spaces, Z° is an open in a product of projective spaces, hence, in particular, smooth. Working fppf locally on Y, we can express $X \to Y$ as an intersection of relative quadrics in $\mathbb{P}^2$, in which case $X \to Y$ is pulled back from $U^\circ \to Z^\circ$ via a map $Y \to Z^\circ$. Hence, it suffices to show that $U^\circ \to Z^\circ$ itself is flat. In this case, since Z° is reduced, flatness follows from constancy of the degree.

To conclude the construction of the map from (1) to (2), we will show it is possible to choose identifications $\mathscr{E}^X \simeq \mathscr{E}, \mathscr{F}^X \simeq \mathscr{F}$ so that we obtain an associated section $\eta^X \in H^0(Y, \mathscr{F}^\vee \otimes \operatorname{Sym}^2 \mathscr{E}) \simeq H^0(Y, (\mathscr{F}^X)^\vee \otimes \operatorname{Sym}^2 \mathscr{E}^X)$.

First we show $\mathscr{E}^X \simeq \mathscr{E}$. Indeed, there is the following Koszul complex.

$$0 \longrightarrow \pi^* \det \mathscr{F} \otimes \mathscr{O}_{\mathbb{P}\mathscr{E}}(-4) \longrightarrow \pi^* \mathscr{F} \otimes \mathscr{O}_{\mathbb{P}\mathscr{E}}(-2) \longrightarrow \mathscr{O}_{\mathbb{P}\mathscr{E}} \longrightarrow \mathscr{O}_X. \tag{3.14}$$

It also follows from [Eis95, Theorem 20.15] (using the comments on [Eis95, p. 503] and the fact that Gorenstein schemes are Cohen–Macaulay) that (3.14) yields a minimal free resolution of X_y in $\mathbb{P}\mathscr{E}_y$ for every $y \in Y$. Because $\det \mathscr{F} \simeq \det \mathscr{E}$ by assumption, Theorem 3.2(iv) implies $\mathscr{E} \simeq \mathscr{E}^X$.

Using the isomorphism $\mathscr{E} \simeq \mathscr{E}^X$, we also verify $\mathscr{F} \simeq \mathscr{F}^X$. Let $i : X \to \mathbb{P}\mathscr{E}$ and $i^X : X \to \mathbb{P}\mathscr{E}^X$ denote the two embeddings. By pushing forward the twist of (3.14) by $\mathscr{O}_{\mathbb{P}\mathscr{E}}(2)$ along π,

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we find $\mathscr{F}^X \simeq \ker(\operatorname{Sym}^2 \mathscr{E}^X \to \pi_*(i_*^X \mathscr{O}_X \otimes \mathscr{O}_{\mathbb{P}\mathscr{E}^X}(2)))$. Similarly, the analogous resolution from Theorem 3.2 for X in terms of $\mathscr{E}^X$ and $\mathscr{F}^X$ yields $\mathscr{F} \simeq \ker(\operatorname{Sym}^2 \mathscr{E} \to \pi_*(i_* \mathscr{O}_X \otimes \mathscr{O}_{\mathbb{P}\mathscr{E}}(2)))$. Hence, the isomorphism $\mathscr{E} \simeq \mathscr{E}^X$ induces the desired isomorphism $\mathscr{F} \simeq \mathscr{F}^X$.

The isomorphism $\mathscr{E} \simeq \mathscr{E}^X$ is compatible with the above restriction map, and so induces an isomorphism $\mathscr{F} \simeq \mathscr{F}^X$. This concludes the verification that the map we have produced indeed goes from (1) to (2).

It remains to prove the compositions of the above maps in both directions are equivalent to the identity. As in the degree 3 case, if we start with a cover, and produce the associated section η^X, $\Psi_4(\eta^X)$ is isomorphic to X via the construction. For showing the reverse composition is equivalent to the identity, start with some section η. Let X denote the resulting cover $\Psi_4(\eta)$.

Given the above identifications $\mathscr{E}^X \simeq \mathscr{E}, \mathscr{F}^X \simeq \mathscr{F}$, we wish to show η^X is related to η by automorphisms of $\mathscr{E}$ and $\mathscr{F}$. Note also here that any automorphism of $\mathscr{E}$ and $\mathscr{F}$ sends η to another section defining an isomorphic cover. Using Theorem 3.2, there is an automorphism of $\mathbb{P}\mathscr{E}$ taking the subscheme $\Psi_4(\eta^X)$ to $\Psi_4(\eta)$. From Theorem 3.2(iv) and the fact that the leftmost term of the resolution (3.8) is $\pi^* \det \mathscr{E}(-4)$, we find $\mathscr{E}$ is isomorphic to $\ker(\rho_* \omega_{X/Y} \to \mathscr{O}_Y)$. By Corollary 3.5, the above automorphism of $\mathbb{P}\mathscr{E}$ is induced by an automorphism of $\mathscr{E}$. By composing with the inverse of this automorphism, we may assume the resulting map is the identity on $\mathbb{P}\mathscr{E}$, and so the automorphism of $\mathbb{P}\mathscr{E}$ is then induced by some automorphism of $\mathscr{E}$ via multiplication by a section $s \in \mathscr{O}_Y(Y)$. After composing with multiplication by s^{-1}, we may reduce to the case s is the identity. Since $\mathscr{F}$ is a subsheaf of $\operatorname{Sym}^2 \mathscr{E}$ by Theorem 3.2(v), the image of the induced map $\mathscr{F} \to \operatorname{Sym}^2 \mathscr{E}$ is uniquely determined by X, but the precise map is only determined up to automorphism of $\mathscr{F}$. Upon composing with such an automorphism, we may identify not just the images of $\mathscr{F}$ in $\operatorname{Sym}^2 \mathscr{E}$, but further we may identify the maps. Under these identifications, η agrees with η^X, when viewed as maps $\mathscr{F} \to \operatorname{Sym}^2 \mathscr{E}$. □

We next state and prove the analogous parametrization in degree 5. As preparation, we will need the following application of the structure theorem for codimension-3 Gorenstein algebras due to Buchsbaum and Eisenbud.

Lemma 3.15. *Let Y be a scheme, and let $\mathscr{E}$ and $\mathscr{F}$ be locally free sheaves on Y of ranks 3 and 5. A finite locally free Gorenstein map $\rho : X \to Y$ of degree 5, described as $\Psi_5(\eta)$ for $\eta \in H^0(Y, \wedge^2 \mathscr{F} \otimes \mathscr{E} \otimes \det \mathscr{E}^\vee)$, has a resolution of the form*

$$0 \longrightarrow \pi^* \det \mathscr{E}^\vee \otimes \pi^* \det \mathscr{F}(-5) \xrightarrow{\beta_3} \pi^* \det \mathscr{E}^\vee \otimes \pi^* \mathscr{F}^\vee(-3) \xrightarrow{\beta_2}$$

$$\xrightarrow{\beta_2} \pi^* \mathscr{F}(-2) \xrightarrow{\beta_1} \mathscr{O}_{\mathbb{P}\mathscr{E}} \longrightarrow \mathscr{O}_X, \tag{3.15}$$

which restricts to a minimal free resolution over each $y \in Y$, where β_2 is alternating and β_3 is identified with the dual of β_1 tensored with $\pi^* \det \mathscr{E}^\vee \otimes \pi^* \det \mathscr{F}(-5)$.

Proof. In (3.15), the map β_2 is obtained from η, interpreted as a section of $H^0(Y, \wedge^2 \mathscr{F} \otimes \mathscr{E} \otimes \det \mathscr{E}^\vee) \simeq H^0(\mathbb{P}\mathscr{E}, \pi^*(\wedge^2 \mathscr{F} \det \mathscr{E}^\vee)(1))$. The map β_3 is obtained by taking five 4×4 Pfaffians of β_2. To make sense of this, one may first construct β_3 locally upon choosing trivializations of $\mathscr{F}$ and $\mathscr{E}$. One then obtains a global map $\mathscr{F}(-2) \to \mathscr{O}_{\mathbb{P}\mathscr{E}}$ because the formation of the Pfaffians are compatible with restriction to an open subscheme of Y. Finally, β_1 is obtained as the dual to β_3, tensored with $\pi^* \det \mathscr{E}^\vee \otimes \pi^* \det \mathscr{F}(-5)$.

Since we have constructed the maps in (3.15) globally over $\mathbb{P}\mathscr{E}$, it is enough to verify they furnish a minimal free resolution on geometric fibers. To this end, we may work locally on Y and choose a trivialization $u : \det \mathscr{E} \simeq \mathscr{O}_Y$. Upon choosing this trivialization and composing with the

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isomorphism u for the two left nonzero sheaves in (3.15), we obtain a sequence

$$0 \longrightarrow \pi^* \det \mathscr{F}(-5) \xrightarrow{\beta_3'} \pi^* \mathscr{F}^\vee(-3) \xrightarrow{\beta_2'} \pi^* \mathscr{F}(-2) \xrightarrow{\beta_1'} \mathscr{O}_{\mathbb{P}\mathscr{E}} \longrightarrow \mathscr{O}_X, \tag{3.16}$$

where β_2' is still alternating, i.e. it corresponds to an element of $H^0(Y, \wedge^2 \mathscr{F} \otimes \mathscr{E})$, and β_3' remains identified with the dual of β_1', now tensored with $\pi^* \det \mathscr{F}(-5)$. Since the sequence (3.16) commutes with base change on Y, we may further restrict to a geometric point $y \in Y$ and, hence, assume Y is the spectrum of an algebraically closed field.

We wish to show (3.16) is a minimal locally free resolution. To do so, we wish to apply [BE77], and so we translate the above to the setting of commutative algebra. By Theorem 3.2(iii), $X \to \mathbb{P}\mathscr{E}$ is an arithmetically Gorenstein subscheme. Writing $\mathbb{P}\mathscr{E} = \operatorname{Proj} \kappa(y)[x_0, x_1, x_2, x_3]$, the cone over X defines a Gorenstein subscheme of $\operatorname{Spec} \kappa(y)[x_0, x_1, x_2, x_3]_{(x_0,x_1,x_2,x_3)}$, the localization of $\mathbb{A}^4_{\kappa(y)}$ at the origin. Taking $R := \kappa(y)[x_0, x_1, x_2, x_3]_{(x_0,x_1,x_2,x_3)}$, we can identify $\pi^*\mathscr{F}$ with a rank 5 free R-module F. Let J denote the ideal of the cone over X in R. The resolution (3.16) can then be reexpressed in the form

$$0 \longrightarrow R \xrightarrow{\beta_3''} F^\vee \xrightarrow{\beta_2''} F \xrightarrow{\beta_1''} R \longrightarrow R/JR, \tag{3.17}$$

with $\beta_2'' \in \wedge^2 F$ alternating and β_3'' the dual of β_1''. By the definition of $\Psi_5(\eta)$ this sequence is exact at R, so J is the image of β_1''. Since X has codimension 3 in $\mathbb{P}\mathscr{E}$ by assumption, J is of grade 3. Hence, (3.17) satisfies the hypotheses of [BE77, Theorem 2.1(1)]. It is stated that any such resolution satisfying these hypotheses is a minimal free resolution of R/JR in the bottom paragraph of [BE77, p. 463] and the proof is given in [BE77, p. 464]. $\square$

Theorem 3.16 (Generalization of [Cas96, Theorem 3.8]). *Fix a scheme Y, a rank-4 locally free sheaf $\mathscr{E}$ on Y, and a rank-5 locally free sheaf $\mathscr{F}$ on Y such that there exists an unspecified isomorphism $\det \mathscr{F} \simeq (\det \mathscr{E})^{\otimes 2}$. The map $\eta \mapsto \Psi_5(\eta)$ induces a bijection between:*

(1) *sections $\eta \in H^0(Y, \wedge^2 \mathscr{F} \otimes \mathscr{E} \otimes \det \mathscr{E}^\vee)$ having the right Hilbert polynomial at every $y \in Y$, up to automorphisms of $\mathscr{E}$ and $\mathscr{F}$; and*

(2) *finite locally free Gorenstein maps $\rho : X \to Y$ of degree 5 with associated sheaves $\mathscr{E}^X, \mathscr{F}^X$ as in § 3.11 which are isomorphic to $\mathscr{E}$ and $\mathscr{F}$.*

Proof. To start, we construct the map from (2) to (1). Beginning with a cover $X \to Y$, we obtain a resolution (3.9). Upon choosing isomorphisms $\mathscr{E}^X \simeq \mathscr{E}$ and $\mathscr{F}^X \simeq \mathscr{F}$ we obtain a section $\eta \in H^0(Y, \mathscr{F}^{\otimes 2} \otimes \mathscr{E} \otimes \det \mathscr{E}^\vee)$ having the right Hilbert polynomial at every $y \in Y$. We wish to check next that this section actually lies in $H^0(Y, \wedge^2 \mathscr{F} \otimes \mathscr{E} \otimes \det \mathscr{E}^\vee)$. Viewing this as a map $\pi^*\mathscr{F}^\vee \otimes \pi^* \det \mathscr{E} \to \pi^*\mathscr{F}(1)$ via (3.9), it is enough to verify the map is alternating locally on the base. Therefore, for this verification, we may assume Y is the spectrum of a local ring and $\mathscr{E}$ is trivial. After this reduction, $X \subset \mathbb{P}\mathscr{E}$ is codimension 3 and arithmetically Gorenstein, and so the Buchsbaum–Eisenbud parametrization for codimension-3 Gorenstein schemes [BE77, Theorem 2.1(2)] applies. This produces a resolution of $X \subset \mathbb{P}\mathscr{E}$ as in (3.15) which by Theorem 3.2 must agree with (3.9). Since the map corresponding to $\pi^*\mathscr{F}^\vee \otimes \pi^* \det \mathscr{E} \to \pi^*\mathscr{F}(1)$ is alternating in the resolution of [BE77, Theorem 2.1(2)] it follows that $\pi^*\mathscr{F}^\vee \otimes \pi^* \det \mathscr{E} \to \pi^*\mathscr{F}(1)$ is also alternating.

We next construct the map from (1) to (2). This map will send η to $\Psi_5(\eta)$. To show this is indeed a well-defined map, we wish to verify $\Psi_5(\eta)$ is a finitely presented Gorenstein cover of Y. The finite presentation condition follows from the resolution given in (3.9). We may check the remaining conditions locally on Y, and hence assume Y is the spectrum of a local ring. Observe that $X \to \mathbb{P}\mathscr{E}$ is arithmetically Gorenstein and of codimension 3, using the assumption

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that η has the right Hilbert polynomial at each $y \in Y$. Using [BE77, Theorem 2.1(1)], we find that X is Gorenstein and is cut out scheme theoretically by the five 4×4 Pfaffians associated to η, thought of as a map $\pi^*\mathscr{F}^\vee \otimes \pi^* \det \mathscr{E} \to \pi^*\mathscr{F}(1)$. On fibers, $\Psi_5(\eta)$ is described as the vanishing of the five 4×4 Pfaffians of an alternating linear map. The resolution [BE77, Theorem 2.1(1)], can be identified with one of the form (3.9), from which one may calculate that the Hilbert polynomial of every fiber is 5. Therefore, the resulting scheme $\Phi_5(\eta)$ is finite and each fiber has degree 5.

We next deduce flatness of $\Psi_5(\eta)$ over Y. The idea is to reduce to the universal case, where we can verify flatness using constancy of Hilbert polynomial. Let $Z \simeq \mathbb{A}^{40}$ denote the affine space parameterizing alternating 5×5 matrices of linear forms in $\mathbb{P}^3$. Let $\pi : U \to Z$ denote the universal family of intersections of the five 4×4 Pfaffians of the corresponding matrix, so that the fiber of π over a point $[M] \in Z$ is the intersection of the five 4×4 Pfaffians of the alternating matrix of linear forms M. There is an open subset $Z^\circ \subset Z$ parameterizing the locus where the fiber of π is zero-dimensional and degree at most 5. One may verify that every fiber of π has degree at least 5, and so this open Z° parameterizes subschemes of degree exactly 5. Let $U^\circ := \pi^{-1}(Z^\circ)$. Since Z is smooth Z° is as well. Working fppf locally on Y, we can assume $X \to Y$ is a pullback of $U^\circ \to Z^\circ$ along a map $Y \to Z$. Hence, it suffices to show that $U^\circ \to Z^\circ$ itself is flat. In this case, since Z is reduced, flatness follows from constancy of the degree.

In order to show the above constructed map indeed takes (1) to (2), we must demonstrate identifications $\mathscr{E}^X \simeq \mathscr{E}$ and $\mathscr{F}^X \simeq \mathscr{F}$. To obtain the first identification, we use Lemma 3.15. Since $\det \mathscr{E}^{\otimes 2} \simeq \det \mathscr{F}$, the leftmost nonzero term of the resolution in Lemma 3.15 becomes $\det \pi^*\mathscr{E}^\vee \otimes \pi^* \det \mathscr{F}(-5) \simeq \pi^* \det \mathscr{E}(-5)$. Hence, Theorem 3.2(iv) implies $\mathscr{E} \simeq \mathscr{E}^X$. Let $i : X \to \mathbb{P}\mathscr{E}, i^X : X \to \mathbb{P}\mathscr{E}^X$ denote the inclusions. By twisting (3.15) by $\mathscr{O}_{\mathbb{P}\mathscr{E}}(2)$ and pushing forward, we find $\mathscr{F} \simeq \ker(\operatorname{Sym}^2 \mathscr{E} \to \pi_*(i_*\mathscr{O}_X \otimes \mathscr{O}_{\mathbb{P}\mathscr{E}}(2)))$. The analogous resolution from Theorem 3.2 for X in terms of $\mathscr{E}^X$ and $\mathscr{F}^X$ yields $\mathscr{F}^X \simeq \ker(\operatorname{Sym}^2 \mathscr{E}^X \to \pi_*(i^X_*\mathscr{O}_X \otimes \mathscr{O}_{\mathbb{P}\mathscr{E}^X}(2)))$. Hence, the isomorphism $\mathscr{E} \simeq \mathscr{E}^X$ induces the desired identification $\mathscr{F} \simeq \mathscr{F}^X$. This completes the construction of the map from (1) to (2).

It remains to prove the compositions of the above maps between (1) and (2) are equivalent to the identity. As in the degree 3 case, if we start with a cover, produce the associated section η^X, $\Psi_5(\eta^X)$ is isomorphic to X via the construction.

For the reverse composition, start with some section η and let X denote the resulting cover $\Psi_5(\eta)$. Now, choose identifications $\mathscr{E}^X \simeq \mathscr{E}, \mathscr{F}^X \simeq \mathscr{F}$ as above so that we obtain an associated section $\eta^X \in H^0(Y, \wedge^2\mathscr{F}^\vee \otimes \det \mathscr{E} \to \mathscr{E}) \simeq H^0(Y, \wedge^2(\mathscr{F}^X)^\vee \otimes \det \mathscr{E}^X \to \mathscr{E}^X)$. We wish to show η^X is related to η by automorphisms of $\mathscr{E}$ and $\mathscr{F}$. Note also here that any automorphism of $\mathscr{E}$ and $\mathscr{F}$ sends η to another section defining an isomorphic cover. Using Theorem 3.2, there is an automorphism of $\mathbb{P}\mathscr{E}$ taking $\Psi_5(\eta^X)$ to $\Psi_5(\eta)$. From Theorem 3.2(iv) and the fact that the leftmost term of the resolution (3.8) is $\pi^* \det \mathscr{E}(-5)$, we find $\mathscr{E}$ is isomorphic to $\ker(\rho_*\omega_{X/Y} \to \mathscr{O}_Y)$. By Corollary 3.5, this automorphism of $\mathbb{P}\mathscr{E}$ is induced by an automorphism of $\mathscr{E}$. By composing with the inverse of this automorphism, we may assume η and η^X define the same subscheme of $\mathbb{P}\mathscr{E}$. Hence we may assume the automorphism of $\mathbb{P}\mathscr{E}$ is then induced by multiplication by a section $s \in \mathscr{O}_Y(Y)$. After composing with multiplication by s^{-1}, we may therefore reduce to the case that s is the identity section. By Theorem 3.2(iii), we obtain a unique isomorphism between the two resolutions of X in $\mathbb{P}\mathscr{E}$ (3.9) determined by η and η^X. This isomorphism can be specified as a tuple of 5 maps between the nonzero terms of (3.9).

We next show we can apply an automorphism of $\mathscr{F}$ so as to assume the map $\pi^*\mathscr{F}(-2) \to \pi^*\mathscr{F}(-2)$ is the identity. Since $\mathscr{F}$ is a subsheaf of $\operatorname{Sym}^2 \mathscr{E}$ by Theorem 3.2(v), the image of the induced map $\mathscr{F} \to \operatorname{Sym}^2 \mathscr{E}$ coming from the Pfaffians associated to η is uniquely determined by

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X, but the precise map is only determined up to automorphism of $\mathscr{F}$. Upon composing with such an automorphism, we may identify not just the images of $\mathscr{F}$ in $\operatorname{Sym}^2 \mathscr{E}$, but further we may identify the maps. Under these identifications, η agrees with η^X, when viewed as maps $\mathscr{F} \to \operatorname{Sym}^2 \mathscr{E}$.

So far, we have constructed a map of the two resolutions (3.9) associated to η and η_X. Upon choosing identifications $\mathscr{E} \simeq \mathscr{E}^X$ and $\mathscr{F} \simeq \mathscr{F}^X$ as above, we have enforced that the map of resolutions is given by the identity on the terms $\mathscr{O}_X \to \mathscr{O}_X, \mathscr{O}_{\mathbb{P}} \to \mathscr{O}_{\mathbb{P}}$, and $\pi^*\mathscr{F}(-2) \to \pi^*\mathscr{F}(-2)$. When we write the second nonzero term of (3.9) as $\pi^*\mathscr{F}^\vee \otimes \pi^* \det \mathscr{E}(-3)$, we have identified it via Grothendieck duality as pairing with the third nonzero term $\pi^*\mathscr{F}(-3)$ into $\pi^*\mathscr{E}(-5)$, and therefore the induced automorphism of $\pi^*\mathscr{F}^\vee \otimes \pi^* \det \mathscr{E}(-3)$ must respect this duality. In particular, since we have reduced to the case where the automorphism of $\pi^*\mathscr{F}(-2)$ is the identity, we also obtain the induced automorphism of $\pi^*\mathscr{F}^\vee \otimes \pi^* \det \mathscr{E}(-3)$ is the identity. Using Theorem 3.2(v) to guarantee that the maps η and η^X from $\mathscr{F}^\vee \otimes \det \mathscr{E} \to \mathscr{F} \otimes \mathscr{E}$ are injective, we obtain the desired identification of η with η^X. □

Finally, we recall a rather elementary criterion for when $\Psi_d(\eta)$ is geometrically connected.

Theorem 3.17 (Part of [CE96, Theorem 3.6], [CE96, Theorem 4.5], and [Cas96, Theorem 4.4]). *Keeping notation as in Notation 3.8, assume that Y is a geometrically connected and geometrically reduced projective scheme over a field k. If $h^0(Y, \mathscr{E}^\vee) = 0$, then $\Psi_d(\eta)$ is geometrically connected.*

Proof. The proof is essentially given in [CE96, Theorem 3.6], and we repeat it for the reader's convenience. Let $X := \Psi_d(\eta)$. If $h^0(Y, \mathscr{E}^\vee) = 0$ the exact sequence

$$0 \longrightarrow \mathscr{O}_Y \longrightarrow \rho_*\mathscr{O}_X \longrightarrow \mathscr{E}^\vee \longrightarrow 0 \tag{3.18}$$

induces an isomorphism $H^0(Y, \mathscr{O}_Y) \simeq H^0(Y, \rho_*\mathscr{O}_X) = H^0(X, \mathscr{O}_X)$. Since Y is geometrically connected and geometrically reduced, we have $h^0(Y, \mathscr{O}_Y) = 1$. From this we find $H^0(X, \mathscr{O}_X) = 1$ as well, and therefore X is necessarily geometrically connected. □

4. Describing stacks of low-degree covers as quotients

In this section, we give a description of the stack of degree d Gorenstein covers as a global quotient stack for $3 \leq d \leq 5$. We now introduce the groups we will be quotienting by. Since the Hurwitz stack is closely related to the Weil restriction of the stack of degree d covers along $\mathbb{P}^1 \to \operatorname{Spec} k$, we will simultaneously define these automorphism groups along Weil restrictions.

Remark 4.1. We are about to define an automorphism sheaf $\operatorname{Aut}^{Y/B}_{\mathscr{E},\mathscr{F}_\bullet}$ for $Y \to B$ a morphism of schemes and $\mathscr{E}, \mathscr{F}_\bullet$ locally free sheaves on Y. Before giving the formal definition, we give an intuitive description.

Consider first the case that $d = 4$, or $d = 5$, $Y = B = \operatorname{Spec} k$, and additionally assume there is an isomorphism $\det \mathscr{E}^{\otimes d-3} \simeq \det \mathscr{F}_1$. Then, the points of $\operatorname{Aut}^{\operatorname{Spec} k/\operatorname{Spec} k}_{\mathscr{E},\mathscr{F}_\bullet}$ corresponds to automorphisms of $\mathscr{E}$ and $\mathscr{F}_1$ which preserve the above isomorphism. However, in what follows we do not require such an isomorphism $\det \mathscr{E}^{\otimes d-3} \simeq \det \mathscr{F}_1$ exists, and so the definition we give is somewhat more general. Namely, we instead work with automorphisms $(M, N) \in \operatorname{Aut}_{\mathscr{E}} \times \operatorname{Aut}_{\mathscr{F}_1}$ so that $\det M^{d-3} = \det N$.

Another important case is that where $Y = D, B = \operatorname{Spec} k$, and again $d = 4$ or 5. If, in addition, there is an isomorphism $\det(\mathscr{E}^{\otimes d-3})_D \simeq (\det \mathscr{F}_1)_D$, $\operatorname{Aut}^{D/\operatorname{Spec} k}_{\mathscr{E},\mathscr{F}_\bullet}$ can be thought of as parameterizing automorphisms of $\mathscr{E}$ and $\mathscr{F}_1$ over D which preserve the isomorphism

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$\det(\mathscr{E}^{\otimes d-3})_D \simeq (\det \mathscr{F}_1)_D$. Again, we have the caveat that this is only correct when such an isomorphism exists.

DEFINITION 4.2. Given a scheme Y over a base B and an integer d, let *resolution data* for Y and d denote a tuple of locally free sheaves $(\mathscr{E}, \mathscr{F}_\bullet)$ on Y, where $\mathscr{E}$ is a locally free sheaf of rank $d-1$ and $\mathscr{F}_\bullet$ denotes the sequence $\mathscr{F}_1, \dots, \mathscr{F}_{\lfloor (d-2)/2 \rfloor}$ where $\operatorname{rk} \mathscr{F}_i = \beta_i$ as in (3.2). Let $3 \leq d \leq 5$, fix a scheme Y over a field, and fix resolution data $(\mathscr{E}, \mathscr{F}_\bullet)$ for a degree d cover of Y. For $\mathscr{G}$ a locally free sheaf on Y, let $\Delta_{\mathscr{G}}^{Y/B} := \mathbb{G}_m \to \operatorname{Res}_{Y/B}(\operatorname{Aut}_{\mathscr{G}/Y})$ denote the map adjoint to the central inclusion $(\mathbb{G}_m \times_B Y) \to \operatorname{Aut}_{\mathscr{G}/Y}$ on Y. We denote by $\left(\Delta_{\mathscr{G}}^{Y/B}\right)^i$ the composition $\mathbb{G}_m \xrightarrow{x \mapsto x^i} \mathbb{G}_m \xrightarrow{\Delta_{\mathscr{G}}^{Y/B}} \operatorname{Res}_{Y/B}(\operatorname{Aut}_{\mathscr{G}/Y})$ and define

$$\begin{aligned} ((\Delta_{\mathscr{E}}^{Y/B})^i, (\Delta_{\mathscr{F}_1}^{Y/B})^j) : \mathbb{G}_m &\to \operatorname{Res}_{Y/B}(\operatorname{Aut}_{\mathscr{E}/Y}) \times \operatorname{Res}_{Y/B}(\operatorname{Aut}_{\mathscr{F}_1/Y}) \\ x &\mapsto (\Delta_{\mathscr{E}}^{Y/B}(x^i), \Delta_{\mathscr{E}}^{Y/B}(x^j)). \end{aligned}$$

Finally, we use

$$\operatorname{coker}((\Delta_{\mathscr{E}}^{Y/B})^i, (\Delta_{\mathscr{F}_1}^{Y/B})^j) := \frac{(\operatorname{Res}_{Y/B}(\operatorname{Aut}_{\mathscr{E}/Y}) \times \operatorname{Res}_{Y/B}(\operatorname{Aut}_{\mathscr{F}_1/Y}))}{((\Delta_{\mathscr{E}}^{Y/B})^i, (\Delta_{\mathscr{F}_1}^{Y/B})^j)(\mathbb{G}_m)}.$$

Then, define the *automorphism sheaf* of this resolution data to be the B-scheme

$$\operatorname{Aut}_{\mathscr{E},\mathscr{F}_\bullet}^{Y/B} := \begin{cases} \operatorname{Res}_{Y/B}(\operatorname{Aut}_{\mathscr{E}/Y}) & \text{if } d=3, \\ \operatorname{coker}(\Delta_{\mathscr{E}}^{Y/B}, (\Delta_{\mathscr{F}_1}^{Y/B})^2) & \text{if } d=4, \\ \operatorname{coker}((\Delta_{\mathscr{E}}^{Y/B})^2, (\Delta_{\mathscr{F}_1}^{Y/B})^3) & \text{if } d=5. \end{cases} \tag{4.1}$$

In the case $Y = B$, we notate $\operatorname{Aut}_{\mathscr{E},\mathscr{F}_\bullet}^{B/B}$ simply by $\operatorname{Aut}_{\mathscr{E},\mathscr{F}_\bullet}$. When $d = 4$ or 5, we will often denote $\mathscr{F}_1$ by $\mathscr{F}$.

Throughout much of the remainder of the paper, we will typically work over the base $B = \operatorname{Spec} k$ for k a field. There are notable exceptions, such as Proposition 4.8, where we take $B = \operatorname{Spec} \mathbb{Z}$.

Remark 4.3. Concretely, $\mathscr{E}$ and $\mathscr{F}_\bullet$ in Definition 4.2 are (sequences of) sheaves of the following ranks. For $d = 3$, $\mathscr{E}$ is locally free of rank 2 and $\mathscr{F}_\bullet$ is trivial (i.e. the sequence of sheaves has length 0). When $d = 4$, $\mathscr{E}$ is locally free of rank 3 and $\mathscr{F}_\bullet = \mathscr{F}$ is locally free of rank 2. When $d = 5$, $\mathscr{E}$ is locally free of rank 4 and $\mathscr{F}_\bullet = \mathscr{F}$ is locally free of rank 5.

In order be able to calculate the class of quotients by the groups of Definition 4.2 in the Grothendieck ring, it will be useful to know these groups are often special. The following description of these quotients will allow us later, in Lemma 7.12, to easily deduce these groups are special.

LEMMA 4.4. *Maintaining the notation of Definition 4.2, we have an isomorphism of functors*

$$\operatorname{Aut}_{\mathscr{E},\mathscr{F}_\bullet}^{Y/B} \simeq \begin{cases} \ker(\det, \det^{-1}) : \operatorname{Res}_{Y/B}(\operatorname{Aut}_{\mathscr{E}/Y}) \times \operatorname{Res}_{Y/B}(\operatorname{Aut}_{\mathscr{F}/Y}) \to \operatorname{Res}_{Y/B}(\mathbb{G}_m) & \text{if } d=4, \\ \ker(\det^2, \det^{-1}) : \operatorname{Res}_{Y/B}(\operatorname{Aut}_{\mathscr{E}/Y}) \times \operatorname{Res}_{Y/B}(\operatorname{Aut}_{\mathscr{F}/Y}) \to \operatorname{Res}_{Y/B}(\mathbb{G}_m) & \text{if } d=5. \end{cases} \tag{4.2}$$

Here, by determinant we mean the map adjoint to the corresponding determinant map on Y.

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Proof. We produce the claimed isomorphisms by constructing a section to the quotient map $q : \mathrm{Res}_{Y/B}(\mathrm{Aut}_{\mathscr{E}/Y}) \times \mathrm{Res}_{Y/B}(\mathrm{Aut}_{\mathscr{F}/Y}) \to \mathrm{Aut}^{Y/B}_{\mathscr{E},\mathscr{F}_\bullet}$ defining $\mathrm{Aut}^{Y/B}_{\mathscr{E},\mathscr{F}_\bullet}$.

To start, we cover the case $d = 4$. Given $(M, N) \in \mathrm{Res}_{Y/B}(\mathrm{Aut}_{\mathscr{E}/Y}) \times \mathrm{Res}_{Y/B}(\mathrm{Aut}_{\mathscr{F}/Y})$, for $\lambda \in \mathbb{G}_m$, we can identify $q(M, N) = q(\lambda M, \lambda^2 N)$. For any such (M, N) the key observation is that there is a unique $\lambda \in \mathbb{G}_m$ such that $\det(\lambda M) = \det(\lambda^2 N)$. Indeed, $\det(\lambda M) = \lambda^3 \det M$ while $\det(\lambda^2 N) = \lambda^4 \det N$ and so the unique such λ is $\lambda = \det M / \det N$. This gives the desired splitting realizing $\mathrm{Aut}^{Y/B}_{\mathscr{E},\mathscr{F}_\bullet}$ as a subgroup of $\mathrm{Res}_{Y/B}(\mathrm{Aut}_{\mathscr{E}/Y}) \times \mathrm{Res}_{Y/B}(\mathrm{Aut}_{\mathscr{F}/Y})$ because the composition $\ker(\det, \det^{-1}) \to \mathrm{Res}_{Y/B}(\mathrm{Aut}_{\mathscr{E}/Y}) \times \mathrm{Res}_{Y/B}(\mathrm{Aut}_{\mathscr{F}/Y}) \to \mathrm{Aut}_{\mathscr{E},\mathscr{F}_\bullet}$ is an isomorphism.

The $d = 5$ case is quite similar to the $d = 4$ case. Namely, in this case, for $(M, N) \in \mathrm{Res}_{Y/B}(\mathrm{Aut}_{\mathscr{E}/Y}) \times \mathrm{Res}_{Y/B}(\mathrm{Aut}_{\mathscr{F}/Y})$, there is again a unique $\lambda \in \mathbb{G}_m$ so that $(\det(\lambda^2 M))^2 = \det(\lambda^3 N)$. Indeed, $(\det(\lambda^2 M))^2 = \lambda^{16} \det M^2$ and $\det(\lambda^3 N) = \lambda^{15} \det N$, so the unique desired λ is $\det N/(\det M)^2$. As in the $d = 4$ case, this provides a section to the given quotient map realizing $\mathrm{Aut}^{Y/B}_{\mathscr{E},\mathscr{F}_\bullet}$ as the subgroup of $\mathrm{Res}_{Y/B}(\mathrm{Aut}_{\mathscr{E}/Y}) \times \mathrm{Res}_{Y/B}(\mathrm{Aut}_{\mathscr{F}/Y})$ given as those (M, N) with $(\det M)^2 = \det N$. □

We next describe a presentation of the stack parameterizing degree d Gorenstein covers for $3 \le d \le 5$. To make our next definition, we will need to know the Gorenstein locus of a finite locally free map is open.

Lemma 4.5. *Let $f : X \to Y$ be a finite locally free morphism of schemes. The locus of points of Y on which the fiber of f is Gorenstein is an open subscheme of Y.*

Proof. First, by [Sta, Tag 00RH], the condition that the fiber be Cohen–Macaulay is an open condition. After restricting to such an open subscheme, by [Con00, Theorem 3.5.1], a dualizing sheaf exists, and the Gorenstein locus is then the locus where this dualizing sheaf is locally free, which again defines an open subscheme. □

We are now ready to define the relevant Gorenstein loci. With notation as in Definition 4.2, we work over $B = \operatorname{Spec} \mathbb{Z}$.

Definition 4.6. For each $3 \le d \le 5$, fix free sheaves on $Y = B = \operatorname{Spec} \mathbb{Z}$, $\mathscr{E}$ and $\mathscr{F}_\bullet$ as in Definition 4.2 and Remark 4.3. Let $\mathrm{U}_d \subset \operatorname{Spec}(\operatorname{Sym}^\bullet H^0(\operatorname{Spec} \mathbb{Z}, \mathscr{H}(\mathscr{E}, \mathscr{F}_\bullet))^\vee)$ denote the open subscheme (using Lemma 4.5) functorially parameterizing those sections η so that $\Psi_d(\eta)$ defines a degree d locally free Gorenstein cover, for Ψ_d the maps (depending on $3 \le d \le 5$) defined in § 3.6.

In what follows, we use Covers_d to denote the fibered category whose S points are finite locally free covers $X \to S$ of degree d with Gorenstein fibers.

Definition 4.7. For $3 \le d \le 5$, the map Ψ_d over $B = \operatorname{Spec} \mathbb{Z}$ induces a map $\mu_d : \mathrm{U}_d \to \mathrm{Covers}_d$, with Covers_d as defined above. There is a natural action of $\mathrm{Aut}_{\mathscr{E},\mathscr{F}_\bullet}$ on U_d, induced by the action of $\mathrm{Aut}_{\mathscr{E}} \times \mathrm{Aut}_{\mathscr{F}_\bullet}$ on U_d. The map μ_d is invariant under this action, since the resulting abstract degree d cover is unchanged by such re-coordinatizations. We now define the induced map from the quotient stack $\phi_d : [\mathrm{U}_d / \mathrm{Aut}_{\mathscr{E},\mathscr{F}_\bullet}] \to \mathrm{Covers}_d$.

Proposition 4.8. *For $3 \le d \le 5$, the map ϕ_d defined in Definition 4.7 over $B = \operatorname{Spec} \mathbb{Z}$ is an isomorphism.*

When $d = 3, 4$, Proposition 4.8 is the specialization of the isomorphisms of moduli stacks given in [Poo08, Proposition 5.1] and [Woo11, Theorem 1.1] to Gorenstein covers.

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Proof. We will construct an inverse map using Theorem 3.2. Using Theorem 3.2, there is an $\mathrm{Aut}_{\mathscr{E}} \times \mathrm{Aut}_{\mathscr{F}_\bullet}$ torsor T_d over Covers_d whose S-points parameterize covers $X \to S$ together with specified trivializations $\mathscr{E}^X \simeq \mathscr{E}, \mathscr{F}_\bullet^X \simeq \mathscr{F}_\bullet$ of the sheaves $\mathscr{E}^X$ and $\mathscr{F}_\bullet^X$ associated to X coming from Theorem 3.2. Note here that T_d maps surjectively to Covers_d because for any S point, there is an open cover of S on which these vector bundles become isomorphic to trivial bundles. The parametrizations Theorems 3.13, 3.14, and 3.16 then give a section $\eta \in \mathscr{H}(\mathscr{E}, \mathscr{F}_\bullet)$. This induces a map $\mathrm{T}_d \to \mathrm{U}_d$.

We wish to show this induced map $\mathrm{T}_d \to \mathrm{U}_d$ is an isomorphism in degree 3 and a $\mathbb{G}_m$ torsor in degrees 4 and 5, where $\mathbb{G}_m$ is the copy of $\mathbb{G}_m \subset \mathrm{Aut}_{\mathscr{E}} \times \mathrm{Aut}_{\mathscr{F}}$ as in Definition 4.2 whose quotient yields $\mathrm{Aut}_{\mathscr{E},\mathscr{F}_\bullet}$. Once we verify this, the parametrizations Theorems 3.13, 3.14, and 3.16 imply that the composition $\mathrm{T}_d \to \mathrm{U}_d \to [\mathrm{U}_d/\,\mathrm{Aut}_{\mathscr{E},\mathscr{F}_\bullet}] \xrightarrow{\phi_d} \mathrm{Covers}_d$ is the structure map for the torsor $\mathrm{T}_d \to \mathrm{Covers}_d$. From this, it follows that the resulting isomorphism $[\mathrm{T}_d/\,\mathrm{Aut}_{\mathscr{E}} \times \mathrm{Aut}_{\mathscr{F}_\bullet}] \to \mathrm{Covers}_d$ factors through an isomorphism $[\mathrm{T}_d/\,\mathrm{Aut}_{\mathscr{E}} \times \mathrm{Aut}_{\mathscr{F}_\bullet}] \to [\mathrm{U}_d/\,\mathrm{Aut}_{\mathscr{E},\mathscr{F}_\bullet}]$ and, hence, $\phi_d : [\mathrm{U}_d/\,\mathrm{Aut}_{\mathscr{E},\mathscr{F}_\bullet}] \to \mathrm{Covers}_d$ is an isomorphism.

First, we verify the map $\mathrm{T}_d \to \mathrm{U}_d$ is invariant under the above mentioned $\mathbb{G}_m$ action in the cases that $d = 4$ and 5. In the degree 4 case, scaling $\mathscr{E}$ by λ and $\mathscr{F}$ by λ^2 scales $\mathscr{F}^\vee \otimes \mathrm{Sym}^2 \mathscr{E}$ by $\lambda^{-2} \cdot \lambda^2 = 1$. In the degree 5 case, scaling $\mathscr{E}$ by λ^2 and $\mathscr{F}$ by λ^3 scales $\wedge^2 \mathscr{F} \otimes \mathscr{E} \otimes \det \mathscr{E}^\vee$ by $\lambda^6 \cdot \lambda^2 \cdot \lambda^{2\cdot(-4)} = 1$.

Therefore, to conclude the verification, it is enough to show the only elements of $\mathrm{Aut}_{\mathscr{E}} \times \mathrm{Aut}_{\mathscr{F}_\bullet}$ fixing a given section are trivial when $d = 3$ and lie in $\mathbb{G}_m$ when $d = 4$ or 5. To start, the map $X \to \mathbb{P}\mathscr{E}$ realizes X as a nondegenerate subscheme of $\mathbb{P}\mathscr{E}$ and, therefore, only the trivial element of $\mathrm{PGL}_{\mathscr{E}}$ fixes X as a subscheme of $\mathbb{P}\mathscr{E}$. In the degree 3 case, scaling by λ in the central $\mathbb{G}_m \subset \mathrm{Aut}_{\mathscr{E}}$ scales the resulting section by λ, and so only the identity element of $\mathrm{Aut}_{\mathscr{E}}$ preserves the section. This establishes the claim when $d = 3$.

We now consider the cases $d = 4$ and $d = 5$. We are seeking automorphisms of $\mathscr{E}$ and $\mathscr{F}$ preserving a given section $\eta \in \mathrm{U}_d$. We have seen above that any such automorphism must act on $\mathscr{E}$ by some element λ in the central $\mathbb{G}_m \subset \mathrm{Aut}_{\mathscr{E}}$. Since we are quotienting by a copy of $\mathbb{G}_m \subset \mathrm{Aut}_{\mathscr{E}} \times \mathrm{Aut}_{\mathscr{F}}$ which maps surjectively to the central $\mathbb{G}_m$ in $\mathrm{Aut}_{\mathscr{E}}$, we may modify our given automorphism so as to assume it is trivial in $\mathrm{Aut}_{\mathscr{E}}$. Note that when $d = 5$, we may have to pass to an fppf cover so as to extract a square root of λ. We may now assume the automorphism is trivial on $\mathscr{E}$ and wish to show it is also trivial on $\mathscr{F}$. However, the given section η induces an injective map $\mathscr{F} \to \mathrm{Sym}^2 \mathscr{E}$, realizing $\mathscr{F}$ as a subsheaf of $\mathrm{Sym}^2 \mathscr{E}$ by Theorem 3.2(v). Since we are assuming the automorphism acts as the identity on $\mathscr{E}$ and it preserves this inclusion, it must also act as the identity on $\mathscr{F}$. □

5. Defining our Hurwitz stacks

In this section, we construct and define the Hurwitz spaces we will be working with. We will ultimately be interested in the Hurwitz space whose geometric points parameterize degree d S_d covers of $\mathbb{P}^1$ which are smooth and connected. When one restricts to simply branched covers, such a Hurwitz scheme was constructed by Fulton [Ful69]. Another good reference is [Deo14, Theorem A], though this reference assumes characteristic 0. Another excellent reference is [BR11, Theorem 6.6.6], which constructs the Hurwitz stacks in the case that the cover is not Galois, but has a fixed Galois closure G, which is invertible on the base. Although we are ultimately primarily interested in counting S_d covers, we will do so by realizing them as a certain proportion of the space of all degree d covers, so this reference again does not quite suffice for our purposes. We were unable to find a reference that allows arbitrary branching and non-Galois covers in

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arbitrary characteristic, and so we give the construction here. To begin, we define a certain Hurwitz stack parameterizing covers of $\mathbb{P}^1$ which are not necessarily S_d covers.

Definition 5.1. For S a base scheme, and $d \geq 0$ an integer, let $\overline{\mathrm{Hur}}_{d,S}$ denote the category fibered in groups over S-schemes whose T points over a given map of schemes $T \to S$ consists of $(T, X, h : X \to T, f : X \to \mathbb{P}^1_T)$

$$\begin{array}{ccc} X & \xrightarrow{\;f\;} & \mathbb{P}^1_T \\ & {\scriptstyle h}\searrow \quad \swarrow & \\ & T & \end{array} \tag{5.1}$$

where X is a scheme, f is a finite locally free map of degree d and h is a smooth proper relative curve. A map $(T, X, h, f) \to (T, X', h', f')$ consists of a T-isomorphism $\alpha : X \to X'$ such that the following commutes.

$$\begin{array}{ccc} X & \xrightarrow{\;\alpha\;} & X' \\ {\scriptstyle f}\searrow & & \swarrow{\scriptstyle f'} \\ {\scriptstyle h}\downarrow & \mathbb{P}^1_T & \downarrow{\scriptstyle h'} \\ & \downarrow & \\ & T & \end{array} \tag{5.2}$$

For $g \geq 0$ an integer, let $\overline{\mathrm{Hur}}_{d,g,S}$ denote the substack parameterizing those T-points of $\overline{\mathrm{Hur}}_{d,S}$ such that $X \to T$ has arithmetic genus g.

Lemma 5.2. *For S a scheme, $\overline{\mathrm{Hur}}_{d,S}$ and $\overline{\mathrm{Hur}}_{d,g,S}$ are algebraic stacks.*

Proof. First, we show $\overline{\mathrm{Hur}}_{d,S}$ is an algebraic stack. It is enough to establish this in the universal case $S = \operatorname{Spec} \mathbb{Z}$. Observe that $\overline{\mathrm{Hur}}_{d,\mathbb{Z}}$ is a stack because descent for finite degree d locally free morphisms is effective. To see it is algebraic, we construct it as a hom stack. Let $\mathfrak{A}_d$ denote the stack parameterizing finite locally free degree d covers, as constructed in [Poo08, Definition 3.2].

Next, we claim the mapping stack $\mathrm{Hom}(\mathbb{P}^1, \mathfrak{A}_d)$ is algebraic. This would follow from [Aok06a, Theorem 1.1], except the theorem there is not stated correctly, as mentioned in the erratum [Aok06b]. This erratum asserts that we only need verify the additional condition that for any complete local noetherian ring A with maximal ideal $\mathfrak{m}$ and $A_n := A/\mathfrak{m}^n$, a collection of compatible maps $\mathrm{Hom}(\mathbb{P}^1_{A_n}, (\mathfrak{A}_d)_{A_n})$ for each n lifts to a map $\mathrm{Hom}(\mathbb{P}^1_A, (\mathfrak{A}_d)_A)$. In our setting, this condition is indeed satisfied because specifying such maps over A_n corresponds to specifying degree d locally free covers $X_n \to \mathbb{P}^1_{A_n}$ over A_n for each n. Then, by Grothendieck's algebraization theorem [FGI$^+$05, Theorem 8.4.10] such a family algebraizes to a family $X \to \mathbb{P}^1_A$ over $\operatorname{Spec} A$, using the pullback of $\mathscr{O}_{\mathbb{P}^1}(1)$ to X as the relevant ample line bundle on X.

The stack $\overline{\mathrm{Hur}}_{d,S}$ is then the open substack of the mapping stack $\mathrm{Hom}(\mathbb{P}^1, \mathfrak{A}_d)$ corresponding to those finite locally free covers $X \to \mathbb{P}^1$ which are smooth over the base.

Finally, $\overline{\mathrm{Hur}}_{d,g,S}$ is an open and closed substack of $\overline{\mathrm{Hur}}_{d,S}$ because the genus is locally constant in flat families. □

Having constructed the Hurwitz stack parameterizing all degree d covers of $\mathbb{P}^1$, we next construct an open substack parameterizing S_d covers, over geometric fibers. For the following definition, recall that B_n, the nth Bell number, is the number of ways to partition a set of n elements into subsets. Thus, for example, $B_1 = 1$, $B_2 = 2$, $B_3 = 5$ and $B_4 = 15$.

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DEFINITION 5.3. Let S be a scheme with $d!$ invertible on S. Let $\mathrm{Hur}_{d,g,S}$ denote the substack of $\overline{\mathrm{Hur}}_{d,g,S}$ parameterizing those $(T, X, h : X \to T, f : X \to \mathbb{P}^1_T)$ such that $X^d := \underbrace{X \times_{\mathbb{P}^1_T} X \times_{\mathbb{P}^1_T} \cdots \times_{\mathbb{P}^1_T} X}_{\text{d times}}$ has B_d irreducible components in each geometric fiber over T, where B_d is the dth Bell number.

The above definition is a bit opaque, but the point is that it parameterizes degree d covers $X \to \mathbb{P}^1$ so that the Galois closure of $K(X) \leftarrow K(\mathbb{P}^1)$ is an S_d Galois extension, as we now verify.

LEMMA 5.4. *The fiber product X^d as in Definition 5.3 always has at least B_d irreducible components in each geometric fiber over T.*

Further, it has exactly B_d components if and only if $X \to \mathbb{P}^1_T$ is a degree d cover whose Galois closure has Galois group S_d on geometric fibers over T.

Proof. We may reduce to the case T is a geometric point. First, we check X^d has at least B_d irreducible components. To see this, for any partition $U = \{S_1, \ldots, S_{\#U}\}$ of $\{1, \ldots, d\}$ into $\#U$ many subsets, let $X^U \subset X^d$ denote the subscheme of X^d given as the image $X^{\#U} \to X^d$ sending the ith copy of X via the identity to those copies of X indexed by elements of S_i. For each partition V of $\{1, \ldots, d\}$ such that U refines V, the closure of $X^U - \bigcup_{V, U \text{ refines } V} X^V$ defines a nonempty union of irreducible components of X^d. We have therefore produced B_d irreducible components of X_d, showing there are always at least B_d irreducible components.

Conversely, X^d has exactly B_d geometric components if and only if each of the B_d subschemes described in the previous paragraph are irreducible. Let us focus on the subscheme Y corresponding to the partition $U = \{\{1\}, \{2\}, \ldots, \{d\}\}$ into singletons, which has degree $d!$ over $\mathbb{P}^1$ and is the closure of the complement of the 'fat diagonal' in X^d. Observe that $X \to \mathbb{P}^1$ is generically étale because X is smooth and we are assuming the characteristic of T does not divide $d!$. Therefore, $Y \to \mathbb{P}^1$ is also generically étale, and contains a component whose function field is the Galois closure of the extension of function fields $K(X) \leftarrow K(\mathbb{P}^1)$. Therefore, Y is irreducible if and only if $K(Y)$ is the Galois closure of $K(X) \leftarrow K(\mathbb{P}^1)$. As $Y \to \mathbb{P}^1$ has degree $d!$, this, in turn, is equivalent to $X \to \mathbb{P}^1$ having Galois closure with Galois group S_d. In particular, for any cover $X \to \mathbb{P}^1$ whose Galois closure is smaller than S_d X^d has strictly more than B_d irreducible components.

Finally, we check that for any S_d cover, each of the B_d components described above are irreducible. As we have shown, even the component Y of degree $d!$ over $\mathbb{P}^1$ is irreducible. Because all the other components correspond to intermediate extensions between Y and $\mathbb{P}^1$, they are also irreducible. □

We next carry out the surprisingly tricky verification that $\mathrm{Hur}_{d,g,S}$ is an open substack of $\overline{\mathrm{Hur}}_{d,g,S}$.

PROPOSITION 5.5. *For any integers $d, g \geq 0$, and $d!$ invertible on S, $\mathrm{Hur}_{d,g,S}$ is an open substack of $\overline{\mathrm{Hur}}_{d,g,S}$, hence an algebraic stack. Further, if we have a family of curves $X \to \mathbb{P}^1_T \to T$ corresponding to a T-point of $\mathrm{Hur}_{d,g,S}$, all fibers of X over T are geometrically irreducible.*

Proof. It is enough to demonstrate $\mathrm{Hur}_{d,g,S}$ is an open substack of $\overline{\mathrm{Hur}}_{d,g,S}$. Let $X \to \mathbb{P}^1_T \to T$ be a family of smooth curves, corresponding to a point of $\overline{\mathrm{Hur}}_{d,g,S}$. Let X^d denote the d-fold fiber product of X over $\mathbb{P}^1_T$. By Lemma 5.4, any such point corresponds to an S_d cover of $\mathbb{P}^1$ on geometric fibers, and therefore these geometric fibers are irreducible, verifying the final statement.

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It remains to show that the locus where X^d has B_d irreducible fibers in geometric fibers is open on T. First, we will see in Lemma 5.7 that the geometric fibers of X^d over T have no embedded points.

Because the fibers have no embedded points, we may apply [Gro66, 12.2.1(xi)], which says that the total multiplicity (in the sense defined in [Gro65, p. 77], following [Gro65, 4.7.4], where total multiplicity is defined for integral schemes) is upper semicontinuous. From this, we conclude that the locus of geometric points in T where the total multiplicity of X^d is at most B_d is open. By Lemma 5.4, the total multiplicity of any geometric fiber is always at least B_d and, hence, the locus where the total multiplicity is exactly B_d is also open. To conclude, it remains to verify the total multiplicity of any geometric fiber is equal to the number of its irreducible components. Note that the radicial multiplicity of any fiber is 1 because X^d is generically reduced, since it has a generically separable map to $\mathbb{P}^1$ by assumption that $d! \nmid \operatorname{char}(k)$. It follows that the total multiplicity is equal to the separable multiplicity. By definition, the separable multiplicity of a finite-type scheme over a field is equal to 1 if and only if the scheme is geometrically irreducible, as desired. □

Remark 5.6. Later, in Lemma 9.6, we will appeal to [Wew98] to construct substacks of $\overline{\mathrm{Hur}}_{d,g,S}$ parameterizing covers with specified Galois group $G \subset S_d$. One can also see using the method of proof of Proposition 5.5 that these form locally closed substacks, with partial ordering given by the partial ordering along inclusion of subgroups in S_d.

Lemma 5.7. *Let $X \to \mathbb{P}^1$ be a degree d map of smooth proper curves over an algebraically closed field k. If the characteristic of k does not divide $d!$, then $X^d := \underbrace{X \times_{\mathbb{P}^1} X \times_{\mathbb{P}^1} \cdots \times_{\mathbb{P}^1} X}_{d \text{ times}}$ is Cohen–Macaulay and, hence, has no embedded points.*

Proof. It is enough to show X^d is Cohen–Macaulay, as one-dimensional Cohen–Macaulay schemes have no embedded points. To verify X^d is Cohen–Macaulay, we may do so étale locally on $\mathbb{P}^1$ and, hence, we may freely base change to the strict henselization of $\mathbb{P}^1$ at any given closed point. Using the assumption on the characteristic of k and the classification of prime to $\operatorname{char}(k)$ covers of the strict henselization of $k[t]$, we may assume our cover is given by extracting roots of the uniformizer. Equivalently, it is enough to verify Cohen–Macaulayness in the case X^d is locally described as a localization of $k[x_1] \otimes_{k[t]} k[x_2] \otimes_{k[t]} \cdots \otimes_{k[t]} k[x_m]$ where the maps $k[t] \to k[x_i]$ are given by $t \mapsto x_i^{s_i}$, for $s_i \leq d$. We can equivalently write this tensor product as $k[x_1] \otimes_{k[t]} k[x_2] \otimes_{k[t]} \cdots \otimes_{k[t]} k[x_m] \simeq k[x_1, x_2, \ldots, x_m]/(x_1^{s_1} - x_2^{s_2}, \ldots, x_1^{s_1} - x_m^{s_m}) =: R$. We wish to verify R is Cohen–Macaulay. Observe that R is a one-dimensional scheme, being a finite cover of $k[t]$. Since it is defined by $m-1$ equations in $\mathbb{A}^m$, it is a complete intersection and, therefore, Cohen–Macaulay. □

The following remark will not be used in the remainder of the paper, but may be nice for the reader to keep in mind.

Remark 5.8. For $d > 2$ and $g \geq 1$, $\mathrm{Hur}_{d,g,S}$ is a scheme when $d!$ is invertible on S. We have seen above it is an algebraic stack. In order to see it is a scheme, one may first verify it is an algebraic space by checking any degree d cover of $\mathbb{P}^1$ with Galois group S_d for $d > 2$ has no nontrivial automorphisms [Sta, Tag 04SZ]. Indeed, if such a cover did have automorphisms, it would factor through an intermediate cover obtained by quotienting by some such nontrivial automorphism, forcing the Galois group to be smaller than S_d.

Having established $\mathrm{Hur}_{d,g,S}$ is an algebraic space, we next wish to explain why it is a scheme. Observe this Hurwitz space has a map to the symmetric power $\mathrm{Sym}^{2g-2+2d}_{\mathbb{P}^1}$ of $2g-2+2d$

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points on $\mathbb{P}^1$ given by 'taking the branch locus'. This uses that $d!$ is invertible on S and Riemann–Hurwitz. One may verify this map is separated (for example, using the valuative criterion) and quasi-finite (since the inertia data around the branch points determines the cover), hence quasi-affine [Sta, Tag 082J]. Therefore, it is quasi-affine over a scheme, and therefore a scheme.

6. Defining the Casnati–Ekedahl stratification of Hurwitz stacks

For this section, we now fix a positive integer d and a base field k with $d!$ invertible on k. We parenthetically note that much of the following can be generalized to work over arbitrary base schemes. For T a k-scheme, given a Gorenstein finite locally free degree d cover $X \to \mathbb{P}^1_T$, from Theorem 3.2, we obtain a canonical sequence of vector bundles $(\mathscr{E}^X, \mathscr{F}_1^X, \mathscr{F}_2^X, \ldots, \mathscr{F}_{d-2}^X)$ on $\mathbb{P}^1_T$. We next aim to define certain locally closed substacks of $\overline{\mathrm{Hur}}_{d,g,S}$ corresponding to those covers $X \to \mathbb{P}^1_T$ whose associated vector bundles are isomorphic to some specified sequence $(\mathscr{E}, \mathscr{F}_1, \mathscr{F}_2, \ldots, \mathscr{F}_{d-2})$. To define this substack, we first define the corresponding stack of these vector bundles.

Recall that the stack of locally free rank-n sheaves on $\mathbb{P}^1_k$ is an algebraic stack, as is well known; see, for example, [Beh91, Proposition 4.4.6].

DEFINITION 6.1. Let $\mathrm{Vect}^n_{\mathbb{P}^1_k}$ denote the moduli stack of locally free rank-n sheaves on $\mathbb{P}^1_k$. For $\vec{a} = (a_1, a_2, \ldots, a_n)$, with $a_i \in \mathbb{Z}$, let $\mathscr{O}_{\mathbb{P}^1_k}(\vec{a}) := \bigoplus_{i=1}^n \mathscr{O}_{\mathbb{P}^1_k}(a_i)$ and let $\mathrm{Vect}^{\vec{a}}_{\mathbb{P}^1_k}$ denote the residual gerbe at the point corresponding to the vector bundle $\mathscr{O}_{\mathbb{P}^1_k}(\vec{a})$.

Remark 6.2. Note that this residual gerbe is indeed a locally closed substack by [Ryd11, Theorem B.2]. Alternatively, the residual gerbe is given concretely as the quotient stack $B(\mathrm{Res}_{\mathbb{P}^1_k/k}(\mathrm{Aut}_{\mathscr{O}_{\mathbb{P}^1_k}(\vec{a})}))$.

In order to relate the genus of a cover of $\mathbb{P}^1$ to the associated vector bundle $\mathscr{E}$ we need the following standard lemma.

LEMMA 6.3. *Suppose $\rho : X \to \mathbb{P}^1_k$ is a degree d Gorenstein finite locally free cover and let $\mathscr{E} := \ker(\rho_*\omega_X \to \mathscr{O}_{\mathbb{P}^1_k})$. If $h^0(X, \mathscr{O}_X) = 1$, such as in the case that X is smooth and geometrically connected, then $\deg(\det \mathscr{E}) = g + d - 1$.*

Proof. First, we claim $\rho_*\mathscr{O}_X \simeq \mathscr{O}_{\mathbb{P}^1_k} \oplus \mathscr{E}^\vee$. Indeed, by duality, we have a short exact sequence $\mathscr{O}_{\mathbb{P}^1_k} \to \rho_*\mathscr{O}_X \to \mathscr{E}^\vee$. Because all vector bundles on $\mathbb{P}^1$ split, and $h^0(\mathbb{P}^1_k, \rho_*\mathscr{O}_X) = h^0(X, \mathscr{O}_X) = 1$, we find that $\mathscr{E}^\vee \simeq \bigoplus_{i=1}^{d-1} \mathscr{O}_{\mathbb{P}^1_k}(-a_i)$ for $a_i > 0$. Because there are no extensions of $\mathscr{O}_{\mathbb{P}^1_k}(-a_i)$ by $\mathscr{O}_{\mathbb{P}^1_k}$, the above exact sequence splits, yielding $\rho_*\mathscr{O}_X \simeq \mathscr{O}_{\mathbb{P}^1_k} \oplus \mathscr{E}^\vee \simeq \mathscr{O}_{\mathbb{P}^1_k} \oplus \bigoplus_{i=1}^{d-1} \mathscr{O}_{\mathbb{P}^1_k}(-a_i)$. Then, for n sufficiently large and $\mathscr{L}$ a degree n line bundle on $\mathbb{P}^1_k$, Riemann Roch on the curve X implies $h^0(\mathbb{P}^1_k, \mathscr{O}_{\mathbb{P}^1_k}(n) \oplus \bigoplus_{i=1}^{d-1} \mathscr{O}_{\mathbb{P}^1_k}(-a_i + n)) = h^0(\mathbb{P}^1_k, \rho_*\mathscr{O}_X \otimes \mathscr{L}) = h^0(X, \rho^*\mathscr{L}) = dn - g + 1$. For n larger than the maximum of the a_i, the left-hand side is equal to $dn + d - \sum_{i=1}^{d-1} a_i$, and so we obtain $-\sum_{i=1}^{d-1} a_i = -g - d + 1$. Therefore, $\deg(\det \mathscr{E}) = -\deg(\det \mathscr{E}^\vee) = \sum_{i=1}^{d-1} a_i = g + d - 1$. $\square$

With the relation between g and $\mathscr{E}$ of Lemma 6.3 established, we are ready to define the Casnati–Ekedahl strata. For the next definition, we will fix vectors $\vec{a}^{\mathscr{E}}, \vec{a}^{\mathscr{F}_1}, \ldots, \vec{a}^{\mathscr{F}_{d-2}}$ and vector bundles $\mathscr{E}, \mathscr{F}_1, \ldots, \mathscr{F}_{\lfloor (d-2)/2 \rfloor}$ on $\mathbb{P}^1$ given by $\mathscr{E} \simeq \mathscr{O}_{\mathbb{P}^1_k}(\vec{a}^{\mathscr{E}})$ and $\mathscr{F}_i \simeq \mathscr{O}_{\mathbb{P}^1_k}(\vec{a}^{\mathscr{F}_i})$. Note that although $d-2$ vector bundles appear in Theorem 3.2, the isomorphism classes of vector bundles

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$\mathscr{F}_i$ for $i > \lfloor (d-2)/2 \rfloor$ are, in fact, determined by those with $i \leq \lfloor (d-2)/2 \rfloor$ because duality enforces the relation $\mathscr{F}_{d-2} \simeq \det \mathscr{E}$ and for $1 \leq i \leq d-3$, $\mathscr{F}_{d-2-i} \simeq \det \mathscr{E} \otimes \mathscr{F}_i^\vee$.

Definition 6.4. Let k be a field with $d!$ invertible on k, and fix a tuple of vectors $(\vec{a}^{\mathscr{E}}, \vec{a}^{\mathscr{F}_1}, \dots, \vec{a}^{\mathscr{F}_{\lfloor (d-2)/2 \rfloor}})$. Let $g := 1 - d + \sum_{i=1}^{d-1} a_i^{\mathscr{E}}$ and define the *Casnati–Ekedahl stratum* $\mathscr{M}(\vec{a}^{\mathscr{E}}, \vec{a}^{\mathscr{F}_1}, \dots, \vec{a}^{\mathscr{F}_{\lfloor (d-2)/2 \rfloor}})$ as the locally closed substack of $\overline{\mathrm{Hur}}_{d,g,k}$ given as the fiber product

$$\overline{\mathrm{Hur}}_{d,g,k} \times_{\mathrm{Vect}^{d-1}_{\mathbb{P}^1_k} \times \prod_{i=1}^{\lfloor (d-2)/2 \rfloor} \mathrm{Vect}^{\beta_i}_{\mathbb{P}^1_k}} \mathrm{Vect}^{\vec{a}^{\mathscr{E}}}_{\mathbb{P}^1_k} \times \prod_{i=1}^{\lfloor (d-2)/2 \rfloor} \mathrm{Vect}^{\vec{a}^{\mathscr{F}_i}}_{\mathbb{P}^1_k}.$$

Here, β_i are as in Theorem 3.2, and the map $\overline{\mathrm{Hur}}_{d,g,k} \to \mathrm{Vect}^{d-1}_{\mathbb{P}^1_k} \times \prod_{i=1}^{\lfloor (d-2)/2 \rfloor} \mathrm{Vect}^{\beta_i}_{\mathbb{P}^1_k}$ is induced by Theorem 3.2. In other words, $\mathscr{M}(\vec{a}^{\mathscr{E}}, \vec{a}^{\mathscr{F}_1}, \dots, \vec{a}^{\mathscr{F}_{\lfloor (d-2)/2 \rfloor}})$ is the locally closed substack of the Hurwitz stack such that the associated morphism $T \to \mathrm{Vect}^{d-1}_{\mathbb{P}^1_k} \times \prod_{i=1}^{\lfloor (d-2)/2 \rfloor} \mathrm{Vect}^{\beta_i}_{\mathbb{P}^1_k}$ factors through a map $T \to \mathrm{Vect}^{\vec{a}^{\mathscr{E}}}_{\mathbb{P}^1_k} \times \prod_{i=1}^{\lfloor (d-2)/2 \rfloor} \mathrm{Vect}^{\vec{a}^{\mathscr{F}_i}}_{\mathbb{P}^1_k}$.

Remark 6.5. There is a natural generalization of the construction of Casnati–Ekedahl strata of covers of $\mathbb{P}^1$ to a version for covers of genus-g curves C in place of the genus-0 curve $\mathbb{P}^1$. Namely, given a finite locally free cover $C' \to C$ over a base T, using Theorem 3.2, one can associate a sequence of vector bundles on the relative curve $C \to T$. A given Casnati–Ekedahl stratum would naturally be defined as the locus where these bundles have specific Harder–Narasimhan filtration, generalizing the notion of splitting type.

Remark 6.6. Since the substacks $\mathrm{Vect}^{\vec{a}}_{\mathbb{P}^1_k}$ form a stratification of $\mathrm{Vect}^{n}_{\mathbb{P}^1_k}$, it follows that the Casnati–Ekedahl strata, varying over all tuples $(\vec{a}^{\mathscr{E}}, \vec{a}^{\mathscr{F}_1}, \dots, \vec{a}^{\mathscr{F}_{\lfloor (d-2)/2 \rfloor}})$ form a stratification of $\overline{\mathrm{Hur}}_{d,g,k}$. This will enable us to write the class of $\overline{\mathrm{Hur}}_{d,g,k}$ in the Grothendieck ring as the sum of the classes of $\mathscr{M}(\vec{a}^{\mathscr{E}}, \vec{a}^{\mathscr{F}_1}, \dots, \vec{a}^{\mathscr{F}_{\lfloor (d-2)/2 \rfloor}})$ for $(\vec{a}^{\mathscr{E}}, \vec{a}^{\mathscr{F}_1}, \dots, \vec{a}^{\mathscr{F}_{\lfloor (d-2)/2 \rfloor}})$ varying over all integer tuples of vectors.

To conclude this section, we introduce some notation for objects we will associate with a Casnati–Ekedahl stratum of the Hurwitz stack.

Notation 6.7. For $3 \leq d \leq 5$, $\mathscr{M}(\vec{a}^{\mathscr{E}}, \vec{a}^{\mathscr{F}_1}, \dots, \vec{a}^{\mathscr{F}_{\lfloor (d-2)/2 \rfloor}}) \subset \overline{\mathrm{Hur}}_{d,g,k}$ a Casnati–Ekedahl stratum, for $\mathscr{E} := \bigoplus_j \mathscr{O}(\vec{a}_i^{\mathscr{E}})$, $\mathscr{F}_\bullet := \bigoplus_j \mathscr{O}(\vec{a}_j^{\mathscr{F}_\bullet})$, define $\mathrm{Aut}_{\mathscr{M}} := \mathrm{Aut}^{\mathbb{P}^1_k/k}_{\mathscr{E},\mathscr{F}_\bullet}$, as defined in Definition 4.2, depending on the value of d. In addition, for $f : T \to \mathbb{P}^1_k$ denote $\mathrm{Aut}_{f^*\mathscr{M}} := \mathrm{Aut}^{T/k}_{f^*\mathscr{E}, f^*\mathscr{F}_\bullet}$. When the map f is understood, we also use $\mathrm{Aut}_{\mathscr{M}|_T}$ as notation for $\mathrm{Aut}_{f^*\mathscr{M}}$.

Remark 6.8. The construction $\mathrm{Aut}_{\mathscr{M}|_T}$ at the end of Notation 6.7 will primarily be used when $T = D$, the dual numbers, mapping to a point of $\mathbb{P}^1_k$. Note that, in this case $\mathscr{E}|_D$ and $\mathscr{F}_\bullet|_T$ are free vector bundles because all locally free bundles over D are free.

We conclude this section with a general discussion about the moduli stack of vector bundles on $\mathbb{P}^1_k$. This will be useful in later sections, specifically in Lemma 9.11.

6.9 Discussion of the moduli stack of vector bundles on $\mathbb{P}^1_k$

Recall that, for k a field, every vector bundle $\mathscr{V}$ on $\mathbb{P}^1_k$ of rank r and degree δ can be written as $\mathscr{V} \simeq \bigoplus_{i=1}^r \mathscr{O}_{\mathbb{P}^1_k}(a_i)$ where $\sum_{i=1}^r a_i = \delta$. The moduli stack of vector bundles of rank r and degree δ on $\mathbb{P}^1_k$ is smooth and connected. The generic point of this moduli stack is given by a balanced bundle. Formally, a vector bundle $\mathscr{V}$ on $\mathbb{P}^1$ is *balanced* if it can be written as $\mathscr{V} \simeq \bigoplus_{i=1}^r \mathscr{O}(a_i)$ with $|a_i - a_j| \leq 1$ for all $1 \leq i \leq j \leq r$.

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We can now describe when one degree δ, rank-r vector bundle $\mathscr{V}$, viewed as a point on the moduli stack, lies in the closure of a point corresponding to another degree δ, rank-r vector bundle $\mathscr{W}$. See [EH16, Theorem 14.7(a)] for a proof of the following description. Suppose $\mathscr{V} \simeq \bigoplus_{i=1}^r \mathscr{O}_{\mathbb{P}^1_k}(a_i) = \mathscr{O}_{\mathbb{P}^1_k}(\vec{a})$, and there are some a_i, a_j with $a_i \leq a_j - 2$. Let $\sigma_{i,j}(\vec{a}) := (a_1, \dots, a_i + 1, \dots, a_j - 1, \dots, a_r)$ so that $\sigma_{i,j}(\vec{a})$ agrees with $\vec{a}$ except in positions i and j. Then $\mathscr{O}_{\mathbb{P}^1_k}(\sigma_{i,j}(\vec{a}))$ lies in the closure of $\mathscr{O}_{\mathbb{P}^1_k}(\vec{a})$. Informally, one bundle lies in the closure of another if one can find a sequence of moves as above relating one to the other. More precisely, $\mathscr{O}_{\mathbb{P}^1_k}(\vec{b})$ lies in the closure of $\mathscr{O}_{\mathbb{P}^1_k}(\vec{a})$ if we can write $\vec{b} = \sigma_{i_1,j_1} \circ \sigma_{i_2,j_2} \cdots \circ \sigma_{i_m,j_m}(\vec{a})$ for some non-negative integer m and integers $i_1, \dots, i_m, j_1, \dots, j_m$. In particular, if one starts with any vector bundle of rank r and degree δ, one can sequentially move the entries of $\vec{a}$ closer together, which shows that a balanced bundle correspond to the generic point of the moduli stack.

7. Presentations of the Casnati–Ekedahl strata

We next aim to use the parametrizations from § 3 in order to describe each of the $\mathscr{M}(\mathscr{E}, \mathscr{F}_\bullet)$ for $3 \leq d \leq 5$ as the quotient of an open in affine space by an appropriate group action. Because we will also want to parameterize simply branched covers, it will be useful to restrict the possible ramification types of these covers. We now introduce the notion of ramification profile, which describes the possible ramification types of a finite cover of $\mathbb{P}^1$ by a smooth curve.

Definition 7.1 (Ramification profile). Fix a positive integer d and let $R = (r_1^{t_1}, r_2^{t_2}, \dots, r_n^{t_n})$ denote a partition of d, i.e. a collection of integers with $t_1, \dots, t_n \geq 1$ so that $\sum_{i=1}^n t_i r_i = d$. Here, we think of r_i as the part sizes appearing in the partition and t_i as the corresponding multiplicity. A *ramification profile of degree* d is a partition of d. For $X \to S$ a scheme, we say X *has ramification profile* R if for every geometric point $\operatorname{Spec} k \in S$, the base change $X_k := X \times_S \operatorname{Spec} k$ is isomorphic to $\coprod_{i=1}^n (\coprod_{j=1}^{t_i} \operatorname{Spec} k[x]/(x^{r_i}))$. We let $r(R) := \sum_{i=1}^n (r_i - 1)t_i$ denote the associated *ramification order*.

One way to think about ramification profiles as defined above is to think of each fiber X_k of $X \to S$ having a partition into curvilinear schemes (i.e. schemes with cotangent spaces of dimension at most 1 at every point) of degrees determined by the partition R.

We next introduce the notion of an allowable collection of ramification profiles. The point of allowable collections is that covers of $\mathbb{P}^1$ whose ramification profiles lie in an allowable collection define an open substack of the Hurwitz stack with closed complement of high codimension. We use the notation $\lambda \vdash n$ to indicate that λ is a partition of n.

Definition 7.2. Fix an integer d. Let $\mathcal{R}$ denote a collection of ramification profiles of degree d. We say $\mathcal{R}$ is an *allowable* collection of ramification profiles of degree d if:

(1) $\mathcal{R}$ includes (1^d) and $(2, 1^{d-2})$;
(2) whenever $\lambda \vdash d$ lies in $\mathcal{R}$, and $\lambda' \vdash d$ is a partition refining λ, then λ' also lies in $\mathcal{R}$.

In the remainder of this section, we first define certain open substacks of Hurwitz stacks with restricted ramification, lying in an allowable collection $\mathcal{R}$. Following this, we define a certain space of sections of a vector bundle on $\mathbb{P}^1$ parameterizing smooth degree d covers (for $3 \leq d \leq 5$) with specified ramification profiles in an allowable collection.

Definition 7.3. Suppose k is a field with $d!$ invertible on k. For $\mathcal{R}$ an allowable collection of ramification profiles of degree d, let $\overline{\operatorname{Hur}}_{d,g,k}^{\mathcal{R}} \subset \overline{\operatorname{Hur}}_{d,g,k}$ denote the open substack of $\overline{\operatorname{Hur}}_{d,g,k}$ (we prove it is open in Lemma 7.5) whose T points parameterize smooth curves $X \to \mathbb{P}^1_T$ over T

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so that for each geometric point $\operatorname{Spec}\kappa \to \mathbb{P}^1_T$, X_κ has ramification profile in $\mathcal{R}$. Let $\mathrm{Hur}^{\mathcal{R}}_{d,g,k} \subset \mathrm{Hur}_{d,g,k}$ denote the restriction of $\overline{\mathrm{Hur}}^{\mathcal{R}}_{d,g,k}$ along $\mathrm{Hur}_{d,g,k} \subset \overline{\mathrm{Hur}}_{d,g,k}$.

Similarly, for $\mathscr{M} \subset \overline{\mathrm{Hur}}_{d,g,k}$ a Casnati–Ekedahl stratum, let $\mathscr{M}^{\mathcal{R}} \subset \mathscr{M}$ denote the open substack $\mathscr{M} \times_{\overline{\mathrm{Hur}}_{d,g,k}} \overline{\mathrm{Hur}}^{\mathcal{R}}_{d,g,k} \subset \mathscr{M}$. We use $\mathscr{M}^{\mathcal{R},S_d}$ to denote the pullback of $\mathscr{M}^{\mathcal{R}}$ along $\mathrm{Hur}^{\mathcal{R}}_{d,g,k} \subset \overline{\mathrm{Hur}}^{\mathcal{R}}_{d,g,k}$.

Remark 7.4. In the case $d = 2$, the only allowable $\mathcal{R}$ is $\mathcal{R} = \{(1^2), (2)\}$ and in this case $\mathrm{Hur}^{\mathcal{R}}_{2,g,k} = \mathrm{Hur}_{2,g,k}$.

We now verify that the locus of $\overline{\mathrm{Hur}}^{\mathcal{R}}_{d,g,k} \subset \overline{\mathrm{Hur}}_{d,g,k}$ is an open substack.

Lemma 7.5. *With notation as in Definition 7.3, $\overline{\mathrm{Hur}}^{\mathcal{R}}_{d,g,k} \subset \overline{\mathrm{Hur}}_{d,g,k}$ is an open substack.*

Proof. As a first step, we will show this is a constructible subset. To do so, we can define certain substacks of $\overline{\mathrm{Hur}}_{d,g,k}$ parameterizing covers $f : C \to \mathbb{P}^1_k$ so that the multiset of ramification profiles over the geometric branch points of f is equal to some fixed multiset S, which we call $(\overline{\mathrm{Hur}}_{d,g,k})^S$. One can show $(\overline{\mathrm{Hur}}_{d,g,k})^S$ is an algebraic stack. (See [BR11, Theorem 6.6.6] for a very closely related construction.) Therefore, the image of any of these stacks in $\overline{\mathrm{Hur}}_{d,g,k}$ is constructible. Since the underlying set of $\overline{\mathrm{Hur}}^{\mathcal{R}}_{d,g,k}$ is a finite union $\coprod_S (\overline{\mathrm{Hur}}_{d,g,k})^S$ for all possible multisets S producing genus-g covers which only include ramification lying in $\mathcal{R}$, we obtain that $\overline{\mathrm{Hur}}^{\mathcal{R}}_{d,g,k}$ is a constructible subset of $\overline{\mathrm{Hur}}_{d,g,k}$.

To conclude, we wish to show this constructible subset is in fact an open subset. To do so, we only need show it is closed under generization. However, if a point of $(\overline{\mathrm{Hur}}_{d,g,k})^S$ has a generization which is a point of $(\overline{\mathrm{Hur}}_{d,g,k})^{S'}$ then all ramification profiles appearing in S' must be refinements of those appearing in S. Therefore, condition (2) from the definition of allowable collection of ramification profiles, Definition 7.2, shows that $\overline{\mathrm{Hur}}^{\mathcal{R}}_{d,g,k}$ is indeed closed under generization, and so defines an open substack of $\overline{\mathrm{Hur}}_{d,g,k}$. ☐

We next give analogs of the restricted ramification loci above for spaces of sections.

Definition 7.6. For $3 \leq d \leq 5$, suppose $d!$ is invertible on k. Fix a choice of Casnati–Ekedahl stratum $\mathscr{M} := \mathscr{M}(\vec{a}^{\mathscr{E}}, \vec{a}^{\mathscr{F}_1}, \ldots, \vec{a}^{\mathscr{F}_{\lfloor (d-2)/2 \rfloor}})$, with associated locally free sheaves on $\mathbb{P}^1_k$ given by $\mathscr{E}_{\mathscr{M}} := \bigoplus \mathscr{O}(\vec{a}^{\mathscr{E}}_i)$, and, if $4 \leq d \leq 5$, $\mathscr{F}_{\mathscr{M}} := \bigoplus \mathscr{O}(\vec{a}^{\mathscr{F}_1}_j)$. Let $g := \deg \det \mathscr{E}_{\mathscr{M}} - d + 1$. Let $\mathscr{H}_{\mathscr{M}}$ denote the associated locally free sheaf on $\mathbb{P}^1$ defined in (3.6). Let $\mathcal{R}$ denote an allowable collection of ramification profiles. Then, define $\mathrm{U}^{\mathcal{R}}_{\mathscr{M}}$ to be the open subscheme (we prove openness in Lemma 7.7) of $\operatorname{Spec} \operatorname{Sym}^\bullet H^0(\mathbb{P}^1_k, \mathscr{H}_{\mathscr{M}})^\vee$ parameterizing T-points η so that $\Psi_d(\eta)$ defines a smooth proper curve over T with geometrically connected fibers such that over each geometric point $\operatorname{Spec}\kappa \to \mathbb{P}^1_T$, the pullback of $\Phi_d(\eta) \subset \mathbb{P}\mathscr{E} \to \mathbb{P}^1_T$ along $\operatorname{Spec}\kappa \to \mathbb{P}^1_T$ has ramification profile lying in $\mathcal{R}$.

In addition, define $\mathrm{U}^{\mathcal{R},S_d}_{\mathscr{M}} \subset \mathrm{U}^{\mathcal{R}}_{\mathscr{M}}$ as the open subscheme parameterizing those sections η for which $\Psi_d(\eta)$ is a smooth curve X with geometrically connected fibers, such that over each fiber, the cover $X \to \mathbb{P}^1$ of degree d has Galois closure which is an S_d cover. In other words, $\mathrm{U}^{\mathcal{R},S_d}_{\mathscr{M}}$ is the subset of $\mathrm{U}^{\mathcal{R}}_{\mathscr{M}}$ for which the map Ψ_d defines a point of $\mathrm{Hur}_{d,g,k}$.

In the above definition, we claimed $\mathrm{U}^{\mathcal{R}}_{\mathscr{M}} \subset \operatorname{Spec} \operatorname{Sym}^\bullet H^0(\mathbb{P}^1_k, \mathscr{H}_{\mathscr{M}})^\vee$ is an open subscheme. We now justify this.

Lemma 7.7. *The subset $\mathrm{U}^{\mathcal{R}}_{\mathscr{M}} \subset \operatorname{Spec} \operatorname{Sym}^\bullet H^0(\mathbb{P}^1_k, \mathscr{H}_{\mathscr{M}})^\vee$ naturally has the structure of an open subscheme.*

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Proof. Let $W_d \subset \operatorname{Spec} \operatorname{Sym}^\bullet H^0(\mathbb{P}^1_k, \mathscr{H}_{\mathscr{M}})^\vee$ denote the open subscheme parameterizing those sections η for which $\Psi_d(\eta)$ has degree d on all fibers. There is a map $W_d \to \overline{\mathrm{Hur}}_{d,g,k}$, induced by Ψ_d sending $\eta \mapsto \Psi_d(\eta)$. Under this map, $\mathrm{U}^{\mathcal{R}}_{\mathscr{M}}$ is the preimage of $\overline{\mathrm{Hur}}^{\mathcal{R}}_{d,g,k}$, which is open by Lemma 7.5. Hence, $\mathrm{U}^{\mathcal{R}}_{\mathscr{M}} \subset \operatorname{Spec} \operatorname{Sym}^\bullet H^0(\mathbb{P}^1_k, \mathscr{H}_{\mathscr{M}})^\vee$ is open. □

Example 7.8. If we take $\mathcal{R}$ in Definition 7.6 to range over all possible ramification profiles (i.e. all partitions of d) then $\mathrm{U}^{\mathcal{R}}_{\mathscr{M}}$ corresponds to all sections η as in Definition 7.6 with $\Phi_d(\eta)$ a smooth geometrically connected degree d cover of $\mathbb{P}^1_k$.

On the other hand, if we take $\mathcal{R}$ to be the union of two ramification profiles, the first given by 1^d and the second given by $(2, 1^{d-2})$, we obtain all sections η with $\Phi_d(\eta)$ a smooth geometrically connected curve which is simply branched over $\mathbb{P}^1$.

7.9 Writing the class as a sum over Casnati–Ekedahl strata

Our goal for the remainder of the section is to express the class of the Hurwitz stack as a sum over the Casnati–Ekedahl strata, which will be somewhat more manageable due to their descriptions as quotients of opens in affine spaces by relatively simple algebraic groups.

PROPOSITION 7.10. *For $3 \le d \le 5$, and $\mathcal{R}$ an allowable collection of ramification profiles of degree d, we have an equality in $K_0(\mathrm{Stacks}_k)$*

$$\{\mathrm{Hur}^{\mathcal{R}}_{d,g,k}\} = \sum_{\text{Casnati–Ekedahl strata } \mathscr{M}} \frac{\{\mathrm{U}^{\mathcal{R},S_d}_{\mathscr{M}}\}}{\{\mathrm{Aut}_{\mathscr{M}}\}}.$$

Proof assuming Proposition 7.11 and Lemma 7.12. We claim

$$\begin{aligned}
\{\mathrm{Hur}^{\mathcal{R}}_{d,g,k}\} &= \sum_{\text{Casnati–Ekedahl strata } \mathscr{M}} \{\mathscr{M}^{\mathcal{R},S_d}\} \\
&= \sum_{\text{Casnati–Ekedahl strata } \mathscr{M}} \left\{\left[\frac{\mathrm{U}^{\mathcal{R},S_d}_{\mathscr{M}}}{\mathrm{Aut}_{\mathscr{M}}}\right]\right\} \\
&= \sum_{\text{Casnati–Ekedahl strata } \mathscr{M}} \frac{\{\mathrm{U}^{\mathcal{R},S_d}_{\mathscr{M}}\}}{\{\mathrm{Aut}_{\mathscr{M}}\}}.
\end{aligned}$$

The first equality holds because the Casnati–Ekedahl strata form a stratification of $\overline{\mathrm{Hur}}_{d,g,k}$ by locally closed substacks. The second holds by Proposition 7.11. The final equality holds by Lemma 7.12, using both that $\mathrm{Aut}_{\mathscr{M}}$ is special so $\{\mathrm{Aut}_{\mathscr{M}}\}\{[\mathrm{U}^{\mathcal{R},S_d}_{\mathscr{M}} / \mathrm{Aut}_{\mathscr{M}}]\} = \{\mathrm{U}^{\mathcal{R},S_d}_{\mathscr{M}}\}$ by [Eke09a, Proposition 1.4(i)], and that $\{\mathrm{Aut}_{\mathscr{M}}\}$ is invertible. □

To conclude our proof of Proposition 7.10, we need to verify Proposition 7.11 and Lemma 7.12. We omit the proof of Proposition 7.11 since it is analogous to Proposition 4.8, where we additionally fix isomorphisms to fixed bundles $\mathscr{E}, \mathscr{F}_\bullet$ on $\mathbb{P}^1_k$ (as opposed to trivial bundles on $\operatorname{Spec} \mathbb{Z}$) and add in conditions associated to the ramification profiles in $\mathcal{R}$ and lying in $\mathrm{Hur}_{d,g,k}$ appropriately.

PROPOSITION 7.11. *For $3 \le d \le 5$, fix a choice of Casnati–Ekedahl stratum $\mathscr{M} := \mathscr{M}(\vec{a}^{\mathscr{E}}, \vec{a}^{\mathscr{F}_1}, \dots, \vec{a}^{\mathscr{F}_{\lfloor (d-2)/2 \rfloor}})$ with associated sheaves $\mathscr{E}_{\mathscr{M}}$ and, if $4 \le d \le 5$, $\mathscr{F}_{\mathscr{M}}$ as in Definition 7.6. There are isomorphisms $[\mathrm{U}^{\mathcal{R}}_{\mathscr{M}} / \mathrm{Aut}_{\mathscr{M}}] \simeq \mathscr{M}^{\mathcal{R}}$ and $[\mathrm{U}^{\mathcal{R},S_d}_{\mathscr{M}} / \mathrm{Aut}_{\mathscr{M}}] \simeq \mathscr{M}^{\mathcal{R},S_d}$.*

We now verify the relevant automorphism groups are special. Because later we will have to deal with an analogous construction over the dual numbers D, we include that setting in the following lemma as well.

Lemma 7.12. *For $\mathscr{V}$ any vector bundle on Y, for $Y = \mathbb{P}^1_k$ or $Y = D$, $\operatorname{Res}_{Y/k}(\operatorname{Aut}_{\mathscr{V}})$ and $\operatorname{Res}_{Y/k}(\ker(\det : \operatorname{Aut}_{\mathscr{E}} \to \mathbb{G}_m))$ are special and their classes are invertible in $K_0(\mathrm{Stacks}_k)$.*

When $Y = \mathbb{P}^1$ or $Y = D$, the three group schemes appearing in (4.1) in the cases $d = 3$, 4, and 5 are special. Further, the classes of these groups are invertible in $K_0(\mathrm{Stacks}_k)$.

Proof. We only explicate the proof in the case $Y = \mathbb{P}^1_k$, since the proof when $Y = D$ is analogous but simpler (noting that all vector bundles are trivial over D).

First we show that for any vector bundle $\mathscr{G}$ on $\mathbb{P}^1_k$, $\operatorname{Res}_{\mathbb{P}^1_k/k}(\operatorname{Aut}_{\mathscr{G}})$ is special. The reason for this is as follows. Write $\mathscr{G} = \bigoplus_{i=1}^m \mathscr{O}_{\mathbb{P}^1_k}(a_i)^{n_i}$ with $a_1 \leq \cdots \leq a_m$. We can express $\operatorname{Res}_{\mathbb{P}^1_k/k}(\operatorname{Aut}_{\mathscr{G}}) \simeq \prod_i \mathrm{GL}_{n_i} \ltimes \prod_{i<j} V_{ij}$ where V_{ij} is the vector group $V_{ij} = \operatorname{Res}_{\mathbb{P}^1_k/k}(\operatorname{Hom}(\mathscr{O}_{\mathbb{P}^1_k}(a_i)^{n_i}, \mathscr{O}_{\mathbb{P}^1_k}(a_j)^{n_j})) \simeq \mathbb{G}_a^{(a_j - a_i + 1)n_i n_j}$. It will also be useful to note that $\ker(\det) : \operatorname{Res}_{\mathbb{P}^1_k/k}(\operatorname{Aut}_{\mathscr{G}}) \to \operatorname{Res}_{\mathbb{P}^1_k/k}(\mathbb{G}_m)$ is special, since it can be expressed as an extension of a power of $\mathbb{G}_m$ by $\prod_i \mathrm{SL}_{n_i} \ltimes \prod_{i<j} V_{ij}$, both of which are special. These statements imply the first part of the lemma.

We now check the groups $\operatorname{Aut}^{\mathbb{P}^1_k/k}_{\mathscr{E},\mathscr{F}_\bullet}$ are special when $d = 3$, 4, and 5. The above observations immediately implies the claim when $d = 3$. To deal with the cases $d = 4$ and $d = 5$, we use Lemma 4.4. In both cases, the composition coming from Lemma 4.4 $\operatorname{Aut}^{\mathbb{P}^1_k/k}_{\mathscr{E},\mathscr{F}} \to \operatorname{Res}_{\mathbb{P}^1_k/k}(\operatorname{Aut}_{\mathscr{E}/\mathbb{P}^1_k}) \times \operatorname{Res}_{\mathbb{P}^1_k/k}(\operatorname{Aut}_{\mathscr{F}/\mathbb{P}^1_k}) \to \operatorname{Res}_{\mathbb{P}^1_k/k}(\operatorname{Aut}_{\mathscr{E}/\mathbb{P}^1_k})$ is surjective. From the description in Lemma 4.4, the kernel of this composition is identified with $\ker(\det) : \operatorname{Res}_{\mathbb{P}^1_k/k}(\operatorname{Aut}_{\mathscr{F}/\mathbb{P}^1_k}) \to \mathbb{G}_m$. As mentioned above, this is special, and so $\operatorname{Aut}^{\mathbb{P}^1_k/k}_{\mathscr{E},\mathscr{F}_\bullet}$ is an extension of special group schemes, hence special.

By the above explicit description of $\operatorname{Aut}^{\mathbb{P}^1_k/k}_{\mathscr{E},\mathscr{F}_\bullet}$ in terms of classes of special linear groups, general linear groups, and vector groups, we conclude that $\operatorname{Aut}^{\mathbb{P}^1_k/k}_{\mathscr{E},\mathscr{F}_\bullet}$ has class which is a product of powers of $\mathbb{L}$, and expressions of the form $\mathbb{L}^s - 1$ for varying s. Therefore, $\operatorname{Aut}^{\mathbb{P}^1_k/k}_{\mathscr{E},\mathscr{F}_\bullet}$ is invertible in $K_0(\mathrm{Stacks}_k)$. □

8. Computing the local classes

The goal of this section is to compute the classes of sections over the dual numbers in Theorem 8.9. These classes can be thought of as describing the 'probability' that a curve is smooth at a point and has a certain ramification profile. We will then use these classes to sieve for smoothness and ramification conditions by employing the work of Bilu and Howe [BH21] in Proposition 9.10. The condition of smoothness can be rephrased as a local condition over an infinitesimal neighborhood of the point in $\mathbb{P}^1$. We will first prove Theorem 8.3 which computes this 'probability' for abstract covers, and from this deduce Theorem 8.9, which computes this 'probability' for sections of $\mathscr{H}(\mathscr{E}, \mathscr{F}_\bullet)$. Theorem 8.3 can be thought of as a motivic analog of Bhargava's mass formulas for counting local fields [Bha07], though we note that the interesting part of [Bha07] is when there is wild ramification, and our hypothesis eliminates that possibility. On the other hand, it is still interesting to upgrade even the (much easier) tame mass formula to a motivic statement.

The idea for computing these local classes seems one of the main new insights of this paper. In the arithmetic analogs of this work, one is able to directly count the number of sections over $\mathbb{Z}/p^2\mathbb{Z}$, see [BST13, Lemma 18] for the degree 3 case, [Bha04, Lemma 23] for the degree 4 case, and [Bha08, Lemma 20] for the degree 5 case. In the Grothendieck ring, when working over infinite fields, there are infinitely many sections, and so to determine the relevant class, direct counting is no longer possible. We relate computing the classes of these sections to computing the classes of the classifying stacks of abstract automorphism groups of the corresponding schemes. These classes can, in turn, be computed using stacky symmetric powers Symm^n (see Definition 8.15) and the class of BS_n. An observation which is the key to the proof of Theorem 8.9 is that for G a group scheme, we have an isomorphism of stacks $\mathrm{Symm}^n(BG) \simeq B(G \wr S_n)$.

Throughout this section, we fix $d \in \mathbb{Z}_{\geq 1}$ an integer and let k be a field with $\mathrm{char}(k) \nmid d!$. For later explicit calculations, it will be convenient to work with the following explicit scheme $X_{(R)}$ over D which has ramification profile $R = (r_1^{t_1}, \ldots, r_n^{t_n})$, with $\sum_{i=1}^n r_i t_i = d$, over the closed point of D. Define

$$X_{(R)} := \coprod_{i=1}^{n} \left(\coprod_{j=1}^{t_i} \operatorname{Spec} k[x, \varepsilon]/(x^{r_i} - \varepsilon, \varepsilon^2) \right). \tag{8.1}$$

Thus, $X_{(R)}$ is a disjoint union of curvilinear schemes flat over the dual numbers, which have degrees over the dual numbers corresponding to elements of the partition. In particular, the total degree of $X_{(R)}$ over the dual numbers is d. We use the parentheses around R in $X_{(R)}$ to distinguish it from the base change of X to R.

Recall we defined Covers_d prior to Definition 4.7 as the algebraic stack parameterizing degree d finite locally free covers over a base field k.

Definition 8.1. We let $\mathscr{X}_{R,d} \subset \mathrm{Res}_{D/k}(\mathrm{Covers}_d \times_{\operatorname{Spec} k} D)$ denote the residual gerbe at the k-point of $\mathrm{Res}_{D/k}(\mathrm{Covers}_d \times_{\operatorname{Spec} k} D)$ corresponding to the D-point of Covers_d given by $X_{(R)}$.

Remark 8.2. Since we have an induced monomorphism $B(\mathrm{Res}_{D/k}(\mathrm{Aut}_{X_{(R)}/D})) \to \mathrm{Covers}_d$ and an epimorphism $\operatorname{Spec} k \to B(\mathrm{Res}_{D/k}(\mathrm{Aut}_{X_{(R)}/D}))$, it follows that $\mathscr{X}_{R,d}$ is equivalent to $B(\mathrm{Res}_{D/k}(\mathrm{Aut}_{X_{(R)}/D}))$ from the universal property for residual gerbes.

Our main result of this section is to compute the class of $\mathscr{X}_{R,d}$ in $K_0(\mathrm{Stacks}_k)$, and we complete the proof at the end of the section in § 8.17.

Theorem 8.3. *Let R be a ramification profile which is a partition of d. Let $r(R)$ be the ramification order associated to the ramification profile R, as defined in Definition 7.1. Then, for k a field with $\mathrm{char}(k) \nmid d!$, we have*

$$\{\mathscr{X}_{R,d}\} = \mathbb{L}^{-r(R)}$$

in $K_0(\mathrm{Stacks}_k)$.

The plan for the rest of the section is to first use Theorem 8.3 to deduce the local condition for a section of $\mathscr{H}(\mathscr{E}, \mathscr{F}_\bullet)$ to be smooth in Theorem 8.9. Following this, we devote the remainder of the section to proving Theorem 8.3. The main idea is to directly compute the automorphism group of $X_{(R)}$ in terms of its combinatorial data starting in § 8.10 and culminating in Corollary 8.12. Using this, we will then be able to compute the class of the classifying stack of the resulting affine (but typically quite disconnected) group scheme in § 8.13. For this, we appeal to a result of Ekedahl on stacky symmetric powers and another result of Ekedahl showing $\{BS_d\} = 1$. We complete the proof of Theorem 8.3 in § 8.17.

Remark 8.4. With some additional work, one can also prove a variant of Theorem 8.3 which computes the class of the locally closed subscheme $\mathscr{Z}_{R,d}$ of the Hilbert scheme $\mathrm{Res}_{D/k}(\mathrm{Hilb}^d_{\mathbb{P}^{d-2}_D/D})$ parameterizing curvilinear nondegenerate subschemes with ramification profile R so that on any geometric fiber, no degree-$(d-1)$ subscheme is contained in a hyperplane. One can show, $\{\mathscr{Z}_{R,d}\} = \{\mathrm{PGL}_{d-1}\}\mathbb{L}^{\dim \mathrm{PGL}_{d-1} - r(R)}$. Note there is some subtlety in verifying this because this Hilbert scheme is naturally a $\mathrm{Res}_{D/k}(\mathrm{PGL}_{d-1})$ torsor over Covers_d, and PGL_{d-1} is not a special

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group. Nevertheless, one may prove this by 'linearizing the action' so as to construct this as a quotient of a $\operatorname{Res}_{D/k}(\mathrm{GL}_{d-1})$ torsor by $\operatorname{Res}_{D/k}(\mathbb{G}_m)$, both of which are special.

8.5 Using Theorem 8.3 to compute smooth sections

Before proving Theorem 8.3, we will see how it can be used to determine local conditions for a section in a given Casnati–Ekedahl stratum to be smooth. In order to apply Theorem 8.3 to our problem of computing the classes of Hurwitz stacks we want to relate it to sections of the sheaf $\mathscr{H}(\mathscr{E},\mathscr{F}_\bullet)$ on D (for $\mathscr{E}$ and $\mathscr{F}_\bullet$ trivial sheaves on D of appropriate ranks as in Notation 3.8, depending on d with $3 \le d \le 5$). For this we need a generalization of Proposition 4.8 where we take a Weil restriction from the dual numbers. More precisely, for $3 \le d \le 5$, the map $\mu_d : \mathrm{U}_d \to \mathrm{Covers}_d$ defined in Definition 4.7 induces a map $\operatorname{Res}_{D/k}(\mu_d) : \operatorname{Res}_{D/k}((\mathrm{U}_d)_D) \to \operatorname{Res}_{D/k}((\mathrm{Covers}_d)_D)$. Since μ_d is invariant for the action of $\mathrm{Aut}_{\mathscr{E},\mathscr{F}_\bullet}$ as in Definition 4.7, we obtain a map $\phi_d^{D/k} : [\operatorname{Res}_{D/k}((\mathrm{U}_d)_D)/\operatorname{Res}_{D/k}(\mathrm{Aut}_{\mathscr{E}|_D,\mathscr{F}_\bullet|_D})] \to \operatorname{Res}_{D/k}((\mathrm{Covers}_d)_D)$ induced by sending a section to its vanishing locus.

Lemma 8.6. *For $3 \le d \le 5$, the map $\phi_d^{D/k} : [\operatorname{Res}_{D/k}((\mathrm{U}_d)_D)/\operatorname{Res}_{D/k}(\mathrm{Aut}_{\mathscr{E}|_D,\mathscr{F}_\bullet|_D})] \to \operatorname{Res}_{D/k}((\mathrm{Covers}_d)_D)$ is an isomorphism.*

This is proven via a nearly identical argument to Proposition 4.8 and we omit the proof. The one minor difference one must note is that, in order to show $\phi_d^{D/k}$ is surjective, for any $T \to \operatorname{Spec} k$ and any vector bundle on $T \times_k D$, one may replace T by an open cover which trivializes the bundle.

Definition 8.7. Let $\mathscr{Y}_{R,d} \subset \operatorname{Res}_{D/k}((\mathrm{U}_d)_D)$ denote the preimage under the composition

$$\operatorname{Res}_{D/k}((\mathrm{U}_d)_D) \to [\operatorname{Res}_{D/k}((\mathrm{U}_d)_D)/\operatorname{Res}_{D/k}(\mathrm{Aut}_{\mathscr{E}|_D,\mathscr{F}_\bullet|_D})] \xrightarrow{\phi_d^{D/k}} \operatorname{Res}_{D/k}((\mathrm{Covers}_d)_D)$$

of $\mathscr{X}_{R,d} \subset \operatorname{Res}_{D/k}((\mathrm{Covers}_d)_D)$.

Remark 8.8. We will implicitly use the following geometric description of the residual gerbe $\mathscr{X}_{R,d}$ and its preimage $\mathscr{Y}_{R,d}$ in $\operatorname{Res}_{D/k}((\mathrm{U}_d)_D)$. As a fibered category, $\mathscr{X}_{R,d}$ has T points given by finite locally free degree d Gorenstein covers $Z \to T \times_k D$ satisfying the following properties:

(1) Z has ramification profile R over each geometric point $\operatorname{Spec} \kappa \to T_D$;
(2) Z is curvilinear in the sense that for each geometric point $\operatorname{Spec} \kappa \to T$, the resulting scheme $Z \times_T \operatorname{Spec} \kappa$ has 1-dimensional Zariski tangent space at each point.

Similarly, when $3 \le d \le 5$, we can describe $\mathscr{Y}_{R,d}$ as those sections $\eta \in \operatorname{Res}_{D/k}((\mathrm{U}_d)_D)(T)$ for which the associated degree d cover of $T \times_k D$, $\Psi_d(\eta)$ (as defined in § 3.11) has the above properties. We note that $\mathscr{Y}_{R,d}$ is a locally closed subscheme of $\operatorname{Res}_{D/k}((\mathrm{U}_d)_D)$ since the same holds for the residual gerbe $\mathscr{X}_{R,d}$ in $\operatorname{Res}_{D/k}(\mathrm{Covers}_d)$ (see [Ryd11, Theorem B.2]). One may also deduce this is locally closed directly from the above functorial description.

By combining Theorem 8.3 with Lemma 8.6, we can easily deduce the following.

Theorem 8.9. *Let R be a ramification profile which is a partition of d and let $\mathscr{Y}_{R,d}$ be the scheme defined in Definition 8.7 (with associated free sheaves $\mathscr{E}, \mathscr{F}_\bullet$ on D). Let $r(R)$ denote the ramification order associated to the ramification profile R, as defined in Definition 7.1. Then,*

$$\{\mathscr{Y}_{R,d}\} = \{\mathrm{Aut}_{\mathscr{E},\mathscr{F}_\bullet}^{D/k}\}\mathbb{L}^{-r(R)}.$$

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Proof of Theorem 8.9 assuming Theorem 8.3. Using Lemma 8.6 and Remark 8.2,

$$\{[\mathscr{Y}_{R,d}/\operatorname{Aut}^{D/k}_{\mathscr{E},\mathscr{F}_\bullet}]\} = \{B(\operatorname{Res}_{D/k}(\operatorname{Aut}_{X_{(R)}/D}))\} = \mathscr{X}_{R,d}.$$

Since $\operatorname{Aut}^{D/k}_{\mathscr{E},\mathscr{F}_\bullet}$ is special and has invertible class in $K_0(\mathrm{Stacks}_k)$ by Lemma 7.12, [Eke09a, Propositions 1.4(i), 1.1(ix)] implies that

$$\{[\mathscr{Y}_{R,d}/\operatorname{Aut}^{D/k}_{\mathscr{E},\mathscr{F}_\bullet}]\} = \frac{\{\mathscr{Y}_{R,d}\}}{\{\operatorname{Aut}^{D/k}_{\mathscr{E},\mathscr{F}_\bullet}\}}.$$

Then, by Theorem 8.3,

$$\{\mathscr{Y}_{R,d}\} = \{\operatorname{Aut}^{D/k}_{\mathscr{E},\mathscr{F}_\bullet}\} \cdot \{\mathscr{X}_{R,d}\} = \{\operatorname{Aut}^{D/k}_{\mathscr{E},\mathscr{F}_\bullet}\} \cdot \mathbb{L}^{-r(R)}. \tag{8.2}$$

□

8.10 Computing the algebraic group $\operatorname{Res}_{D/k}(\operatorname{Aut}_{X_{(R)}/D})$

Let $X_{(R)}$ denote the scheme as defined in (8.1). Our next goal is to compute the group scheme $\operatorname{Res}_{D/k}(\operatorname{Aut}_{X_{(R)}/D})$, which we will carry out in Corollary 8.12. In order to do so, we first deal with the case that $X_{(R)}$ is connected.

LEMMA 8.10. *Let $d \in \mathbb{Z}_{\geq 1}$, and let k be a field with $\operatorname{char}(k) \nmid d!$. Let $W := \operatorname{Spec} k[y,\varepsilon]/(\varepsilon^2, \varepsilon - y^d)$. For $\operatorname{Aut}_{W/D}$ the automorphism scheme of W over D, we have $\operatorname{Res}_{D/k}(\operatorname{Aut}_{W/D}) \simeq \mathbb{G}_a^{d-1} \rtimes \mu_d$, explicitly given by $\alpha \in \mu_d$ sending $y \mapsto \alpha y$ and $(a_1, \ldots, a_{d-1}) \in \mathbb{G}_a^{d-1}$ sending $y \mapsto y + \sum_{i=1}^{d-1} a_i y^{d+i}$.*

Proof. For T a k algebra, a functorial T point of $\operatorname{Res}_{D/k}(\operatorname{Aut}_{W/D})$ corresponds (upon taking global sections) to an isomorphism of T algebras

$$\phi : T[y,\varepsilon]/(\varepsilon^2, \varepsilon - y^d) \simeq T[y,\varepsilon]/(\varepsilon^2, \varepsilon - y^d)$$

over $T[\varepsilon]/(\varepsilon^2)$. Such an automorphism is uniquely determined by where it sends y. To conclude the proof, it suffices to verify that any such ϕ is of the form $y \mapsto \alpha y + \sum_{i=1}^{d-1} a_i y^{d+i}$ for $\alpha \in \mu_d(T)$ and $a_i \in \mathbb{G}_a(T)$, and conversely that any map of this form determines an automorphism.

Let $\phi_{\alpha,a_1,\ldots,a_{d-1}}$ denote the map of T algebras sending $y \mapsto \alpha y + \sum_{i=1}^{d-1} a_i y^{d+i}$ as above. Under the isomorphism $T[y,\varepsilon]/(\varepsilon^2, \varepsilon - y^d) \simeq T[y]/y^{2d}$, any automorphism ϕ must induce an isomorphism on cotangent spaces, and hence send y to some polynomial $p_\phi(y) = b_1 y + b_2 y^2 + \cdots + b_{2d-1} y^{2d-1}$, with $b_1 \neq 0$ and $b_i \in T$. The condition that ϕ determines a map of $T[\varepsilon]/(\varepsilon^2)$ algebras precisely corresponds to $y^d = p_\phi(y)^d$. Comparing the coefficients of y^d in this equation implies $b_1 \in \mu_d(T)$. Since $\operatorname{char}(k) \nmid d!$, comparing the coefficients of $y^{d+1}, \ldots, y^{2d-1}$ in the equation $y^d = p_\phi(y)^d$ implies $b_2 = b_3 = \cdots = b_d = 0$. However, the coefficients $b_{d+1}, \ldots, b_{2d-1}$ can be arbitrary and $y^d = p_\phi(y)^d$ will be satisfied. So, any automorphism ϕ must be of the form $\phi_{\alpha,a_1,\ldots,a_{d-1}}$ (where we take $a_i = b_{d+i}$ in the above notation).

To see any map $\phi_{\alpha,a_1,\ldots,a_{d-1}}$ determines an automorphism of T algebras, note first that it is well defined, because $(\alpha y + \sum_{i=1}^{d-1} a_i y^{d+i})^d = y^d$, using that $y^{2d} = 0$. It is an automorphism as its inverse is explicitly given by $\phi_{(\alpha^{-1}, -\alpha^{-2}a_1, -\alpha^{-3}a_2, \ldots, -\alpha^{-d}a_{d-1})}$ □

COROLLARY 8.12. *Choose a partition $(r_1^{t_1}, \ldots, r_n^{t_n})$ of d, i.e. $d = \sum_{i=1}^n t_i \cdot r_i$. For $i = 1, \ldots, n$, let $W_i := \operatorname{Spec} \prod_{j=1}^{t_i} k[y,\varepsilon]/(\varepsilon^2, \varepsilon - y^{r_i})$. Let $W := \coprod_{i=1}^n W_i$, so that $W \simeq X_{(R)}$ when R is the ramification profile associated to the above partition. We have an isomorphism $\operatorname{Res}_{D/k}(\operatorname{Aut}_{W/D}) \simeq \prod_{i=1}^n (\mathbb{G}_a^{r_i-1} \rtimes \mu_{r_i}) \wr S_{t_i}$, where each $\mathbb{G}_a^{r_i-1} \rtimes \mu_{r_i}$ is explicitly realized acting on each component of W_i as in Lemma 8.11, and the action of the wreath product with S_{t_i} is obtained by permuting the t_i components of W_i.*

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Proof. To compute the automorphism group of W, first observe that any automorphism must permute all connected components of a fixed degree, and therefore $\mathrm{Aut}_{W/D} = \prod_{i=1}^n \mathrm{Aut}_{W_i/D}$ and, consequently,

$$\mathrm{Res}_{D/k}(\mathrm{Aut}_{W/D}) = \mathrm{Res}_{D/k}\left(\prod_{i=1}^n \mathrm{Aut}_{W_i/D}\right) = \prod_{i=1}^n \mathrm{Res}_{D/k}(\mathrm{Aut}_{W_i/D}).$$

It therefore suffices to show $\mathrm{Res}_{D/k}(\mathrm{Aut}_{W_i/D}) \simeq (\mathbb{G}_a^{r_i-1} \rtimes \mu_{r_i}) \wr S_{t_i}$. As all connected components of W_i are isomorphic, any automorphism is realized as the composition of an automorphism preserving each connected component, followed by some permutation of the connected components. Since there are t_i connected components, the group of permutations of the components is the symmetric group S_{t_i}, while for Z_i a connected component of W_i, we established $\mathrm{Res}_{D/k}\,\mathrm{Aut}_{Z_i/D} \simeq \mathbb{G}_a^{r_i-1} \rtimes \mu_{r_i}$ in Lemma 8.11. It follows that

$$\mathrm{Res}_{D/k}(\mathrm{Aut}_{W_i/D}) = (\mathrm{Res}_{D/k}(\mathrm{Aut}_{Z_i/D})) \wr S_{t_i} = (\mathbb{G}_a^{r_i-1} \rtimes \mu_{r_i}) \wr S_{t_i}. \qquad \square$$

8.13 Computing $\{B\,\mathrm{Res}_{D/k}(\mathrm{Aut}_{X_{(R)}/D})\}$

Our next goal is to prove Theorem 8.3 by computing the class of $\{B\,\mathrm{Res}_{D/k}(\mathrm{Aut}_{X_{(R)}/D})\}$ in $K_0(\mathrm{Stacks}_k)$, which we carry out at the end of this section in § 8.17. Of course, we will use our computation of $\mathrm{Res}_{D/k}(\mathrm{Aut}_{X_{(R)}/D})$ from Corollary 8.12. In order to set up our computation we need the following lemma.

Lemma 8.14. *For x and y positive integers, $\{B(\mathbb{G}_a^x \rtimes \mu_y)\} = \{B(\mathbb{G}_a^x)\} = \mathbb{L}^{-x}$, where μ_y acts on $\mathbb{G}_a^x$ by the scaling action $(\alpha, (a_1, \dots, a_x)) \mapsto (\alpha a_1, \dots, \alpha a_x)$.*

Proof. Indeed, we have an inclusion

$$(\mathbb{G}_a^x \rtimes \mu_y) \hookrightarrow (\mathbb{G}_a^x \rtimes \mathbb{G}_m),$$

where the semidirect product $\mathbb{G}_a^x \rtimes \mathbb{G}_m$ is defined similarly to that in Lemma 8.11 so that $\mathbb{G}_m$ acts on $\mathbb{G}_a^x$ by

$$\begin{aligned} \mathbb{G}_m \times \mathbb{G}_a^x &\to \mathbb{G}_a^x \\ (\alpha, (a_1, \dots, a_x)) &\mapsto (\alpha a_1, \dots, \alpha a_x). \end{aligned}$$

The natural inclusion $\mu_y \to \mathbb{G}_m$ then respects the constructed group structures. For simplicity of notation, temporarily define $K := \mathbb{G}_a^x \rtimes \mu_y$ and $L := \mathbb{G}_a^x \rtimes \mathbb{G}_m$.

Since L is special, and special groups are closed under extensions, it follows from [Eke09a, Proposition 1.1(ix)] that $\{BK\} = \{L/K\}\{BL\}$. However, since $\mathbb{G}_a^x$ is a normal subgroup of both L and K, the quotient L/K is identified with

$$L/K \simeq \frac{L/\mathbb{G}_a^x}{K/\mathbb{G}_a^x} \simeq \mathbb{G}_m/\mu_y \simeq \mathbb{G}_m.$$

Since L is special, using [Eke09a, Proposition 1.4(i)] and [Eke09a, Proposition 1.1(v)], we obtain that $\{BL\} = 1/\{L\} = \mathbb{L}^{-x} 1/(\mathbb{L} - 1)$. Therefore,

$$\{BK\} = \{L/K\}\{BL\} = (\mathbb{L} - 1)\mathbb{L}^{-x}\frac{1}{\mathbb{L} - 1} = \mathbb{L}^{-x} = \{B(\mathbb{G}_a^x)\},$$

using again that $L = \mathbb{G}_a^x \rtimes \mathbb{G}_m$ is special. $\square$

Using Lemma 8.14, we next compute the class of $B\,\mathrm{Res}_{D/k}(\mathrm{Aut}_{X_{(R)}/D})$ in the case that the partition R has a single part. To continue our computation, we need the notion of stacky symmetric powers.

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Definition 8.15. For $\mathscr{X}$ a stack, define the *stacky symmetric power* $\operatorname{Symm}^n \mathscr{X} := [\mathscr{X}^n/S_n]$ (where $[\mathscr{X}^n/S_n]$ denotes the stack quotient for S_n acting on $\mathscr{X}^n$ by permuting the factors).

The key input to our next computation will be that taking the stacky symmetric powers is a well-defined operation on the Grothendieck ring of stacks by [Eke09b, Proposition 2.5].

Lemma 8.16. *For integers s and t,*

$$B((\mathbb{G}_a^{s-1} \rtimes \mu_s) \wr S_t) = \mathbb{L}^{-(s-1)\cdot t} \in K_0(\mathrm{Stacks}_k).$$

Proof. By definition,

$$\{B((\mathbb{G}_a^{s-1} \rtimes \mu_s) \wr S_t)\} = \{B(\mathbb{G}_a^{s-1} \rtimes \mu_s) \wr B(S_t)\} = \{\operatorname{Symm}^t(B(\mathbb{G}_a^{s-1} \rtimes \mu_s))\}.$$

Having computed $\{B(\mathbb{G}_a^{s-1} \rtimes \mu_s)\} = \mathbb{L}^{-s+1}$ in Lemma 8.14, we therefore wish to next compute $\{\operatorname{Symm}^t(B(\mathbb{G}_a^{s-1} \rtimes \mu_s))\}$. Since $\{\operatorname{Symm}^n \mathscr{X}\}$ only depends on $\{\mathscr{X}\}$ by [Eke09b, Proposition 2.5], and we have shown $\{B(\mathbb{G}_a^{s-1} \rtimes \mu_s)\} = \{B(\mathbb{G}_a^{s-1})\}$ in Lemma 8.14 it follows that

$$\{\operatorname{Symm}^t(B(\mathbb{G}_a^{s-1} \rtimes \mu_s))\} = \{\operatorname{Symm}^t(B\mathbb{G}_a^{s-1})\}.$$

Next, by [Eke09b, Lemma 2.4], we have

$$\begin{aligned}\{\operatorname{Symm}^t(\mathbb{A}^{s-1} \times B\mathbb{G}_a^{s-1})\} &= \{\operatorname{Symm}^t(B\mathbb{G}_a^{s-1}) \times \mathbb{A}^{(s-1)t}\} \\ &= \{\operatorname{Symm}^t(B\mathbb{G}_a^{s-1})\} \cdot \mathbb{L}^{(s-1)t}.\end{aligned}$$

However, since $\{\mathbb{A}^{s-1} \times B\mathbb{G}_a^{s-1}\} = 1$, and $\{\operatorname{Symm}^n \mathscr{X}\}$ only depends on $\{\mathscr{X}\}$ by [Eke09b, Proposition 2.5], we obtain

$$\begin{aligned}\{\operatorname{Symm}^t(B\mathbb{G}_a^{s-1})\} &= \{\operatorname{Symm}^t(\mathbb{A}^{s-1} \times B\mathbb{G}_a^{s-1})\}\mathbb{L}^{-(s-1)t} \\ &= \{\operatorname{Symm}^t(1)\}\mathbb{L}^{-(s-1)t} \\ &= \{BS_t\}\mathbb{L}^{-(s-1)t} \\ &= \mathbb{L}^{-(s-1)t}.\end{aligned}$$

For the last step, we used Theorem A.1, which says that $\{BS_t\} = 1$. □

8.17 Completing the calculation of the local class

We now complete the proof of Theorem 8.3.

Proof of Theorem 8.3. By Remark 8.2 $\{\mathscr{X}_{R,d}\} = \{B(\operatorname{Res}_{D/k}(\operatorname{Aut}_{X_{(R)}/D}))\}$, and so we will now compute the latter. By Corollary 8.12 we equate

$$\operatorname{Res}_{D/k}(\operatorname{Aut}_{X_{(R)}/D}) = \prod_{i=1}^{n} (\mathbb{G}_a^{r_i-1} \rtimes \mu_{r_i}) \wr S_{t_i}.$$

Factoring this as a product, it suffices to compute the class of $B(\mathbb{G}_a^{r_i-1} \rtimes \mu_{r_i}) \wr S_{t_i}$. Using that $\sum_{i=1}^{n}(r_i-1)t_i = r(R)$, the result follows from Lemma 8.16. □

9. Codimension bounds for the main result

In this section, we establish various bounds on the codimension or certain bad loci we will want to weed out when computing the class of Hurwitz stacks in the Grothendieck ring.

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9.1 Weeding out the strata of unexpected codimension

In order to compute the classes of Hurwitz stacks, we will stratify the Hurwitz stacks by Casnati–Ekedahl strata. The following lemma computes the codimension of these loci in the Hurwitz stack. For the following statement, recall our notation for $\mathscr{H}(\mathscr{E}, \mathscr{F}_\bullet)$ from (3.6).

Lemma 9.2. *Fix some d, resolution data $(\mathscr{E}, \mathscr{F}_\bullet)$, and define g by $\deg \det \mathscr{E} = g + d - 1$. Letting $\mathscr{H} := \mathscr{H}(\mathscr{E}, \mathscr{F}_\bullet)$, the codimension of $\mathscr{M} := \mathscr{M}(\mathscr{E}, \mathscr{F}_\bullet)$ in $\overline{\mathrm{Hur}}_{d,g,k}$, assuming it is nonempty, is*

$$\begin{cases} h^1(\mathbb{P}^1_k, \operatorname{End} \mathscr{E}) & \text{if } d = 3, \\ h^1(\mathbb{P}^1_k, \operatorname{End} \mathscr{E}) + h^1(\mathbb{P}^1_k, \operatorname{End} \mathscr{F}) - h^1(\mathbb{P}^1_k, \mathscr{H}) & \text{if } d = 4 \text{ or } 5. \end{cases}$$

Proof. The case $d = 3$ is proven in [Pat13, Proposition 1.4]. Therefore, for the remainder of the proof, we assume $d = 4$ or $d = 5$, in which case we simply write $(\mathscr{E}, \mathscr{F})$ in place of $(\mathscr{E}, \mathscr{F}_\bullet)$. Let $\mathscr{M}^\circ := \mathscr{M}(\mathscr{E}^\circ, \mathscr{F}^\circ)$ denote the dense open stratum, corresponding to vector bundles $\mathscr{E}^\circ$ and $\mathscr{F}^\circ$ which are balanced (see § 6.9), subject to the conditions that $\det \mathscr{F}^\circ \simeq \det \mathscr{E}^\circ$ when $d = 4$ and $\det \mathscr{F}^\circ \simeq \det(\mathscr{E}^\circ)^{\otimes 2}$ when $d = 5$, coming from coming from Theorems 3.14 and 3.16. Similarly, let $\mathscr{H}^\circ := \mathscr{H}(\mathscr{E}^\circ, \mathscr{F}^\circ)$.

We are looking to compute the codimension of $\mathscr{M}$ in $\overline{\mathrm{Hur}}_{d,g,k}$, or equivalently the difference of dimensions $\dim \mathscr{M}^\circ - \dim \mathscr{M}$. Using the description of $\mathscr{M}$ from Proposition 7.11 as a quotient of an open in the affine space associated to $H^0(\mathbb{P}^1_k, \mathscr{H})$ by $\operatorname{Aut}_{\mathscr{M}}$, it follows that the codimension of $\mathscr{M}$ in $\overline{\mathrm{Hur}}_{d,g,k}$ is $\dim \mathscr{M}^\circ - \dim \mathscr{M}$. Since $\dim \mathscr{M} = h^0(\mathbb{P}^1_k, \mathscr{H}) - \dim \operatorname{Aut}_{\mathscr{M}}$, we are looking to compute

$$\begin{aligned} &(h^0(\mathbb{P}^1_k, \mathscr{H}^\circ) - \dim \operatorname{Aut}_{\mathscr{M}^\circ}) - (h^0(\mathbb{P}^1_k, \mathscr{H}) - \dim \operatorname{Aut}_{\mathscr{M}}) \\ &\quad = (h^0(\mathbb{P}^1_k, \mathscr{H}^\circ) - h^0(\mathbb{P}^1_k, \mathscr{H})) + (\dim \operatorname{Aut}_{\mathscr{M}} - \dim \operatorname{Aut}_{\mathscr{M}^\circ}). \end{aligned} \tag{9.1}$$

We will first identify $\dim \operatorname{Aut}_{\mathscr{M}} - \dim \operatorname{Aut}_{\mathscr{M}^\circ}$ with $h^1(\mathbb{P}^1_k, \operatorname{End} \mathscr{E}) + h^1(\mathbb{P}^1_k, \operatorname{End} \mathscr{F})$. Second, we will show $(h^0(\mathbb{P}^1_k, \mathscr{H}^\circ) - h^0(\mathbb{P}^1_k, \mathscr{H}))$ agrees with $h^1(\mathbb{P}^1_k, \mathscr{H})$. Combining these with (9.1) will complete the proof.

To identify $\dim \operatorname{Aut}_{\mathscr{M}} - \dim \operatorname{Aut}_{\mathscr{M}^\circ}$, we may identify $\dim \operatorname{Aut}_{\mathscr{M}}$ with the dimension of the tangent space to $\operatorname{Aut}_{\mathscr{M}}$ at the identity, which is given by $H^0(\mathbb{P}^1_k, \operatorname{End} \mathscr{E}) \times H^0(\mathbb{P}^1_k, \operatorname{End} \mathscr{F})$. It is then enough to show that

$$h^0(\mathbb{P}^1_k, \operatorname{End} \mathscr{E}^\circ) - h^0(\mathbb{P}^1_k, \operatorname{End} \mathscr{E}) = h^1(\mathbb{P}^1_k, \operatorname{End} \mathscr{E})$$

and

$$h^0(\mathbb{P}^1_k, \operatorname{End} \mathscr{F}^\circ) - h^0(\mathbb{P}^1_k, \operatorname{End} \mathscr{F}) = h^1(\mathbb{P}^1_k, \operatorname{End} \mathscr{F}).$$

We focus on the case of $\mathscr{E}$, as the case of $\mathscr{F}$ is completely analogous. By Riemann Roch, since the degrees and ranks of $\mathscr{E}$ and $\mathscr{E}^\circ$ are the same, we find

$$h^0(\mathbb{P}^1_k, \operatorname{End} \mathscr{E}^\circ) - h^0(\mathbb{P}^1_k, \operatorname{End} \mathscr{E}) = h^1(\mathbb{P}^1_k, \operatorname{End} \mathscr{E}) - h^1(\mathbb{P}^1_k, \operatorname{End} \mathscr{E}^\circ).$$

Because $\mathscr{E}^\circ$ is balanced, we find $h^1(\mathbb{P}^1_k, \operatorname{End} \mathscr{E}^\circ) = 0$.

To complete the proof, it only remains to show $(h^0(\mathbb{P}^1_k, \mathscr{H}^\circ) - h^0(\mathbb{P}^1_k, \mathscr{H}))$ agrees with $h^1(\mathbb{P}^1_k, \mathscr{H})$. Similarly to our computation above for $h^0(\mathbb{P}^1_k, \operatorname{End} \mathscr{E}^\circ) - h^0(\mathbb{P}^1_k, \operatorname{End} \mathscr{E})$, we find

$$h^0(\mathbb{P}^1_k, \mathscr{H}^\circ) - h^0(\mathbb{P}^1_k, \mathscr{H}) = h^1(\mathbb{P}^1_k, \mathscr{H}) - h^1(\mathbb{P}^1_k, \mathscr{H}^\circ)$$

by Riemann Roch. To complete the proof, we only need verify $h^1(\mathbb{P}^1_k, \mathscr{H}^\circ) = 0$. Indeed, by writing out $\mathscr{E}^\circ$ and $\mathscr{F}^\circ$ as sums of line bundles on $\mathbb{P}^1_k$, and using the relation between $\det \mathscr{E}$ and $\det \mathscr{F}$, the balancedness of $\mathscr{E}^\circ$ and $\mathscr{F}^\circ$ implies $h^1(\mathbb{P}^1_k, \mathscr{H}^\circ) = 0$. □

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With the above lemma in hand, we may note that the codimension of the vector bundles $(\mathscr{E}, \mathscr{F}_\bullet)$ in the stack of vector bundles is $H^1(\mathbb{P}^1, \mathrm{End}(\mathscr{E})) + H^1(\mathbb{P}^1, \mathrm{End}(\mathscr{F}))$ (the latter interpreted as 0 when $d = 3$).

Remark 9.3. We will think of a Casnati–Ekedahl stratum as having the 'expected codimension' when its codimension in the Hurwitz stack agrees with the corresponding codimension of $(\mathscr{E}, \mathscr{F}_\bullet)$ in the stack of tuples of vector bundles. Using Lemma 9.2 and its proof, a stratum is of the expected codimension precisely when $H^1(\mathbb{P}^1, \mathscr{H}(\mathscr{E}, \mathscr{F}_\bullet)) = 0$.

The next lemma bounds the codimension of strata not having the expected codimension.

LEMMA 9.4. *Suppose $3 \leq d \leq 5$ and $\mathscr{M} := \mathscr{M}(\mathscr{E}, \mathscr{F}_\bullet)$ is a Casnati–Ekedahl stratum containing a curve $C \to \mathbb{P}^1$ which does not factor through some intermediate cover $C' \to \mathbb{P}^1$ of positive degree. If $H^1(\mathbb{P}^1, \mathscr{H}(\mathscr{E}, \mathscr{F}_\bullet)) \neq 0$ or $H^0(\mathbb{P}^1, \mathscr{E}^\vee) \neq 0$, $\mathrm{codim}_{\overline{\mathrm{Hur}}_{d,g,k}} \mathscr{M} \geq (g + d - 1)/d - 4^{d-3}$.*

Proof. If $H^1(\mathbb{P}^1, \mathscr{H}(\mathscr{E}, \mathscr{F}_\bullet)) \neq 0$, then we have $\mathrm{codim}_{\overline{\mathrm{Hur}}_{d,g,k}} \mathscr{M} \geq (g + d - 1)/d - 4^{d-3}$ by [CL24, Lemma 5.8] when $d = 4$, [CL24, Lemma 5.12] when $d = 5$, and [Mir85, (6.2)] for the cases that $d = 3$ (see also [BV12, Proposition 2.2]). It therefore remains to show that if $H^1(\mathbb{P}^1, \mathscr{H}(\mathscr{E}, \mathscr{F}_\bullet)) = 0$ but $H^0(\mathbb{P}^1, \mathscr{E}^\vee) \neq 0$, we will also have $\mathrm{codim}_{\overline{\mathrm{Hur}}_{d,g,k}} \mathscr{M} \geq (g + d - 1)/d - 4^{d-3}$. In the case $H^1(\mathbb{P}^1, \mathscr{H}(\mathscr{E}, \mathscr{F}_\bullet)) = 0$, the codimension of $\mathscr{M}$ in $\overline{\mathrm{Hur}}_{d,g,k}$ is simply $h^1(\mathbb{P}^1, \mathrm{End}(\mathscr{E})) + h^1(\mathbb{P}^1, \mathrm{End}(\mathscr{F}_\bullet))$, by Lemma 9.2. We will only have $H^0(\mathbb{P}^1, \mathscr{E}^\vee) \neq 0$ when some summand of $\mathscr{E}$ is non-positive. Recall from Lemma 6.3 $\deg \mathscr{E} = g + d - 1$. Therefore, if $\mathscr{E} = \bigoplus_{i=1}^{d-1} \mathscr{O}(e_i)$ with $e_1 \leq 0$, then $\sum_{i=2}^{d-1} (e_i - e_1) \geq g + d - 1$ and, hence,

$$\begin{aligned} h^1(\mathbb{P}^1, \mathrm{End}(\mathscr{E})) &\geq h^1\left(\mathbb{P}^1, \oplus \bigoplus_{i=2}^{d-1} \mathscr{O}_{\mathbb{P}^1}(e_1 - e_i)\right) \\ &\geq g + d - 1 - (d - 2) \\ &= g + 1 \\ &> \frac{g + d - 1}{d} - 4^{d-3}. \end{aligned}$$

□

9.5 Weeding out covers with smaller Galois groups

In the next few results, culminating in Lemma 9.8, we establish bounds on the codimension of degree d covers of $\mathbb{P}^1$ whose Galois closure has Galois group G strictly contained in S_d.

For a group G and a base S, with $\#G$ invertible on S, we use $\mathrm{Hur}_{G,S}$ to denote the stack whose T-points are given by $(T, X, h : X \to T, f : X \to \mathbb{P}^1_T)$ where X is a scheme, h is a smooth proper relative curve, f is a finite locally free map of degree $\#G$ so that G acts on X over $\mathbb{P}^1_T$, together with an isomorphism $G \simeq \mathrm{Aut}\, f$. Note that $\mathrm{Hur}_{G,S}$ is an algebraic stack with an étale map to the configuration space of points in $\mathbb{P}^1$ given by taking the branch divisor, as follows from [Wew98, Theorem 4], (the key point of the construction being the algebraicity criterion in [Wew98, Theorem 1.3.3]). Upon specifying an embedding $G \subset S_d$ for some d, there is a natural map $\mathrm{Hur}_{G,S} \to \overline{\mathrm{Hur}}_{d,S}$ sending a given cover $(T, X, h : X \to T, f : X \to \mathbb{P}^1_T)$ to an associated cover $\coprod_{h \in G \backslash S_d} (hX)/S_{d-1} \to \mathbb{P}^1_T$ where we take the disjoint union over cosets of $G \backslash S_d$ and then quotienting the resulting S_d cover by S_{d-1}. The image of this map is a substack of $\overline{\mathrm{Hur}}_{d,S}$ whose geometric points parameterize degree d covers whose Galois group is G with the specified embedding $G \subset S_d$. We note that we could have alternatively constructed $\mathrm{Hur}_{G,S}$ directly, as mentioned in Remark 5.6, without appealing to [Wew98].

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Lemma 9.6. *Suppose $G \subset S_d$ is a subgroup not containing a transposition. Then the closure of the image* $(\mathrm{Hur}_{G,S} \to \overline{\mathrm{Hur}}_{d,S}) \cap \overline{\mathrm{Hur}}_{d,g,S}$ *has dimension at most $g-1+d$.*

Proof. By [Wew98, Theorem 4], if the image $(\mathrm{Hur}_{G,S} \to \overline{\mathrm{Hur}}_{d,S}) \cap \overline{\mathrm{Hur}}_{d,g,S}$ parameterizes curves with n branch points, it has dimension n. We therefore use n for the number of branch points. It is possible this image has multiple components, but because the Galois closure of a genus g degree d cover of $\mathbb{P}^1_k$ is a curve of bounded genus, there can only be finitely many components. We now fix one of these components and wish to show $n \leq g-1+d$.

Let $X \to \mathbb{P}^1$ be a degree d genus g cover corresponding to a point on this component with n branch points. If $G \subset S_d$ has no transpositions, the inertia at any point of $\mathbb{P}^1$, which is tame by assumption, does not act as a transposition. Therefore, the cover is not simply branched over that point, i.e. the ramification partition is not (1^d) or $(2, 1^{d-2})$. Hence, the fiber over that point has total ramification degree at least 2. It follows from Riemann–Hurwitz that $2g-2 \geq -2d+2n$ so $n \leq g-1+d$. □

Corollary 9.7. *Suppose $2 \leq d \leq 5$ and $G \subset S_d$ acts transitively on $\{1, \dots, d\}$ with G not isomorphic to D_4. Then, the image* $(\mathrm{Hur}_{G,S} \to \overline{\mathrm{Hur}}_{d,S}) \cap \overline{\mathrm{Hur}}_{d,g,S}$ *has dimension at most $g-1+d$.*

Proof. If $2 \leq d \leq 5$, we claim the only proper conjugacy class of subgroups $G \subset S_d$ acting transitively on $\{1, \dots, d\}$ and containing a transposition is $D_4 \subset S_4$, the dihedral group of order 8. Indeed, this claim follows by a straightforward check of all subgroups of S_d. The corollary then follows from Lemma 9.6 □

The next lemma shows that in any Casnati–Ekedahl stratum having the expected codimension (see Remark 9.3) the locus of non-S_d covers has high codimension in the Hurwitz stack.

Lemma 9.8. *Suppose $3 \leq d \leq 5$ and $\mathscr{M}(\mathscr{E}, \mathscr{F}_\bullet)$ is a Casnati–Ekedahl stratum with $H^1(\mathbb{P}^1, \mathscr{H}(\mathscr{E}, \mathscr{F}_\bullet)) = 0$. Suppose further that $[\mathrm{U}_{\mathscr{M}(\mathscr{E},\mathscr{F}_\bullet)}/\mathrm{Aut}_{\mathscr{M}(\mathscr{E},\mathscr{F}_\bullet)}]$ contains some geometrically connected cover $X \to \mathbb{P}^1_k$ whose Galois closure is not S_d. Then the codimension of this locus of covers in $\overline{\mathrm{Hur}}_{d,g,k}$ is at least $(g+3)/2$.*

Note that the space of D_4 covers is typically of codimension 2 in $\overline{\mathrm{Hur}}_{d,g,k}$, but these covers will typically lie in a Casnati–Ekedahl stratum with $H^1(\mathbb{P}^1, \mathscr{H}(\mathscr{E}, \mathscr{F}_\bullet)) \neq 0$.

Proof. The most difficult case is when $d = 4$ and the Galois closure is D_4, the dihedral group of order 8, this was verified in [CL24, Lemma 5.5]. Note here we are using that whenever the Galois group of a degree 4 cover is D_4, $C \to \mathbb{P}^1$ necessarily factors through an intermediate degree 2 cover.

It remains to verify that if we have a smooth geometrically connected curve $C \to \mathbb{P}^1$ whose Galois closure is not D_4, the codimension of such curves is at least $(g+3)/2$. The geometric connectedness condition guarantees that the action of G on $\{1, \dots, d\}$ is transitive. Note that the dimension of such a stratum is at most $g-1+d$ by Corollary 9.7, and hence also codimension $g-1+d$ in the $(2g+2d-2)$-dimensional stack $\mathrm{Hur}_{d,g,k}$. The lemma follows because $g-1+d > (g+3)/2$. □

9.9 Weeding out the singular sections

Our next goal is to show that for any given Casnati–Ekedahl stratum, the sections defining smooth curves can be expressed in terms of a fairly simple motivic Euler product, away from high codimension. This is, in some sense, the key input to our approach, and draws heavily on the

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work of [BH21] while also making use of our computations of classes associated to sections with given ramification profiles over the dual numbers from § 8. It will turn out that this codimension is the dominant term, in the sense that for large g, the codimension bound we obtain on these singular sections agrees with the codimension bound we find in our main result Theorem 10.5. At this point, it may be useful to recall notation for motivic Euler products from § 2.10.

PROPOSITION 9.10. *Let $3 \le d \le 5$, and let $\mathcal{R}$ be an allowable collection of ramification profiles of degree d. Suppose $s \ge 0$ and $\mathscr{M} := \mathscr{M}(\mathscr{E}, \mathscr{F}_\bullet)$ is a Casnati–Ekedahl stratum for which $\mathscr{H}(\mathscr{E}, \mathscr{F}_\bullet)(-s)$ is globally generated and each entry of $\vec{a}^{\mathscr{E}}$ is positive. Then,*

$$\{\mathrm{U}^{\mathcal{R}}_{\mathscr{M}}\} \equiv \mathbb{L}^{\dim \mathrm{U}^{\mathcal{R}}_{\mathscr{M}}} \prod_{x\in\mathbb{P}^1_k} \left(1 - \left(1 - \frac{(\sum_{R\in\mathcal{R}} \mathbb{L}^{-r(R)})\{\mathrm{Aut}_{\mathscr{M}|_D}\}}{\mathbb{L}^{h^0(D,\mathscr{H}(\mathscr{E}|_D,\mathscr{F}_\bullet|_D))}}\right) t\right)\Bigg|_{t=1} \tag{9.2}$$

modulo codimension $\lfloor (s+1)/2 \rfloor$ in $\widehat{\widetilde{K_0}}(\mathrm{Spaces}_k)$.

In the above product, the restriction to D is understood to take place at any subscheme $D \subset \mathbb{P}^1_k$, noting that $h^0(D, \mathscr{H}(\mathscr{E}|_D, \mathscr{F}_\bullet|_D))$ and $\{\mathrm{Aut}_{\mathscr{M}|_D}\}$ are independent of the choice of such D.

Proof. We first explain the final statement that $h^0(D, \mathscr{H}(\mathscr{E}|_D, \mathscr{F}_\bullet|_D))$ and $\{\mathrm{Aut}_{\mathscr{M}|_D}\}$ are independent of the choice of D. Indeed, $h^0(D, \mathscr{H}(\mathscr{E}|_D, \mathscr{F}_\bullet|_D))$ is independent of choice of $D \subset \mathbb{P}^1_k$ because it only depends on the rank of $\mathscr{H}(\mathscr{E}, \mathscr{F}_\bullet)$. Similarly, $\{\mathrm{Aut}_{\mathscr{M}|_D}\}$ only depends on d and the ranks of the sheaves $\mathscr{E}, \mathscr{F}_\bullet$, and not the specific choice of $D \subset \mathbb{P}^1_k$. We therefore focus on proving (9.2).

We will deduce (9.2) by applying [BH21, Theorem 9.3.1] with the local condition determined by the ramification profile R, as determined in Theorem 8.9, as we next explain. In particular, in applying [BH21, Theorem 9.3.1], we will take $r = 1, m = \lfloor (s+1)/2 \rfloor, M = 0$ in their notation.

In some more detail, we take $(f : X \to S, \mathcal{F}, \mathcal{L}, r, M)$ in [BH21, Theorem 9.3.1] to be $(\mathbb{P}^1_k \to \mathrm{Spec}\, k, \mathscr{H}(\mathscr{E}, \mathscr{F}_\bullet), \mathscr{O}_{\mathbb{P}^1}(1), 1, 0)$ and the constructible Taylor conditions T of [BH21, Theorem 9.3.1] which we will define in the next paragraph.

For $\mathscr{F}$ a locally free sheaf on some scheme X, we use $\mathcal{P}^1(\mathscr{F})$ to denote the first-order sheaf of principal parts. This is a locally free sheaf on X whose fiber over $x \in X$ can be identified with $H^0(X, \mathscr{F} \otimes \mathscr{O}_{X,x}/\mathfrak{m}^2_{X,x})$. Thus, in the case $X = \mathbb{P}^1_k$ is a curve and D is the copy of the dual numbers whose closed point maps to x, the fiber of $\mathcal{P}^1(\mathscr{F})$ at x is $H^0(\mathbb{P}^1_k, \mathscr{F}|_D)$. For a definition and standard background on bundles of principal parts, see [EH16, § 7.2]. Let T denote the constructible subset of $\mathrm{Spec}(\mathrm{Sym}^\bullet \mathcal{P}^1(\mathscr{H}(\mathscr{E}, \mathscr{F}_\bullet)))^\vee$ defined as follows: upon identifying the fiber of $\mathrm{Spec}(\mathrm{Sym}^\bullet \mathcal{P}^1(\mathscr{H}(\mathscr{E}, \mathscr{F}_\bullet)))^\vee$ at x with $H^0(\mathbb{P}^1_k, \mathscr{H}(\mathscr{E}, \mathscr{F}_\bullet)|_D)$, we take the subset given by those sections $\eta \in H^0(\mathbb{P}^1_k, \mathscr{H}(\mathscr{E}, \mathscr{F}_\bullet)|_D)$ so that $\Psi_d(\eta)$ defines a curvilinear scheme over D whose ramification profile lies in $\mathcal{R}$. Let T^c denote the complement of T in $\mathrm{Spec}(\mathrm{Sym}^\bullet \mathcal{P}^1(\mathscr{H}(\mathscr{E}, \mathscr{F}_\bullet)))^\vee$.

In order to apply [BH21, Theorem 9.3.1], we need to verify the above conditions are indeed admissible in the sense of [BH21, Definition 9.2.6]. Indeed, to see this, we need to check the Taylor conditions imposed by being smooth with ramification profile lying in $\mathcal{R}$ are the complement of a codimension $2 = 1 + \dim \mathbb{P}^1_k$ subset of the fiber of the first sheaf of principal parts associated to $\mathscr{H}(\mathscr{E}, \mathscr{F}_\bullet)$ over a field valued point of $\mathbb{P}^1_k$. First, one can verify directly (for example, by using an incidence correspondence) that those sections η for which $\Psi_d(\eta)$ are not curvilinear form a locus of codimension at least 2 in $\mathrm{Spec}(\mathrm{Sym}^\bullet H^0(D, \mathscr{H}(\mathscr{E}|_D, \mathscr{F}_\bullet|_D))^\vee)$. (Note that non-curvilinear sections also include sections with $\Psi_d(\eta)$ of positive dimension.) It remains to show those curvilinear sections having ramification profile not lying in $\mathcal{R}$ have codimension at least 2. This follows from knowledge of their class in the Grothendieck ring Theorem 8.9, which shows, in particular, the codimension of those sections having ramification profile R is $r(R)$. Since the

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only ramification profiles with $r(R) \leq 1$ are (1^d) and $(2, 1^{d-2})$, the claim follows from the first constraint in the definition of allowable, Definition 7.2.

We next use [BH21, Ex. 5.4.6] to determine the value of m appearing in [BH21, Theorem 9.3.1]. In place of the value D used in [BH21, Ex. 5.4.6], we use $-s$, since we are reserving D for the dual numbers. Otherwise following the notation of [BH21, Ex. 5.4.6], since $\mathscr{O}(1) = \mathcal{L}$ is very ample and $\mathcal{L}^{\otimes 0} \simeq \mathscr{O}_{\mathbb{P}^1}$ globally generated, we may take $A = 1$ and $B = 0$. It follows that (in their notation except that we use δ in place of d) there is a surjection $\mathscr{O}(s)^N \to \mathscr{F}$, and so $H^0(X, \mathscr{F} \otimes \mathscr{O}(\delta))$ (so, again, we are taking $\mathscr{F}$ to be $\mathscr{H}(\mathscr{E}, \mathscr{F}_\bullet)$) is 1-infinitesimally m-generating whenever $\delta \geq -s + 0 + 1(1 + (m-1) \cdot (1+1)) = -s + 1 + 2(m-1)$. Therefore, taking $\delta = 0$, we find $H^0(X, \mathscr{F})$ is 1-infinitesimally m-generating whenever $s \geq 1 + 2(m-1) = 2m - 1$. Therefore, we may take $m = \lfloor (s+1)/2 \rfloor$.

Using that $\operatorname{rk} \mathcal{P}^1(\mathscr{H}(\mathscr{E}, \mathscr{F}_\bullet)) = h^0(D, \mathscr{H}(\mathscr{E}|_D, \mathscr{F}_\bullet|_D))$, we obtain from [BH21, Theorem 9.3.1] the congruence

$$\{\mathrm{U}^{\mathcal{R}}_{\mathscr{M}}\} \equiv \mathbb{L}^{\dim \mathrm{U}^{\mathcal{R}}_{\mathscr{M}}} \prod_{x \in \mathbb{P}^1_k} \left(1 - \left(\frac{\{T^c\}_x}{\mathbb{L}^{\operatorname{rk} \mathcal{P}^1(\mathscr{H}(\mathscr{E}, \mathscr{F}_\bullet))}}\right) t\right)\bigg|_{t=1} \tag{9.3}$$

$$\equiv \mathbb{L}^{\dim \mathrm{U}^{\mathcal{R}}_{\mathscr{M}}} \prod_{x \in \mathbb{P}^1_k} \left(1 - \left(1 - \frac{\{T\}_x}{\mathbb{L}^{h^0(D, \mathscr{H}(\mathscr{E}|_D, \mathscr{F}_\bullet|_D))}}\right) t\right)\bigg|_{t=1}. \tag{9.4}$$

In order to obtain (9.2), we need to identify (9.4) and the right-hand side of (9.2). By working Zariski locally on $\mathbb{P}^1$ so as to trivialize the bundle $\mathscr{H}(\mathscr{E}, \mathscr{F}_\bullet)$, it is enough to identify the fiber of T over $x \in \mathbb{P}^1$ with the class $\sum_{R \in \mathcal{R}} \mathbb{L}^{-r(R)} \{\mathrm{Aut}_{\mathscr{M}|_D}\}$. Indeed, this identification holds because we showed that the class of the subschemes of $\operatorname{Spec}(\operatorname{Sym}^\bullet H^0(D, \mathscr{H}(\mathscr{E}|_D, \mathscr{F}_\bullet|_D))^\vee)$ having ramification profile R is $\mathbb{L}^{-r(R)}\{\mathrm{Aut}_{\mathscr{M}|_D}\}$ when we computed the class of $\mathscr{Y}_{R,d}$ in Theorem 8.9. $\square$

In order to get a good bound on the codimension up to which Proposition 9.10 holds, we need to show that the value of s defined there is high whenever the codimension of the stratum is low. We now establish this.

Lemma 9.11. *For any Casnati–Ekedahl stratum $\mathscr{M}(\mathscr{E}, \mathscr{F}_\bullet)$ so that the minimum degree of a line bundle summand of $\mathscr{H}(\mathscr{E}, \mathscr{F}_\bullet)$ is s, and $H^1(\mathbb{P}^1, \mathscr{H}(\mathscr{E}, \mathscr{F}_\bullet)) = 0$, we have $\operatorname{codim}_{\overline{\mathrm{Hur}}_{d,g,k}} \mathscr{M} + (s+1)/2 \geq (g + c_d)/\kappa_d$, where $c_3 = 0$, $c_4 = -2$, $c_5 = -23$, $\kappa_3 = 4$, $\kappa_4 = 12$, and $\kappa_5 = 40$.*

Proof. We can verify this in the case that $d = 3$ directly. Write $\mathscr{E} = \mathscr{O}(s) \oplus \mathscr{O}(g+2-s)$ with $s \leq g + 2 - s$, so that $\operatorname{codim}_{\overline{\mathrm{Hur}}_{3,g,k}} \mathscr{M} = h^1(\mathbb{P}^1, \operatorname{End}(\mathscr{E})) \geq (g+2) - 2s - 1$. Then, $\operatorname{codim}_{\overline{\mathrm{Hur}}_{3,g,k}} \mathscr{M} + (s+1)/2 \geq (g+2) - 2s - 1 + (s+1)/2 = (2g + 3 - 3s)/2$. This is minimized when s is maximized. Since we must have $s \leq (g+2)/2$, when $s = (g+2)/2$, we find $(2g + 3 - 3s)/2 = g/4$.

We now concentrate on the cases $d = 4$ and $d = 5$. First, in the case that $\mathscr{E}$ and $\mathscr{F}$ are balanced, so that $\operatorname{codim}_{\overline{\mathrm{Hur}}_{d,g,k}} \mathscr{M} = 0$, we claim that $(s+1)/2 \geq (g + c_d)/\kappa_d$.

When $d = 4$, and $\mathscr{E}$ and $\mathscr{F}$ are balanced, the minimum line bundle summand of $\mathscr{E}$ has degree at least $(g+1)/3$ while the maximum line bundle summand of $\mathscr{F}$ has degree at most $(g+4)/2$ using Lemma 6.3 and the isomorphism $\det \mathscr{E} \simeq \det \mathscr{F}$ from Theorem 3.14. Hence, the minimum line bundle summand of $\mathscr{H}$ has degree $s \geq 2(g+1)/3 - (g+4)/2 = (g-8)/6$. Therefore, $(s+1)/2 \geq (g-2)/12$.

When $d = 5$, and $\mathscr{E}$ and $\mathscr{F}$ are balanced, the minimum line bundle summand of $\mathscr{E}$ has degree at least $(g+1)/4$ by Lemma 6.3 and the minimum line bundle summand of $\mathscr{F}$ has degree at

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least $(2(g+4)-4)/5$ as $\det \mathscr{E}^{\otimes 2} = \det \mathscr{F}$ by Theorem 3.16. Therefore, the minimum degree of a line bundle summand of $\mathscr{H}$ is $s \geq 2((2(g+4)-4)/5) - (g+4) + (g+1)/4 = (g-43)/20$ and $(s+1)/2 \geq (g-23)/40$.

In the case $d=4$ or 5, it remains to see that $\operatorname{codim}_{\overline{\mathrm{Hur}}_{d,g,k}} \mathscr{M} + (s+1)/2 \geq (g+c_d)/\kappa_d$ remains true for non-general strata, supposing still that $H^1(\mathbb{P}^1, \mathscr{H}(\mathscr{E}, \mathscr{F})) = 0$. In this case, by Lemma 9.2, the codimension of $\mathscr{M}(\mathscr{E}, \mathscr{F})$ is $h^1(\mathbb{P}^1, \operatorname{End}(\mathscr{E})) + h^1(\mathbb{P}^1, \operatorname{End}(\mathscr{F}))$. This codimension is also the codimension of the point $[(\mathscr{E}, \mathscr{F})]$ in the moduli stack of vector bundles $\mathrm{Vect}_{\mathbb{P}^1_k}^{\mathrm{rk}\,\mathscr{E}} \times \mathrm{Vect}_{\mathbb{P}^1_k}^{\mathrm{rk}\,\mathscr{F}}$, see [Lar21, (3.1)]. Hence, we wish to verify

$$\operatorname{codim}_{\mathrm{Vect}_{\mathbb{P}^1_k}^{\mathrm{rk}\,\mathscr{E}} \times \mathrm{Vect}_{\mathbb{P}^1_k}^{\mathrm{rk}\,\mathscr{F}}} [(\mathscr{E}, \mathscr{F})] + \frac{s+1}{2} \geq \frac{g+c_d}{\kappa_d},$$

granting that we have established this in the case that $\mathscr{E}, \mathscr{F}$ are both balanced, and so correspond to the generic point of $\mathrm{Vect}_{\mathbb{P}^1_k}^{\mathrm{rk}\,\mathscr{E}} \times \mathrm{Vect}_{\mathbb{P}^1_k}^{\mathrm{rk}\,\mathscr{F}}$, as explained in § 6.9. Following the discussion from § 6.9 where we describe when one vector bundle on $\mathbb{P}^1_k$, viewed as a point in the moduli stack of vector bundles lies in the closure of another, any $(\mathscr{E}, \mathscr{F})$ may be connected to a balanced pair by a sequence of $(\mathscr{E}_i, \mathscr{F}_i)$, each contained in the closure of the next. Further, we can assume that for any two adjacent indices i and $i+1$, one of the following two cases occurs:

(1) $\mathscr{E}_i \simeq \mathscr{E}_{i+1}$ and $\mathscr{F}_i$ differs from $\mathscr{F}_{i+1}$ only in two line bundles summands by a single degree;
(2) $\mathscr{F}_i \simeq \mathscr{F}_{i+1}$ and $\mathscr{E}_i$ differs from $\mathscr{E}_{i+1}$ only in two line bundle summands by a single degree.

In order to show the claimed inequality holds for arbitrary strata, it suffices to show it remains true under such specializations. Because each such stratum has codimension at least 1 in the next, it suffices to show the value of s under such specializations decreases by at most 2. When $d=4$ this is the case because $\mathscr{H}(\mathscr{E}, \mathscr{F}) = \operatorname{Sym}^2 \mathscr{E} \otimes \mathscr{F}^\vee$ and increasing a summand of $\mathscr{F}$ by 1 only decreases all summands of $\mathscr{H}(\mathscr{E}, \mathscr{F})$ by at most 1, while decreasing a summand of $\mathscr{E}$ decreases all summands of $\mathscr{H}(\mathscr{E}, \mathscr{F})$ by at most 2. Similarly, when $d=5$, so $\mathscr{H}(\mathscr{E}, \mathscr{F}) = \wedge^2 \mathscr{F} \otimes \mathscr{E} \otimes \det \mathscr{E}^\vee$, and decreasing a summand of $\mathscr{E}$ by 1 while maintaining $\det \mathscr{E}$ decreases all summands of $\mathscr{H}(\mathscr{E}, \mathscr{F})$ by at most 1, while decreasing a summand of $\mathscr{F}$ by 1 decreases all summands of $\mathscr{H}(\mathscr{E}, \mathscr{F})$ by at most 2. □

9.12 Putting the codimension bounds together

We now merge the bounds on codimension of various bad loci established earlier in this section to obtain the following result.

Proposition 9.13. *For $3 \leq d \leq 5$, k a field of characteristic not dividing $d!$, $\mathcal{R}$ an allowable collection of ramification profiles of degree d, let c_d, κ_d be as in Lemma 9.11. Define $n_{d,g} := \chi(\mathscr{H}(\mathscr{E}, \mathscr{F}_\bullet))$. Then, $\{\mathrm{Hur}^{\mathcal{R}}_{d,g,k}\}$ is equal to*

$$\sum_{\text{Casnati–Ekedahl strata } \mathscr{M}} \frac{1}{\{\mathrm{Aut}_{\mathscr{M}}\}} \mathbb{L}^{n_{d,g}} \prod_{x \in \mathbb{P}^1_k} \frac{(\sum_{R \in \mathcal{R}} \mathbb{L}^{-r(R)})\{\mathrm{Aut}_{\mathscr{M}|_D}\}}{\mathbb{L}^{h^0(D, \mathscr{H}(\mathscr{E}|_D, \mathscr{F}_\bullet|_D))}} \tag{9.5}$$

modulo codimension $r_{d,g} := \min((g+c_d)/\kappa_d, (g+d-1)/d - 4^{d-3})$ in $\widehat{\widetilde{K_0}}(\mathrm{Spaces}_k)$.

Proof. First, by Proposition 7.10, it suffices to show

$$\sum_{\text{Casnati–Ekedahl strata } \mathscr{M}} \frac{\{\mathrm{U}^{\mathcal{R}, S_d}_{\mathscr{M}}\}}{\{\mathrm{Aut}_{\mathscr{M}}\}} \tag{9.6}$$

agrees with (9.5).

We next check (9.6) agrees with

$$\sum_{\text{nonempty Casnati–Ekedahl strata } \mathscr{M}} \frac{\{\mathrm{U}^{\mathcal{R}}_{\mathscr{M}}\}}{\{\mathrm{Aut}_{\mathscr{M}}\}} \tag{9.7}$$

modulo codimension $\min((g+c_d)/\kappa_d, (g+d-1)/d - 4^{d-3})$. Since we are working modulo codimension $(g+d-1)/d-4^{d-3}$, we can assume $\mathscr{M}(\mathscr{E},\mathscr{F}_\bullet)$ has $H^1(\mathbb{P}^1,\mathscr{H}(\mathscr{E},\mathscr{F}_\bullet))=0$ and $H^0(\mathbb{P}^1,\mathscr{E}^\vee)=0$, by Lemma 9.4. Note that the condition $H^0(\mathbb{P}^1,\mathscr{E}^\vee)=0$ ensures all curves defined by sections of $\mathrm{U}^{\mathcal{R}}_{\mathscr{M}}$ are geometrically connected, by Theorem 3.17. Since we have now restricted ourselves to work with strata for which $H^1(\mathbb{P}^1,\mathscr{H}(\mathscr{E},\mathscr{F}_\bullet))=0$, it follows from Lemma 9.11, with notation for s as in Lemma 9.11, that $\operatorname{codim}\mathscr{M}(\mathscr{E},\mathscr{F}_\bullet)+(s+1)/2 \geq (g+c_d)/\kappa_d$. We also obtain from Lemma 9.8 that the smooth geometrically connected curves in $\mathrm{U}^{\mathcal{R}}_{\mathscr{M}}$ which do not lie in $\mathrm{Hur}_{d,g,k}$ (because they do not have Galois closure S_d) have codimension at least $(g+3)/2$ in $\mathrm{Hur}_{d,g,k}$. Hence, as we are working modulo codimension $(g+3)/2$, we can freely ignore these, and so (9.6) agrees with (9.7).

We next claim (9.7) agrees with

$$\sum_{\text{nonempty Casnati–Ekedahl strata } \mathscr{M}} \frac{1}{\{\mathrm{Aut}_{\mathscr{M}}\}} \mathbb{L}^{\dim \mathrm{U}^{\mathcal{R}}_{\mathscr{M}}} \prod_{x\in\mathbb{P}^1_k} \frac{(\sum_{R\in\mathcal{R}}\mathbb{L}^{-r(R)})\{\mathrm{Aut}_{\mathscr{M}|_D}\}}{\mathbb{L}^{h^0(D,\mathscr{H}(\mathscr{E}|_D,\mathscr{F}_\bullet|_D))}}. \tag{9.8}$$

Indeed, this follows from Proposition 9.10 using the bounds on s from Lemma 9.11. Next, we claim that for any $\mathscr{M}(\mathscr{E},\mathscr{F}_\bullet)$ as above, $\dim \mathrm{U}^{\mathcal{R}}_{\mathscr{M}}$ is independent of $\mathscr{M}$ whenever $\operatorname{codim}_{\overline{\mathrm{Hur}}_{d,g,k}}\mathscr{M} \leq \min((g+c_d)/\kappa_d, (g+d-1)/d-4^{d-3})$. Indeed, in this case, because $H^1(\mathbb{P}^1,\mathscr{H}(\mathscr{E},\mathscr{F}_\bullet))=0$, we find $\dim \mathrm{U}^{\mathcal{R}}_{\mathscr{M}} = h^0(\mathbb{P}^1,\mathscr{H}(\mathscr{E},\mathscr{F}_\bullet)) = \chi(\mathscr{H}(\mathscr{E},\mathscr{F}_\bullet))$ and indeed this Euler characteristic only depends on the degrees and ranks of $\mathscr{E}$ and $\mathscr{F}$. For notational convenience, we let $n_{d,g}$ denote this dimension $\chi(\mathscr{H}(\mathscr{E},\mathscr{F}_\bullet))$. Then, up to codimension $\min((g+c_d)/\kappa_d,(g+d-1)/d-4^{d-3})$, in $\widehat{\widetilde{K_0}}(\mathrm{Spaces}_k)$, we can rewrite (9.8) as (9.5).

To conclude the proof, we wish to remove the word 'nonempty' in (9.8). That is, there may be certain strata which contain no S_d covers, and we wish to show they do not contribute to (9.5) in low codimension. The summand in (9.5) associated to such an empty stratum $\mathscr{M}(\mathscr{E},\mathscr{F}_\bullet)$ has codimension equal to the codimension of $(\mathscr{E},\mathscr{F}_\bullet)$, considered as a point in the moduli stack of tuples of vector bundles on $\mathbb{P}^1$. Using Corollary 9.7, this is only potentially an issue in the case $d=4$, where we must deal with strata $\mathscr{M}(\mathscr{E},\mathscr{F}_\bullet)$ so that the generic members of $H^0(\mathbb{P}^1,\mathscr{H}(\mathscr{E},\mathscr{F}_\bullet))$ define D_4 covers. In [CL24, Lemma 5.5], it is shown that such strata are either codimension at least $(g+3)/2$ or else have $H^1(\mathbb{P}^1,\mathscr{H}(\mathscr{E},\mathscr{F}_\bullet))\neq 0$. In the latter case, by [CL24, Lemma 5.4], such strata have codimension at least $(g+3)/4-4$ in the stack of vector bundles on $\mathbb{P}^1$. In either case, we may ignore these contributions up to our codimension bounds, and so (9.8) agrees with (9.5). □

10. Proving the main result

In this section, we prove our main result Theorem 10.5 by massaging the formula for $\{\mathrm{Hur}_{d,g,k}\}$ given in Proposition 9.13. We then deduce some corollaries.

In order to prove our main result we will need one of the simplest cases of the 'motivic Tamagawa number conjecture' [BD07, Conjecture 3.4]. To start this Tamagawa number formula, we employ the following notation.

Notation 10.1. For $\mathscr{G}$ a vector bundle on a scheme X, let $\mathrm{Aut}^{\mathrm{SL},X}_{\mathscr{G}}$ denote the SL bundle over X associated to $\mathscr{G}$ (i.e. the kernel of the determinant map of group schemes $\mathrm{Aut}_{\mathscr{G}} \to \mathbb{G}_m$). We use

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$\mathrm{Aut}^{\mathrm{SL}}_{\mathscr{G}}$ as notation for the Weil restriction $\mathrm{Res}_{X/\operatorname{Spec} k}(\mathrm{Aut}^{\mathrm{SL},X}_{\mathscr{G}})$. For $(\mathscr{E},\mathscr{F}_\bullet)$ resolution data, we use $\mathrm{Aut}^{\mathrm{SL}}(\mathscr{F}_\bullet) := \prod_{i=1}^{\lfloor (d-2)/2\rfloor} \mathrm{Aut}^{\mathrm{SL}}(\mathscr{F}_i)$.

Lemma 10.2. *For any positive integer n,*

$$\sum_{\substack{\text{rank-}n\text{ vector bundles }\mathscr{V}\text{ on }\mathbb{P}^1\\ \det\mathscr{V}=\mathscr{O}_{\mathbb{P}^1}}} \frac{1}{\{\mathrm{Aut}^{\mathrm{SL}}_{\mathscr{V}}\}} \prod_{x\in\mathbb{P}^1_k} \frac{\{\mathrm{Aut}^{\mathrm{SL}}_{\mathscr{V}|_D}\}}{\mathbb{L}^{\dim \mathrm{Aut}^{\mathrm{SL}}_{\mathscr{V}|_D}}} = \mathbb{L}^{-\dim \mathrm{SL}_n} \in \widehat{\widetilde{K}}_0(\mathrm{Spaces}_k).$$

Proof. We will deduce this from the motivic Tamagawa number conjecture for SL_n over $\mathbb{P}^1$ proven in [BD07, § 7]. Let $\mathfrak{Bun}_{G,\mathbb{P}^1}$ denote the moduli stack of G-bundles on $\mathbb{P}^1$. It is shown in [BD07, § 7], and also via a different argument in [BD07, § 6], that in $\widehat{\widetilde{K}}_0(\mathrm{Spaces}_k)$ (and even without inverting universally bijective morphisms) we have $\{\mathfrak{Bun}_{\mathrm{SL}_n,\mathbb{P}^1}\} = \mathbb{L}^{-\dim \mathrm{SL}_n}\prod_{i=2}^n Z(\mathbb{P}^1,\mathbb{L}^{-i})$, where $Z(\mathbb{P}^1,t) := \sum_{i=0}^\infty \{\mathrm{Sym}^i_{\mathbb{P}^1}\}t^i = (1/(1-t))(1/(1-\mathbb{L}t))$ is the motivic Zeta function of $\mathbb{P}^1$.

Note that $Z(\mathbb{P}^1,\mathbb{L}^{-i}) = (1/(1-\mathbb{L}^{-i+1}))(1/(1-\mathbb{L}^{-i}))$ is invertible in $\widehat{\widetilde{K}}_0(\mathrm{Spaces}_k)$, with inverse equal to $(1-\mathbb{L}^{-i+1})(1-\mathbb{L}^{-i})$. To complete the proof, it is therefore enough to demonstrate the two equalities

$$\{\mathfrak{Bun}_{\mathrm{SL}_n,\mathbb{P}^1}\} = \sum_{\substack{\text{rank-}n\text{ vector bundles }\mathscr{V}\text{ on }\mathbb{P}^1\\ \det\mathscr{V}=\mathscr{O}_{\mathbb{P}^1}}} \frac{1}{\{\mathrm{Aut}^{\mathrm{SL}}_{\mathscr{V}}\}} \tag{10.1}$$

$$\left(\prod_{i=2}^n Z(\mathbb{P}^1,\mathbb{L}^{-i})\right)^{-1} = \prod_{x\in\mathbb{P}^1_k} \frac{\{\mathrm{Aut}^{\mathrm{SL}}_{\mathscr{V}|_D}\}}{\mathbb{L}^{\dim \mathrm{Aut}^{\mathrm{SL}}_{\mathscr{V}|_D}}}, \tag{10.2}$$

where we note that the right-hand side of (10.2) turns out to be independent of $\mathscr{V}$.

We first verify (10.1). Taking cohomology on $\mathbb{P}^1$ associated to exact sequence $\mathrm{SL}_n \to \mathrm{GL}_n \to \mathbb{G}_m$ defining SL_n shows that SL_n torsors over $\mathbb{P}^1$ are in bijection with GL_n torsors of trivial determinant. We can then stratify $\mathfrak{Bun}_{\mathrm{SL}_n,\mathbb{P}^1} = \coprod B\,\mathrm{Aut}^{\mathrm{SL}}_{\mathscr{V}}$ as a disjoint union of locally closed substacks corresponding to residual gerbes, as is explained for general G in place of SL_n (see [BD07, p. 636]). (Much of this argument can be verified more simply and directly in the case $G = \mathrm{SL}_n$.) Noting that $\mathrm{Aut}^{\mathrm{SL}}_{\mathscr{V}}$ is special with invertible class in the Grothendieck ring by Lemma 7.12, we find $\{B(\mathrm{Aut}^{\mathrm{SL}}_{\mathscr{V}})\} = 1/\{\mathrm{Aut}^{\mathrm{SL}}_{\mathscr{V}}\}$ and (10.1) follows.

It remains only to prove (10.2). First, note that $\mathscr{V}$ is trivial Zariski locally and, hence, trivial over D, so $\mathrm{Aut}^{\mathrm{SL}}_{\mathscr{V}|_D}$ is simply $\mathrm{Res}_{D/\operatorname{Spec} k}(\mathrm{SL}_n)$ which is an extension of SL_n by $\mathbb{G}_a^{\dim \mathrm{SL}_n}$. Therefore, for any vector bundle $\mathscr{V}$, we may re-express

$$\frac{\{\mathrm{Aut}^{\mathrm{SL}}_{\mathscr{V}|_D}\}}{\mathbb{L}^{\dim \mathrm{Aut}^{\mathrm{SL}}_{\mathscr{V}|_D}}} = \frac{\{\mathrm{SL}_n\}}{\mathbb{L}^{\dim \mathrm{SL}_n}} = \{\mathrm{SL}_n\}\mathbb{L}^{-\dim \mathrm{SL}_n} = \left(\prod_{i=2}^n(\mathbb{L}^i-1)\right)\mathbb{L}^{-\dim \mathrm{SL}_n} = \prod_{i=2}^n (1-\mathbb{L}^{-i}).$$

Using multiplicativity of Euler products (Lemma 2.14),

$$\prod_{x\in\mathbb{P}^1_k} \frac{\{\mathrm{Aut}^{\mathrm{SL}}_{\mathscr{V}|_D}\}}{\mathbb{L}^{\dim \mathrm{Aut}^{\mathrm{SL}}_{\mathscr{V}|_D}}} = \prod_{x\in\mathbb{P}^1_k}\prod_{i=2}^n (1-\mathbb{L}^{-i}) = \prod_{i=2}^n\prod_{x\in\mathbb{P}^1_k}(1-\mathbb{L}^{-i}).$$

Hence, to prove (10.2), we only need check $Z(\mathbb{P}^1,\mathbb{L}^{-i})^{-1} = \prod_{x\in\mathbb{P}^1_k}(1-\mathbb{L}^{-i})$ for $2\le i\le n$. The right-hand side is, by definition, $\prod_{x\in\mathbb{P}^1_k}(1-\mathbb{L}^{-i}t)|_{t=1}$. By [Bil17, § 3.8, Property 4], we have $\prod_{x\in\mathbb{P}^1_k}(1-\mathbb{L}^{-i}t)|_{t=1} = \prod_{x\in\mathbb{P}^1_k}(1-t)|_{t=\mathbb{L}^{-i}}$. (As a word of warning, it is important that the substitution we made here was via replacing t by its product with a power of $\mathbb{L}$, see

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[BH21, Remarks 6.5.2 and 6.5.3].) Finally, by [BH21, Ex. 6.1.12] and multiplicativity of Euler products [Bil17, Proposition 3.9.2.4], $\prod_{x\in\mathbb{P}^1_k}(1-t)|_{t=\mathbb{L}^{-i}} = Z(\mathbb{P}^1,\mathbb{L}^{-i})^{-1}$. □

In fact, we will need a slight generalization of the above formula from Lemma 10.2, where we replace bundles of degree 0 with bundles of arbitrary fixed degree.

Lemma 10.3. *For any positive integer n, and any fixed integer δ,*

$$\sum_{\substack{\text{rank-}n\text{ vector bundles }\mathscr{V}\text{ on }\mathbb{P}^1\\ \deg\mathscr{V}=\delta}} \frac{1}{\{\mathrm{Aut}^{\mathrm{SL}}_{\mathscr{V}}\}} \prod_{x\in\mathbb{P}^1_k} \frac{\{\mathrm{Aut}^{\mathrm{SL}}_{\mathscr{V}|_D}\}}{\mathbb{L}^{\dim \mathrm{Aut}^{\mathrm{SL}}_{\mathscr{V}|_D}}} = \mathbb{L}^{-\dim\mathrm{SL}_n} \in \widehat{\widetilde{K}_0}(\mathrm{Spaces}_k).$$

Proof. The case that $\delta=0$ was precisely covered in Lemma 10.2. Therefore, it remains to show that the left-hand side of the statement of the lemma is independent of δ. The left hand side is unchanged upon replacing δ by $\delta\pm n$ because tensoring with the line bundle $\mathscr{O}_{\mathbb{P}^1_k}(1)$ defines a bijection from degree δ vector bundles of rank n to degree $\delta+n$ vector bundles of rank n, which preserves automorphism groups. Therefore, it suffices to show that the left-hand side is independent of which congruence class δ lies in mod$n$.

Next, note that one can express

$$\mathbb{L}^{\dim\mathrm{PGL}_n}\cdot \sum_{\substack{\text{rank-}n\text{ vector bundles }\mathscr{V}\text{ on }\mathbb{P}^1\\ \deg\mathscr{V}=\delta}} \frac{1}{\{\mathrm{Aut}^{\mathrm{SL}}_{\mathscr{V}}\}} \prod_{x\in\mathbb{P}^1_k} \frac{\{\mathrm{Aut}^{\mathrm{SL}}_{\mathscr{V}|_D}\}}{\mathbb{L}^{\dim \mathrm{Aut}^{\mathrm{SL}}_{\mathscr{V}|_D}}},$$

as $f_\delta(\mathbb{L})$, for f_δ a rational function. Hence, for δ,δ' two distinct residue classes mod$n$, it is enough to show $f_\delta(q)=f_{\delta'}(q)$ for infinitely many integers q, as then the two rational functions must agree. The reason for choosing the above expression is that $\sum_{\delta=1}^n f_\delta(q)$ can also be identified with the Tamagawa number of PGL_n over the function field $\mathbb{P}^1_{\mathbb{F}_q}$. Here we are using that if one starts with a vector bundle $\mathscr{V}$ on $\mathbb{P}^1_{F_q}$, $\#\,\mathrm{Aut}\,\mathbb{P}\mathscr{V} = (1/(q-1))\,\mathrm{Aut}\,\mathscr{V}$, and the same expression calculates the number of automorphisms of $\mathscr{V}$ with trivial determinant.

We now use the above description in terms of Tamagawa numbers to show $f_\delta(q)=f_{\delta'}(q)$ for $1\le\delta\le\delta'\le n$. For $x\in\mathbb{P}^1_{\mathbb{F}_q}$ a closed point, let $\widehat{\mathscr{O}}_{x,\mathbb{P}^1_{\mathbb{F}_q}}$ denote the complete local ring at x. We use $K(\mathbb{P}^1_{\mathbb{F}_q})$ to denote the function field of $\mathbb{P}^1_{\mathbb{F}_q}$, and $\mathbb{A} := \prod_{v\in\mathbb{P}^1_{\mathbb{F}_q}\text{ closed points}}(K(\mathbb{P}^1_{\mathbb{F}_q})_v,\widehat{\mathscr{O}}_v)$ to denote the ring of adeles for this function field. Note that the Tamagawa number can be expressed as the Tamagawa measure of $\mathrm{PGL}_n(K(\mathbb{P}^1_{\mathbb{F}_q}))\backslash\mathrm{PGL}_n(\mathbb{A})$.

There is a projection map

$$\alpha:\mathrm{PGL}_n(K(\mathbb{P}^1_{\mathbb{F}_q}))\backslash\mathrm{PGL}_n(\mathbb{A}) \to \mathrm{PGL}_n(K(\mathbb{P}^1_{\mathbb{F}_q}))\backslash\mathrm{PGL}_n(\mathbb{A})/\prod_{\text{places }x\in\mathbb{P}^1_{\mathbb{F}_q}}\mathrm{PGL}_n(\widehat{\mathscr{O}}_{x,\mathbb{P}^1_{\mathbb{F}_q}}). \quad (10.3)$$

We claim one can identify the target with the set of isomorphism classes of PGL_n bundles on $\mathbb{P}^1_{\mathbb{F}_q}$, and moreover, if X is a PGL_n bundle, the Tamagawa measure satisfies

$$\mu_{\mathrm{Tam}}(\alpha^{-1}([X])) = \frac{\mu_{\mathrm{Tam}}\left(\prod_{\text{places }x\in\mathbb{P}^1_{\mathbb{F}_q}}\mathrm{PGL}_n(\widehat{\mathscr{O}}_{x,\mathbb{P}^1_{\mathbb{F}_q}})\right)}{\#\,\mathrm{Aut}\,X}. \quad (10.4)$$

Our claim essentially follows from [GL19, Proposition 1.3.2.11], except the statement there assumes the group G is simply connected, which is not the case for PGL_n. However, the only place in the proof (see the proof of [GL19, Proposition 1.3.2.10]) that the simply connected hypothesis was used was to show there is some dense open of $\mathbb{P}^1_{\mathbb{F}_q}$ on which any PGL_n bundle is trivial. We can instead verify this directly as follows. Note first that the Brauer group of $\mathbb{P}^1_{\mathbb{F}_q}$

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is trivial [Gro68, Remarques 2.5(b)]. Hence, any PGL_n bundle on $\mathbb{P}^1_{\mathbb{F}_q}$ is the projectivization of a GL_n bundle. Since any GL_n bundle is Zariski locally trivial, the same holds for any PGL_n bundle on $\mathbb{P}^1_{\mathbb{F}_q}$.

There is natural map $\pi : \mathrm{PGL}_n(K(\mathbb{P}^1_{\mathbb{F}_q}))\backslash \mathrm{PGL}_n(\mathbb{A}) \to \mathbb{Z}/n\mathbb{Z}$ which factors through the double quotient map (10.3) parameterizing projective bundles on $\mathbb{P}^1$, and sends a projective bundle to its degree mod$n$. Note that the degree of the projectivization of a vector bundle is not well defined as an integer, but it is well defined mod$n$. In this setup, we obtain $\mu_{\mathrm{Tam}}(\pi^{-1}(\delta)) = f_\delta(q)$, where μ_{Tam} denotes the right-invariant Tamagawa measure, by summing (10.4) over all bundles of degree δ mod n.

Since the Tamagawa measure is translation invariant, we can right-translate by any element of $\mathrm{PGL}_n(\mathbb{A})$ in $\pi^{-1}(\delta' - \delta)$ and this sends $\pi^{-1}(\delta)$ to $\pi^{-1}(\delta')$. Hence, the Tamagawa measures of $\pi^{-1}(\delta)$ and $\pi^{-1}(\delta')$ agree, so $f_\delta(q) = f_{\delta'}(q)$, as desired. □

For our main theorem, we will also need the following elementary dimension comparison.

LEMMA 10.4. *For* $3 \le d \le 5$ *and* $n_{d,g}$ *as in Proposition 9.13,* $n_{d,g} - \dim \mathrm{SL}_{\mathrm{rk}\,\mathscr{E}} - \sum_{i=1}^{\lfloor d-2\rfloor/2} \dim \mathrm{SL}_{\mathrm{rk}\,\mathscr{F}_i} = \dim \mathrm{Hur}_{d,g,k} + 1$.

Proof. Indeed, this can be checked separately in the cases $d = 3$, 4, and 5.

We now check the most difficult case that $d = 5$, leaving the other cases to the reader. In the case $d = 5$, one computes

$$\begin{aligned} n_{d,g} &= \chi(\wedge^2 \mathscr{F} \otimes \mathscr{E} \otimes \det \mathscr{E}^\vee) \\ &= \mathrm{rk}(\wedge^2 \mathscr{F} \otimes \mathscr{E} \otimes \det \mathscr{E}^\vee) + \deg \wedge^2 \mathscr{F} \otimes \mathscr{E} \otimes \det \mathscr{E}^\vee \\ &= \binom{5}{2} \cdot 4 + 16 \deg \mathscr{F} - 30 \deg \mathscr{E} = 40 + 32 \deg \mathscr{E} - 30 \deg \mathscr{E} \\ &= 40 + 2 \deg \mathscr{E} \\ &= 40 + 2g + 2d - 2. \end{aligned}$$

Furthermore, still in the $d = 5$ case, $\dim \mathrm{SL}_{\mathrm{rk}\,\mathscr{E}} = 15$ and $\dim \mathrm{SL}_{\mathrm{rk}\,\mathscr{F}} = 24$. Therefore,

$$\begin{aligned} n_{d,g} - \dim \mathrm{SL}_{\mathrm{rk}\,\mathscr{E}} - \dim \mathrm{SL}_{\mathrm{rk}\,\mathscr{F}} &= 40 + 2g + 2d - 2 - 15 - 24 \\ &= (2g + 2d - 2) + 1 \\ &= \dim \mathrm{Hur}_{d,g,k} + 1 \end{aligned}$$

as claimed. □

We are finally prepared to prove our main theorem. For the statement of our main theorem, recall we defined $r_{d,g} = \min((g + c_d)/\kappa_d, (g + d - 1)/d - 4^{d-3})$ in Proposition 9.13, with $c_3 = 0, c_4 = -2$, and $c_5 = -23$. Note that for $g \gg 0$, $r_{d,g}$ is more than $g/\kappa_d - 1$.

THEOREM 10.5. *Let* $2 \le d \le 5$*,* k *a field of characteristic not dividing* $d!$*,* $\mathcal{R}$ *an allowable collection of ramification profiles of degree* d*. Then,*

$$\{\mathrm{Hur}^{\mathcal{R}}_{d,g,k}\} \equiv \frac{\mathbb{L}^{\dim \mathrm{Hur}_{d,g,k}}}{1 - \mathbb{L}^{-1}} \left(\prod_{x \in \mathbb{P}^1_k} \left(\sum_{R \in \mathcal{R}} \mathbb{L}^{-r(R)} \right) (1 - \mathbb{L}^{-1}) \right) \tag{10.5}$$

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modulo codimension $r_{d,g}$ *in* $\widehat{\widetilde{K}_0}(\mathrm{Spaces}_k)$. *In the case* $d=2$, *the left-hand side and right-hand side of* (10.5) *are actually equal in* $K_0(\mathrm{Stacks}_k)$ *(and not just equivalent in* $\widehat{\widetilde{K}_0}(\mathrm{Spaces}_k)$ *modulo terms of a certain dimension).*

Proof. The proof of the $d=2$ case is of a different nature and we defer it to the end of § 11. We now concentrate on the case $3 \le d \le 5$. By Proposition 9.13, our goal reduces to showing (9.5) agrees with the right hand side of (10.5).

Recall our notation for $n_{d,g}$ from Proposition 9.13. First, we claim we can rewrite (9.5) as

$$\frac{1}{\mathbb{L}-1}\sum_{\mathscr{E},\mathscr{F}_\bullet}\frac{1}{\{\mathrm{Aut}^{\mathrm{SL}}_{\mathscr{E}}\}}\frac{1}{\{\mathrm{Aut}^{\mathrm{SL}}_{\mathscr{F}_\bullet}\}}\mathbb{L}^{n_{d,g}}\prod_{x\in\mathbb{P}^1_k}\frac{(\sum_{R\in\mathcal{R}}\mathbb{L}^{-r(R)})(\mathbb{L}-1)\mathbb{L}\{\mathrm{Aut}^{\mathrm{SL}}_{\mathscr{E}|_D}\}\{\mathrm{Aut}^{\mathrm{SL}}_{\mathscr{F}_\bullet|_D}\}}{\mathbb{L}^{h^0(D,\mathscr{H}(\mathscr{E}|_D,\mathscr{F}_\bullet|_D))}}, \qquad (10.6)$$

with the summation over $\mathscr{E},\mathscr{F}_\bullet$ interpreted as follows: $\mathscr{E}$ ranges over all $\mathbb{P}^1$ bundles of rank $d-1$ and degree $g+d-1$; when $d=3$, $\mathscr{F}_\bullet$ is interpreted as being empty (so all classes associated to it are 1); when $d=4$, $\mathscr{F}_\bullet=\mathscr{F}$ has rank 2 and degree $g+d-1$; when $d=5$, $\mathscr{F}_\bullet=\mathscr{F}$ has rank 5 and degree $2(g+d-1)$. To see this we proceed as follows. For $\mathscr{M}=\mathscr{M}(\mathscr{E},\mathscr{F}_\bullet)$, using the formula for $\mathrm{Aut}^{\mathbb{P}^1/k}_{\mathscr{E},\mathscr{F}_\bullet}$ from Lemma 4.4, we can rewrite

$$\frac{1}{\{\mathrm{Aut}_{\mathscr{M}}\}}=\{\mathrm{Res}_{\mathbb{P}^1_k/k}(\mathbb{G}_m)\}\frac{1}{\{\mathrm{Aut}^{\mathbb{P}^1/k}_{\mathscr{E}}\}}\frac{1}{\{\mathrm{Aut}^{\mathbb{P}^1/k}_{\mathscr{F}_\bullet}\}}=\frac{1}{\mathbb{L}-1}\frac{1}{\{\mathrm{Aut}^{\mathrm{SL}}_{\mathscr{E}}\}}\frac{1}{\{\mathrm{Aut}^{\mathrm{SL}}_{\mathscr{F}_\bullet}\}}, \qquad (10.7)$$

where we interpret $\{\mathrm{Aut}^{\mathrm{SL}}_{\mathscr{F}_\bullet}\}=1$ when $d=3$. Similarly,

$$\{\mathrm{Aut}_{\mathscr{M}|_D}\}=\{\mathrm{Res}_{D/k}(\mathbb{G}_m)\}\{\mathrm{Aut}^{\mathrm{SL}}_{\mathscr{E}|_D}\}\{\mathrm{Aut}^{\mathrm{SL}}_{\mathscr{F}_\bullet|_D}\}=(\mathbb{L}-1)\mathbb{L}\{\mathrm{Aut}^{\mathrm{SL}}_{\mathscr{E}|_D}\}\{\mathrm{Aut}^{\mathrm{SL}}_{\mathscr{F}_\bullet|_D}\}. \qquad (10.8)$$

Hence, using (10.7) and (10.8), we can rewrite (9.5) as (10.6).

We next make a sequence of simplifications of (10.6). Then, summing over the same pairs $(\mathscr{E},\mathscr{F}_\bullet)$ as in (10.6), we can rewrite it as

$$\frac{1}{\mathbb{L}-1}\left(\sum_{\mathscr{E}}\frac{1}{\{\mathrm{Aut}^{\mathrm{SL}}_{\mathscr{E}}\}}\right)\left(\sum_{\mathscr{F}_\bullet}\frac{1}{\{\mathrm{Aut}^{\mathrm{SL}}_{\mathscr{F}_\bullet}\}}\right)\mathbb{L}^{n_{d,g}} \cdot\prod_{x\in\mathbb{P}^1_k}\frac{(\sum_{R\in\mathcal{R}}\mathbb{L}^{-r(R)})(\mathbb{L}-1)\mathbb{L}\{\mathrm{Aut}^{\mathrm{SL}}_{\mathscr{E}|_D}\}\{\mathrm{Aut}^{\mathrm{SL}}_{\mathscr{F}_\bullet|_D}\}}{\mathbb{L}^{h^0(D,\mathscr{H}(\mathscr{E}|_D,\mathscr{F}_\bullet|_D))}}, \qquad (10.9)$$

where the parenthesized sum of $\mathscr{F}_\bullet$ is interpreted as 1 in the case $d=3$, in this line and in the remainder of the proof.

Next, observe that $2+\dim\mathrm{Aut}^{\mathrm{SL}}_{\mathscr{E}|_D}+\dim\mathrm{Aut}^{\mathrm{SL}}_{\mathscr{F}_\bullet|_D}=h^0(D,\mathscr{H}(\mathscr{E}|_D,\mathscr{F}_\bullet|_D))$. Indeed, this can be checked separately in the cases $d=3$, 4, and 5. When $d=3$, both sides equal 8, when $d=4$, both sides equal 24, and when $d=5$, both sides equal 80. Therefore, we can rewrite (10.6) as

$$\frac{1}{\mathbb{L}-1}\left(\sum_{\mathscr{E}}\frac{1}{\{\mathrm{Aut}^{\mathrm{SL}}_{\mathscr{E}}\}}\right)\left(\sum_{\mathscr{F}_\bullet}\frac{1}{\{\mathrm{Aut}^{\mathrm{SL}}_{\mathscr{F}_\bullet}\}}\right)\mathbb{L}^{n_{d,g}} \cdot\prod_{x\in\mathbb{P}^1_k}\left(\sum_{R\in\mathcal{R}}\mathbb{L}^{-r(R)}\right)\frac{(\mathbb{L}-1)\mathbb{L}}{\mathbb{L}^2}\frac{\{\mathrm{Aut}^{\mathrm{SL}}_{\mathscr{E}|_D}\}}{\mathbb{L}^{\dim\mathrm{Aut}^{\mathrm{SL}}_{\mathscr{E}|_D}}}\frac{\{\mathrm{Aut}^{\mathrm{SL}}_{\mathscr{F}_\bullet|_D}\}}{\mathbb{L}^{\dim\mathrm{Aut}^{\mathrm{SL}}_{\mathscr{F}_\bullet|_D}}}. \qquad (10.10)$$

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Using multiplicativity of Euler products, as proven in Lemma 2.14, this becomes

$$\frac{1}{\mathbb{L}-1}\left(\sum_{\mathscr{E}}\frac{1}{\{\mathrm{Aut}^{\mathrm{SL}}_{\mathscr{E}}\}}\right)\left(\sum_{\mathscr{F}_\bullet}\frac{1}{\{\mathrm{Aut}^{\mathrm{SL}}_{\mathscr{F}_\bullet}\}}\right)\mathbb{L}^{n_{d,g}}\left(\prod_{x\in\mathbb{P}^1_k}\left(\sum_{R\in\mathcal{R}}\mathbb{L}^{-r(R)}\right)(1-\mathbb{L}^{-1})\right)$$
$$\cdot\left(\prod_{x\in\mathbb{P}^1_k}\frac{\{\mathrm{Aut}^{\mathrm{SL}}_{\mathscr{E}|_D}\}}{\mathbb{L}^{\dim\mathrm{Aut}^{\mathrm{SL}}_{\mathscr{E}|_D}}}\right)\cdot\left(\prod_{x\in\mathbb{P}^1_k}\frac{\{\mathrm{Aut}^{\mathrm{SL}}_{\mathscr{F}_\bullet|_D}\}}{\mathbb{L}^{\dim\mathrm{Aut}^{\mathrm{SL}}_{\mathscr{F}_\bullet|_D}}}\right). \tag{10.11}$$

Then, by the Tamagawa number formula for SL_n, and its slight generalization from Lemma 10.3,

$$\sum_{\mathscr{E}}\frac{1}{\{\mathrm{Aut}^{\mathrm{SL}}_{\mathscr{E}}\}}\prod_{x\in\mathbb{P}^1_k}\frac{\{\mathrm{Aut}^{\mathrm{SL}}_{\mathscr{E}|_D}\}}{\mathbb{L}^{\dim\mathrm{Aut}^{\mathrm{SL}}_{\mathscr{E}|_D}}}=\mathbb{L}^{-\dim\mathrm{SL}_{\mathrm{rk}\,\mathscr{E}}}, \tag{10.12}$$

$$\sum_{\mathscr{F}_\bullet}\frac{1}{\{\mathrm{Aut}^{\mathrm{SL}}_{\mathscr{F}_\bullet}\}}\prod_{x\in\mathbb{P}^1_k}\frac{\{\mathrm{Aut}^{\mathrm{SL}}_{\mathscr{F}_\bullet|_D}\}}{\mathbb{L}^{\dim\mathrm{Aut}^{\mathrm{SL}}_{\mathscr{F}_\bullet|_D}}}=\mathbb{L}^{-\dim\mathrm{SL}_{\mathrm{rk}\,\mathscr{F}_\bullet}}, \tag{10.13}$$

where $-\dim\mathrm{SL}_{\mathrm{rk}\,\mathscr{F}_\bullet}$ is interpreted as 0 in the case $d=3$. Again, in (10.12) and (10.13), the bundles have degrees as described after (10.6). Therefore, (10.11) simplifies to

$$\frac{1}{\mathbb{L}-1}\mathbb{L}^{n_{d,g}-\dim\mathrm{SL}_{\mathrm{rk}\,\mathscr{E}}-\dim\mathrm{SL}_{\mathscr{F}_\bullet}}\prod_{x\in\mathbb{P}^1_k}\left(\sum_{R\in\mathcal{R}}\mathbb{L}^{-r(R)}\right)(1-\mathbb{L}^{-1}). \tag{10.14}$$

Hence, using Lemma 10.4, (10.15) simplifies to

$$\frac{1}{\mathbb{L}-1}\mathbb{L}^{\dim\mathrm{Hur}_{d,g,k}+1}\prod_{x\in\mathbb{P}^1_k}\left(\sum_{R\in\mathcal{R}}\mathbb{L}^{-r(R)}\right)(1-\mathbb{L}^{-1}), \tag{10.15}$$

which equals the right-hand side of (10.5). □

Specializing Theorem 10.5 to the simply branched case gives the following corollary.

COROLLARY 10.6. *For $2\le d\le 5$, and k a field of characteristic not dividing $d!$, in the case $\mathcal{R}=\{(1^d),(2,1^{d-2})\}$ corresponding to simply branched curves, we have*

$$\{\mathrm{Hur}^{\mathcal{R}}_{d,g,k}\}\equiv\mathbb{L}^{\dim\mathrm{Hur}_{d,g,k}}(1-\mathbb{L}^{-2})$$

in $\widehat{\widehat{K_0}}(\mathrm{Stacks}_k)$ modulo codimension $r_{d,g}$ if $d\neq 2$, and in $K_0(\mathrm{Stacks}_k)$ when $d=2$.

Note that in the case $d=2$, this corollary is equivalent to the statement of Theorem 10.5 and is really proven in § 11.

Proof. Simply plug in $\mathcal{R}=\{(1^d),(2,1^{d-2})\}$ into Theorem 10.5. Then, $\sum_{R\in\mathcal{R}}\mathbb{L}^{-r(R)}=1+\mathbb{L}^{-1}$ and so

$$\prod_{x\in\mathbb{P}^1_k}(1-\mathbb{L}^{-1})\left(\sum_{R\in\mathcal{R}}\mathbb{L}^{-r(R)}\right)=\prod_{x\in\mathbb{P}^1_k}(1-\mathbb{L}^{-2})$$
$$=\prod_{x\in\mathbb{P}^1_k}(1-\mathbb{L}^{-2}t)|_{t=1}$$

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$$\begin{aligned}&= \prod_{x\in\mathbb{P}^1_k}(1-t)|_{t=\mathbb{L}^{-2}} \quad \text{by [Bil17, § 3.8, Property 4]}\\ &= \frac{1}{Z_{\mathbb{P}^1_k}(\mathbb{L}^{-2})} \quad \text{by [BH21, Ex. 6.1.12]}\\ &= (1-\mathbb{L}^{-1})(1-\mathbb{L}^{-2}).\end{aligned}$$

Therefore, modulo codimension $r_{d,g}$ in $\widehat{\widehat{K_0}}(\mathrm{Spaces}_k)$ when $d \neq 2$ (and in $K_0(\mathrm{Stacks}_k)$ when $d = 2$)

$$\begin{aligned}\{\mathrm{Hur}^{\mathcal{R}}_{d,g,k}\} &\equiv \frac{\mathbb{L}^{\dim \mathrm{Hur}_{d,g,k}}}{(1-\mathbb{L}^{-1})} \prod_{x\in\mathbb{P}^1_k}(1-\mathbb{L}^{-1})\left(\sum_{R\in\mathcal{R}}\mathbb{L}^{-r(R)}\right)\\ &= \frac{\mathbb{L}^{\dim \mathrm{Hur}_{d,g,k}}}{1-\mathbb{L}^{-1}}(1-\mathbb{L}^{-1})(1-\mathbb{L}^{-2})\\ &= \mathbb{L}^{\dim \mathrm{Hur}_{d,g,k}}(1-\mathbb{L}^{-2}). \qquad \square\end{aligned}$$

When we allow the ramification profile to be arbitrary in Theorem 10.5 we obtain the following corollary counting all degree d S_d Galois covers of $\mathbb{P}^1$. In the cases $d = 4$ and $d = 5$, there does not seem to be any obvious simplification of the motivic Euler product.

Corollary 10.7. *For k a field of characteristic not dividing $d!$,*

$$\mathrm{Hur}_{d,g,k} \equiv \begin{cases} \mathbb{L}^{\dim \mathrm{Hur}_{2,g,k}}(1-\mathbb{L}^{-2}) & \text{if } d=2,\\ \mathbb{L}^{\dim \mathrm{Hur}_{3,g,k}}(1+\mathbb{L}^{-1})(1-\mathbb{L}^{-3}) & \text{if } d=3,\\ \dfrac{\mathbb{L}^{\dim \mathrm{Hur}_{4,g,k}}}{(1-\mathbb{L}^{-1})}\displaystyle\prod_{x\in\mathbb{P}^1_k}(1+\mathbb{L}^{-2}-\mathbb{L}^{-3}-\mathbb{L}^{-4}) & \text{if } d=4,\\ \dfrac{\mathbb{L}^{\dim \mathrm{Hur}_{5,g,k}}}{(1-\mathbb{L}^{-1})}\displaystyle\prod_{x\in\mathbb{P}^1_k}(1+\mathbb{L}^{-2}-\mathbb{L}^{-4}-\mathbb{L}^{-5}) & \text{if } d=5,\end{cases}$$

in $\widehat{\widehat{K_0}}(\mathrm{Stacks}_k)$ modulo codimension $r_{d,g}$ if $d \neq 2$, and in $K_0(\mathrm{Stacks}_k)$ when $d = 2$.

Proof. The case $d = 2$ is already covered in Corollary 10.6, since $\mathrm{Hur}^{\{(1^2),(2)\}}_{2,g,k} = \mathrm{Hur}_{2,g,k}$. Taking

$$\mathcal{R} = \{(1^4), (2, 1^2), (3, 1), (2^2), (4)\}$$

the $d = 4$ case follows from plugging $\mathcal{R}$ into Theorem 10.5 and using $\mathrm{Hur}^{\mathcal{R}}_{4,g,k} = \mathrm{Hur}_{4,g,k}$. Taking $\mathcal{R} = \{(1^5), (2, 1^3), (2^2, 1), (3, 1^2), (3, 2), (4, 1), (5)\}$ the $d = 5$ case follows from plugging $\mathcal{R}$ into Theorem 10.5 and using $\mathrm{Hur}^{\mathcal{R}}_{5,g,k} = \mathrm{Hur}_{5,g,k}$. Finally, let us check the $d = 3$ case. Here, for $\mathcal{R} = \{(1^3), (2, 1), (3)\}$, we have $\mathrm{Hur}^{\mathcal{R}}_{3,g,k} = \mathrm{Hur}_{3,g,k}$. Thus, by Theorem 10.5, using [Bil17, § 3.8, Property 4] and by [BH21, Ex. 6.1.12] as in the proof of Corollary 10.6,

$$\begin{aligned}\{\mathrm{Hur}^{\mathcal{R}}_{d,g,k}\} &\equiv \frac{\mathbb{L}^{\dim \mathrm{Hur}_{d,g,k}}}{1-\mathbb{L}^{-1}} \prod_{x\in\mathbb{P}^1_k}(1-\mathbb{L}^{-1})\left(\sum_{R\in\mathcal{R}}\mathbb{L}^{-r(R)}\right)\\ &= \frac{\mathbb{L}^{\dim \mathrm{Hur}_{d,g,k}}}{1-\mathbb{L}^{-1}} \prod_{x\in\mathbb{P}^1_k}(1-\mathbb{L}^{-1})(1+\mathbb{L}^{-1}+\mathbb{L}^{-2})\end{aligned}$$

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$$= \frac{\mathbb{L}^{\dim \mathrm{Hur}_{d,g,k}}}{1-\mathbb{L}^{-1}} \prod_{x \in \mathbb{P}^1_k} (1-\mathbb{L}^{-3})$$
$$= \frac{\mathbb{L}^{\dim \mathrm{Hur}_{d,g,k}}}{1-\mathbb{L}^{-1}} \frac{1}{Z_{\mathbb{P}^1_k}(\mathbb{L}^{-3})}$$
$$= \frac{\mathbb{L}^{\dim \mathrm{Hur}_{d,g,k}}}{1-\mathbb{L}^{-1}} (1-\mathbb{L}^{-2})(1-\mathbb{L}^{-3})$$
$$= \mathbb{L}^{\dim \mathrm{Hur}_{d,g,k}}(1+\mathbb{L}^{-1})(1-\mathbb{L}^{-3}),$$

where we work in $\widehat{\widetilde{K_0}}(\mathrm{Stacks}_k)$ modulo codimension $r_{d,g}$. □

11. Degree 2

Following the notation introduced in [AV04], let $\mathbb{A}_{sm}(1,n) \subset \mathrm{Spec}(\mathrm{Sym}^\bullet H^0(\mathbb{P}^1, \mathscr{O}(n)^\vee))$ denote the open subscheme parameterizing those degree n forms on $\mathbb{P}^1$ whose associated closed subscheme is reduced.

Lemma 11.1. *For k a field with $\mathrm{char}\, k \neq 2$, there is an isomorphism of stacks $\mathrm{Hur}_{2,g,k} \simeq [\mathbb{A}_{sm}(1,2g+2)/\mathbb{G}_m]$, for an appropriate action of $\mathbb{G}_m$ on $\mathbb{A}_{sm}(1,2g+2)$.*

Remark 11.2. This can be deduced from the proofs of [AV04, Theorem 4.1, Corollary 4.7], though there the authors work with a further quotient by the PGL_2 action on the base $\mathbb{P}^1$. The $\mathbb{G}_m$ action on $\mathbb{A}_{sm}(1,n)$ in Lemma 11.1 is explicitly given by $\alpha \cdot f(x) = \alpha^{-2} f(x)$, though we will not need this in what follows.

Proof. First, we verify that $\mathrm{Hur}_{2,g,k}$ is equivalent to the fibered category whose S-points parameterize pairs $(\mathscr{L}, i : \mathscr{L}^{\otimes 2} \to \mathscr{O}_{\mathbb{P}^1_S})$, for $\mathscr{L}$ a degree-$(-g-1)$ invertible sheaf on $\mathbb{P}^1_S$, and i an injective homomorphism of sheaves. Indeed, to connect this to our given definition of $\mathrm{Hur}_{2,g,k}$, we follow [AV04, Remark 3.3 and Proposition 3.1]: given a cover $\rho : H \to \mathbb{P}^1_S$, we have a natural action of μ_2 on H over $\mathbb{P}^1$. This comes from the isomorphism $\mu_2 \simeq \mathbb{Z}/2\mathbb{Z}$ as we are assuming $\mathrm{char}(k) \neq 2$. From this action, we obtain an isomorphism $\rho_* \mathscr{O}_H \simeq \mathscr{O}_{\mathbb{P}^1_S} \oplus \mathscr{L}$, for $\mathscr{L}$ the subsheaf on which μ_2 acts by $(t,s) \mapsto t \cdot s$, i.e. $\mathscr{L}$ is the weight-1 eigenspace of μ_2, and $\mathscr{O}_{\mathbb{P}^1_S}$ is the weight-0 eigenspace. The description of $\mathscr{L}$ as the 1 eigenspace for the μ_2 action yields a map $i : \mathscr{L} \otimes \mathscr{L} \to \mathscr{O}$. In the other direction, given $(\mathscr{L}, i : \mathscr{L}^{\otimes 2} \to \mathscr{O}_{\mathbb{P}^1_S})$, we can recover $H = \mathrm{Spec}_{\mathscr{O}_{\mathbb{P}^1_S}}(\mathscr{O}_{\mathbb{P}^1_S} \oplus \mathscr{L})$. The given maps respect automorphisms over $\mathbb{P}^1$, as the only nontrivial automorphism in both cases is given by the hyperelliptic involution. Hence, they define an equivalence of algebraic stacks.

Next, consider the cover $\widetilde{\mathrm{Hur}}_{2,g,k}$ of $\mathrm{Hur}_{2,g,k}$ given as the stackification of the fibered category whose S points parameterize triples $(\mathscr{L}, \phi : \mathscr{L} \simeq \mathscr{O}(-g-1), i : \mathscr{L}^{\otimes 2} \to \mathscr{O})$, with i injective. Note that $\widetilde{\mathrm{Hur}}_{2,g,k} \to \mathrm{Hur}_{2,g,k}$ is indeed surjective because $\mathscr{L} \simeq \mathscr{E}^\vee$ is a degree-$(-g-1)$ line bundle on $\mathbb{P}^1$ by Lemma 6.3. Observe that $\widetilde{\mathrm{Hur}}_{2,g,k}$ has a natural action of $\mathbb{G}_m$ acting by automorphisms of $\mathscr{L}$, so that $\mathrm{Hur}_{d,g,k} = [\widetilde{\mathrm{Hur}}_{d,g,k}/\mathbb{G}_m]$. Said another way, quotienting by $\mathbb{G}_m$ forgets the data of the isomorphism ϕ.

It remains to identify $\widetilde{\mathrm{Hur}}_{2,g,k}$ with $\mathbb{A}_{sm}(1,2g+2)$. Indeed, this was done in the course of the proof of [AV04, Theorem 4.1]. Briefly, given an S-point $(\mathscr{L}, \phi, i)$, associate the map $i \circ (\phi^{-1})^{\otimes 2} : \mathscr{O}_{\mathbb{P}^1_S}(-2g-2) \to \mathscr{O}_{\mathbb{P}^1_S}$ corresponding to a section of $H^0(\mathbb{P}^1_S, \mathscr{O}(2g+2))$. Conversely, given a

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section $f \in H^0(\mathbb{P}^1_S, \mathscr{O}(2g+2))$, associate the triple $(\mathscr{O}(-g-1), \mathrm{id} : \mathscr{O}(-g-1) \to \mathscr{O}(-g-1), f : \mathscr{O}(-g-1)^{\otimes 2} \to \mathscr{O})$. □

We are now ready to prove Theorem 10.5 in the case $d = 2$.

11.3 Proof of $d = 2$ case of Theorem 10.5

Note that the only allowable collection of ramification profiles is $\mathcal{R} = \{(2), (1,1)\}$. Since $[\mathrm{Hur}_{2,g,k} \simeq \mathbb{A}_{sm}(1, 2g+2)/\mathbb{G}_m]$ by Lemma 11.1, and $\mathbb{G}_m$ is special, we have $\{\mathrm{Hur}_{2,g,k}\}\{\mathbb{G}_m\} = \mathbb{A}_{sm}(1, 2g+2)$. Since

$$\{\mathbb{G}_m\}\frac{\mathbb{L}^{\dim \mathrm{Hur}_{2,g,k}}}{1-\mathbb{L}^{-1}}\left(\prod_{x\in\mathbb{P}^1_k}\left(\sum_{R\in\mathcal{R}}\mathbb{L}^{-r(R)}\right)(1-\mathbb{L}^{-1})\right) = \frac{\mathbb{L}-1}{1-\mathbb{L}^{-1}}\cdot\mathbb{L}^{2g+2}\prod_{x\in\mathbb{P}^1_k}(1-\mathbb{L}^{-2})$$
$$= \mathbb{L}^{2g+3}\frac{1}{Z_{\mathbb{P}^1_k}(\mathbb{L}^{-2})}$$
$$= \mathbb{L}^{2g+3}(1-\mathbb{L}^{-1})(1-\mathbb{L}^{-2}),$$

(by [BH21, Ex. 6.1.12] and [Bil17, § 3.8, Property 4], as in the proof of Corollary 10.6) it suffices to verify

$$\mathbb{A}_{sm}(1, 2g+2) = \mathbb{L}^{2g+3}(1-\mathbb{L}^{-1})(1-\mathbb{L}^{-2}).$$

Indeed, this follows from [VW15, Lemma 5.9(a)]. In a bit more detail, taking $a = 2$ in [VW15, Lemma 5.9(a)], the expression $K_{<2}(t)$ there is the generating function for which the coefficient of t^n is the class of w_{1^n} in the notation of [VW15, (5.1)]. Here, w_{1^a} is the class of the space of degree n reduced divisors on $\mathbb{P}^1$. Therefore, $w_{1^n} = \{[\mathbb{A}_{sm}(1,n)/\mathbb{G}_m]\}$, and so we only need check $\{w_{1^n}\} = \mathbb{L}^n - \mathbb{L}^{n-2}$. But, indeed, this is the coefficient of t^n in the expansion of

$$\frac{Z_{\mathbb{P}^1}(t)}{Z_{\mathbb{P}^1}(t^2)} = \frac{(1-t^2\mathbb{L})(1-t^2)}{(1-t\mathbb{L})(1-t)} = (1-t^2\mathbb{L})(1+t)\left(\sum_{i=0}^{\infty}(t\mathbb{L})^i\right).$$

Remark 11.4. The construction above used to compute the class of $\mathrm{Hur}_{2,g,k}$ is admittedly fairly *ad hoc* in the context of this paper. A similar construction, more in line with the themes of this paper could be obtained by realizing a given hyperelliptic curve $\rho : H \to \mathbb{P}^1$ as a subscheme of $\mathbb{P}((\rho_*\mathscr{O}_H)^\vee)$. One can verify that $\mathbb{P}(\rho_*\mathscr{O}_H^\vee)$ is fppf locally isomorphic to $\mathbb{P}(\mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(g+1))$, and use this to deduce that $\mathrm{Hur}_{2,g,k}$ is the quotient of the smooth members of a certain linear series on $\mathbb{P}\big(\mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(g+1)\big)$ by the automorphisms of $\mathbb{P}(\mathscr{O}_{\mathbb{P}^1_k} \oplus \mathscr{O}_{\mathbb{P}^1_k}(g+1))$ preserving the projection to $\mathbb{P}^1_k$, and then use this description to compute $\{\mathrm{Hur}_{2,g,k}\}$, obtaining a formula similar to that of Theorem 10.5. However, such a proof would only calculate the class in $\widehat{\widehat{K_0}}(\mathrm{Stacks}_k)$ modulo a certain codimension, as opposed to the proof we give here, which actually calculates the class in $K_0(\mathrm{Stacks}_k)$.

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Appendix A. A proof of a theorem of Ekedahl

Aaron Landesman and Federico Scavia

The main result of this appendix is a proof of the following Theorem of T. Ekedahl. We retain the notation for the Grothendieck ring of stacks described in § 1.20.

THEOREM A.1 (Ekedahl [Eke09b, Theorem 4.3]). *Let k be a field. Then, for all integers $n \geq 1$, $\{BS_n\} = 1$ in $K_0(\mathrm{Stacks}_k)$.*

Unfortunately, Ekedahl passed away prior to publishing [Eke09b], and so the article was never refereed. There are a number of typos and errors appearing in the proof of [Eke09b, Theorem 4.3]. The objective of this appendix is to point out the fixes necessary.

Let k be a field, let G be a finite group, and let V be a G-representation of dimension $d \geq 0$ over k. If H is a subgroup of G, we denote by V^H the subscheme of V fixed by H, and by V_H the locally closed subscheme parameterizing the locus whose stabilizer is exactly H. If there is a point of V whose stabilizer is exactly H, we call H a *stabilizer subgroup* of G. The normalizer $N_G(H)$ of H acts on V^H and V_H, and V_H is an open subscheme of V^H. By definition, a *stabilizer flag* of length n is a sequence

$$f = (\{e\} =: H_0 \subset H_1 \subset \cdots \subset H_n)$$

of subgroups of G such that, for all $0 \leq i \leq n-1$, H_{i+1} is a stabilizer subgroup of the G-action on V. We say that f is *strict* if $H_i \subsetneq H_{i+1}$ for all i. We set $n_f := n$, $H_f := H_n$, $d_f := \dim V^{H_f}$ and $N_G(f) := \cap_{0 \leq i \leq n} N_G(H_i)$.

Remark A.2. Our definition of stabilizer flag differs from that used by Ekedahl [Eke09b, p. 10], as he required that H_{i+1} be a stabilizer subgroup of the action of $\cap_{j \leq i} N_G(H_i)$ on V^{H_i}. In particular, in our definition it is not necessarily true that $H_f \subset N_G(f)$.

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The conjugation action of G on itself induces a G-action on the collection of all stabilizer flags. We say that two stabilizer flags are *conjugate* to each other if they belong to the same orbit under this action.

Proposition A.3. *Let $K \subset G$ be the kernel of the G-action on V. We have*

$$\{BG\}\mathbb{L}^d = \{[V_K/G]\} - \sum_f (-1)^{n_f} \{BN_G(f)\}\mathbb{L}^{d_f}, \tag{A.1}$$

where f runs over a set of representatives of conjugacy classes of strict stabilizer flags of length $n_f \geq 1$.

Proposition A.3 corrects [Eke09b, Theorem 3.4]. The formula there looks the same as ours (up to signs), but it is wrong as it is claimed with a different definition of stabilizer flag. The error there stems from the falsity of [Eke09b, Lemma 3.3(iv)], as illustrated by Example A.4 below.

We note that the proof of Proposition A.3 follows similar lines to that of [Eke09b, Theorem 3.4]. In particular, it uses results from [Eke09b, Lemma 3.3(i), (ii), and (iii)] even though [Eke09b, Lemma 3.3(iv)] is incorrect. It may be helpful for the reader to consult these statements.

Example A.4. The result [Eke09b, Lemma 3.3(iv)] claims that $V_H = (V^H)_H$, where V^H is considered as an $N_G(H)$ representation. However, when $G = S_3$ and H is the subgroup generated by (12), and G acts as the three-dimensional permutation representation, then $V_H = \{(a, a, b) : a \neq b\}$, while $N_G(H) = H$ and $V^H = \{(a, a, b)\}$. Thus, here, when V^H is considered as an $N_G(H) = H$ representation, we have that H acts trivially and $(V^H)_H = V^H \neq V_H$.

Since [Eke09b, Lemma 3.3(iv)] is implicitly used in the proof of [Eke09b, Theorem 3.4], [Eke09b, Theorem 3.4] is also incorrect. To produce a counterexample to the statement of [Eke09b, Theorem 3.4] (even after correcting the + sign appearing in the statement there to the − sign of (A.1)), we can again take $G = S_3$. Then, the only strict stabilizer flags in the sense of [Eke09b, p. 10] (which are defined in a slightly different way than in this appendix) up to conjugacy are $\{e\}, \{e\} \subset S_2, \{e\} \subset S_3$. In this case, with our knowledge that $\{BS_3\} = 1$, the formula of [Eke09b, Theorem 3.4] claims $\mathbb{L}^3 = (\mathbb{L}^3 - \mathbb{L}^2) + (\mathbb{L}^2) + (\mathbb{L})$. Of course, what is missing from this formula is that we should subtract off a term $\mathbb{L}$ coming from the sequence of subgroups $\{e\} \subset S_2 \subset S_3$, which is a stabilizer flag in the sense of this appendix, but not in the sense of [Eke09b, p. 10].

Proof of Proposition A.3. Let f be a strict stabilizer flag. We have

$$[V^{H_f}/N_G(f)] - [V_{H_f}/N_G(f)] = \left[\coprod_{\substack{H \subset G \\ g \in N_G(f)/(N_G(f) \cap N_G(H))}} V_{gHg^{-1}}/N_G(f) \right]$$

where, on the right-hand side, H runs among a set of representatives of $N_G(f)$-conjugacy classes of subgroups of G acting on V and properly containing H_f.

For any fixed $H \subset G$, we have

$$\left[\coprod_{g \in N_G(f)/(N_G(f) \cap N_G(H))} V_{gHg^{-1}}/N_G(f) \right] = [V_H/N_G(f) \cap N_G(H)].$$

For any such H, construct a strict stabilizer flag f' by appending H at the end of f. Then

$$N_G(f) \cap N_G(H) = N_G(f').$$

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We conclude

$$\{[V_{H_f}/N_G(f)]\} = \{BN_G(f)\}\mathbb{L}^{d_f} - \sum_{f'}\{[V_{H_{f'}}/N_G(f')]\}, \tag{A.2}$$

where f' runs over a set of representatives of conjugacy classes of strict stabilizer flags of length $n_f + 1$ and starting with f.

We now wish to prove by induction on $m \geq 1$ that

$$\{BG\}\mathbb{L}^d = \{[V_K/G]\} - \sum_{0<n_f<m}(-1)^{n_f}\{BN_G(f)\}\mathbb{L}^{d_f} - (-1)^m\sum_{n_f=m}\{[V_{H_f}/N_G(f)]\}, \tag{A.3}$$

where f runs among a set of representatives of conjugacy classes of strict stabilizer flags. When $m = 1$, (A.3) coincides with (A.2) for $f = (K)$. Assume now that (A.3) holds for some $m > 1$. One obtains the formula for $m + 1$, by starting from the formula for m and applying (A.2) to every flag f of length m.

Since G is finite, there are only finitely many strict stabilizer flags. The conclusion follows by choosing m to be larger than the length of every strict stabilizer flag. □

Having replaced [Eke09b, Theorem 3.4] by Proposition A.3, the proof of Theorem A.1 can be completed as in [Eke09b]. From now on, let $G = S_n$ be the group of permutations of $\Sigma := \{1, 2, \ldots, n\}$, and let V be the n-dimensional permutation representation of S_n.

A *flag* is a pair (S, R), where S is a finite set, and R is a sequence $R_1 \subset R_2 \subset \cdots \subset R_n$ of equivalence relations $R_i \subset S \times S$ on S. An isomorphism of flags $(S', R') \to (S, R)$ is a bijection $S' \xrightarrow{\sim} S$ sending R'_i to R_i for all i. We denote by $N_R(S)$ the automorphism group of (S, R).

Lemma A.5. *Assume that $G = S_n$ and that V is the standard n-dimensional representation of S_n. Let f be a strict stabilizer flag, and denote by H_i the stabilizer subgroups appearing in f. For every i, let R_i be the equivalence relation determined by the orbit partition of the H_i-action on Σ, and let R be the flag on Σ given by the R_i.*

(a) *We have $N_{S_n}(f) = N_R(\Sigma)$.*
(b) *If $N_{S_n}(f) = S_n$, then either $f = (\{e\})$ or $f = (\{e\} \subset S_n)$.*
(c) *Assume that $\{BS_m\} = 1$ for all $m < n$ and that $N_{S_n}(f) \neq S_n$. Then $\{BN_{S_n}(f)\} = 1$.*

Proof. (a) This follows from the fact that, for every i, a bijection σ of Σ respects R_i if and only if it normalizes H_i.

(b) If $N_{S_n}(f) = S_n$, then for every i, R_i is respected by every bijection of Σ. It follows that either R_i is the diagonal in $\Sigma \times \Sigma$ or $R_i = \Sigma \times \Sigma$. Now part (b) follows from part (a).

(c) We may assume that $\{BN_{S_n}(f')\} = 1$ for all flags such that $n_{f'} < n_f$. By [Eke09b, Proposition 4.2], $N_{S_n}(f)$ is a direct product of wreath products $N' \wr S_r := (N')^r \rtimes S_r$, where N' is the normalizer of a flag of smaller length, and S_r acts by permutation of the r factors N'.

In what follows, we use the symbol Symm for the stacky symmetric power as introduced in [Eke09b, p. 5]. We also use the symbol $\wr$ for wreath product. This was introduced and notated $\int$ in [Eke09b, p. 5], but we use $\wr$ instead of $\int$ in order to keep our notation consistent with the rest of the paper.

Because, for G and H finite groups, $B(G \times H) \simeq BG \times BH$, it suffices to show $\{B(N' \wr S_r)\} = 1$. We have $B(N' \wr S_r) \simeq BN' \wr BS_r \simeq \mathrm{Symm}^r(BN')$, as explained in [Eke09b, p. 5], By inductive assumption, $\{B(N' \wr S_r)\} = \sigma_s^t(\{BN'\}) = \sigma_s^t(1) = 1$. For the symbol σ_s^t, see [Eke09b, Proposition 2.5]. □

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Proof of Theorem A.1. Let V be the n-dimensional permutation representation of S_n over k, and let $U := V_{\{e\}} \subset V$ be the free locus of the S_n-action. By Proposition A.3,

$$\{BS_n\}\mathbb{L}^n = \{U/S_n\} - \sum_f (-1)^{n_f}\{BN_{S_n}(f)\}\mathbb{L}^{d_f},$$

where f runs among conjugacy classes of strict stabilizer flags. By Lemma A.5(b), we may rewrite this as

$$\{BS_n\}(\mathbb{L}^n - \mathbb{L}) = \{U/S_n\} - \sum_f (-1)^{n_f}\{BN_{S_n}(f)\}\mathbb{L}^{d_f},$$

where now f runs among conjugacy classes of strict stabilizer flags such that $N_{S_n}(f) \neq S_n$. By Lemma A.5(c), we have $\{BN_{S_n}(f)\} = 1$ for all such f.

We claim that $\{U/S_n\}$ is a polynomial in $\mathbb{L}$ with integer coefficients. The stacks $V_H/N_{S_n}(H)$ are isomorphic to parts of a locally closed stratification of V/S_n. This is well known from general principles when $\operatorname{char} k = 0$ or when $\operatorname{char} k > 0$ does not divide n, but Ekedahl gave a proof in arbitrary characteristic in [Eke09b, Proposition 1.1(ii)].

To show $\{U/S_n\}$ is a polynomial in $\mathbb{L}$, let f be a strict stabilizer flag. Then, as in the proof of Proposition A.3, we have

$$\{V_{H_f}/N_{S_n}(f)\} = \{V^{H_f}/N_{S_n}(f)\} - \sum_{f'}\{V_{H_{f'}}/N_{S_n}(f')\},$$

where f' runs among conjugacy classes of strict stabilizer flags starting with f and of length $n_f + 1$.

Applying the previous formula iteratively, we obtain

$$\{V/S_n\} = \{U/S_n\} - \sum_f (-1)^{n_f}\{V^{H_f}/N_{S_n}(f)\},$$

where f runs among conjugacy classes of strict stabilizer flags of positive length. For every flag f, we claim that the quotient $W_f := N_{S_n}(f)/(H_f \cap N_{S_n}(f))$ is a product of symmetric groups, and V^{H_f} is a permutation representation of W_f. To see this, note that $N_{S_n}(f)$ can be identified with $N_{R_f}(\Sigma)$ via Lemma A.5 for a sequence of equivalence relations R_f given as $R_1 \subset R_2 \subset \cdots \subset R_{n_f}$. Under this identification, H_f is identified with the subgroup of permutations acting trivially on the equivalence classes defined by R_{n_f}. Therefore, the action of W_f on V^{H_f} is generated by permutations switching two equivalence classes of R_{n_f} for which there exists an isomorphism of those two classes respecting R. Therefore, W_f is a product of symmetric groups acting by a permutation representation on V^{H_f}. Hence, by the fundamental theorem for symmetric polynomials, $V^{H_f}/N_{S_n}(f) = V^{H_f}/W_f$ is an affine space over k. Since V/S_n is also isomorphic to affine space, we deduce that $\{U/S_n\}$ is a polynomial in $\mathbb{L}$, as claimed. We conclude that $\{BS_n\}$ can be written as a rational function in $\mathbb{L}$ with integer coefficients, and with denominator $\mathbb{L}^n - \mathbb{L}$. By [Eke09b, Lemma 3.5], this implies that $\{BS_n\} = 1$. □

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