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CHARACTERIZATION OF EIGENFUNCTIONS BY BOUNDEDNESS CONDITIONS

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ABSTRACT. Suppose $\{f_k(x)\}_{k=-\infty}^{\infty}$ is a sequence of functions on $\mathbb{R}^n$ with $\Delta f_k = f_{k+1}$ (where Δ is the Laplacian) that satisfies the growth condition: $|f_k(x)| \leq M_k(1 + |x|)^a$ where $a \geq 0$ and the constants have sublinear growth $\frac{M_k}{k} \longrightarrow 0$ as $k \longrightarrow \pm\infty$. Then $\Delta f_0 = -f_0$. This characterizes eigenfunctions f of Δ with polynomial growth in terms of the size of the powers $\Delta^k f$, $-\infty < k < \infty$. It also generalizes results of Roe (where $a = 0$, $M_k = M$, and $n = 1$) and Strichartz (where $a = 0$, $M_k = M$ for n). The analogue holds for formally self-adjoint constant coefficient linear partial differential operators on $\mathbb{R}^n$.

1. **Introduction.** Let $\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$ be the Laplacian operator on $\mathbb{R}^n$. Recently Strichartz [8] has given a characterization of the bounded solutions of $\Delta f = -f$ in terms of bounds on the iterates $\Delta^k f$, where $k \in \mathbb{Z}$; more precisely, if $\langle f_k \rangle_{k=-\infty}^{\infty}$ is a doubly infinite sequence of functions on $\mathbb{R}^n$ with $\Delta f_k = f_{k+1}$ and $|f_k(x)| \leq M$, for some M, then $\Delta f_0 = -f_0$. If $n = 1$, this is Roe's theorem [6]: If $\langle f_k \rangle_{k=-\infty}^{\infty}$ is a sequence of real-valued functions with $\frac{d}{dx}\big(f_k(x)\big) = f_{k+1}(x)$ and $|f_k(x)| \leq M$, then $f_0(x) = A\sin(x + \alpha)$. (See also the paper by Burkill [2] and the paper [3] for generalizations of Roe's theorem.)

This does not characterize all solutions of $\Delta f = -f$ on $\mathbb{R}^n$ because many are unbounded. For example, let $\mathcal{P}_j$ be the vector space of all complex-valued polynomials in x and y of degree at most j. For reasons of dimension the linear map $p \longmapsto p_{xx} + p_{yy} + 2ip_x$ from $\mathcal{P}_j$ to $\mathcal{P}_{j-1}$ has nontrivial kernel. Let p be a solution of degree j to $p_{xx} + p_{yy} + 2ip_x = 0$. Then $u(x, y) = p(x, y)e^{ix}$ satisfies $\Delta u = -u$ and has polynomial growth at infinity. In Fourier analysis the functions with polynomial growth are interesting because they are exactly the ones that can be viewed as tempered distributions (i.e., as elements of the dual of the space $\mathcal{S}(\mathbb{R}^n)$ of rapidly decreasing functions (see [4], Chapter 1)). As such they have Fourier transforms that are also tempered distributions. The following gives a characterization of solutions to $\Delta f = -f$ of (at most) polynomial growth in terms bounds on the power $\Delta^k f$ for $-\infty < k < \infty$.

THEOREM 1. *Let $a \geq 0$ and let $\langle f_k \rangle_{-\infty}^{\infty}$ be a sequence of complex-valued functions on $\mathbb{R}^n$ that satisfy*

$$\Delta f_k = f_{k+1}$$

and

$$|f_k(x)| \leq M_k(1 + |x|)^a$$

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where the constants M_k have sublinear growth:

$$\lim_{k\to\infty}\frac{M_k}{k}=\lim_{k\to\infty}\frac{M_{-k}}{k}=0 \tag{1.1}$$

Then $\Delta f_0=-f_0$.

Conversely if f is of polynomial growth $|f(x)|\le M(1+|x|)^a$ and satisfies $\Delta f=-f$, then $f_k=\Delta^k f=(-1)^k f$ satisfies $|f_k(x)|\le M(1+|x|)^a$. The growth condition (1.1) is as weak as possible if polynomial growth of the functions is to be allowed. For example (in one dimension) if $f_k(x)=(-1)^k(x-2ki)e^{ix}$ then $\frac{d^2}{dx^2}f_k=f_{k+1}$ and $|f_k(x)|\le(1+2|k|)(1+|x|)$ but $\frac{d^2}{dx^2}f_0\ne -f_0$. Even in the one dimensional case this gives the strengthening of Roe's theorem due to Burkill [2]: If $\langle f_k\rangle_{-\infty}^{\infty}$ satisfies $f_k'(x)=f_{k+1}(x)$ and $|f_k(x)|\le M(1+|x|)^a$, then $f_0(x)=A\sin(x+\alpha)$.

In Section 2 we extend Theorem 1 to any formally self-adjoint constant-coefficient differential operator. The proof has the flavor of Roe's original proof—using the growth conditions to show that the support of the Fourier transform of f_0 is contained in the unit sphere—but concluding that f_0 is an eigenfunction requires first showing that f_0 is a generalized eigenfunction of Δ. A nonzero function f is a *generalized eigenfunction* of Δ with eigenfunction λ if and only if $(\Delta-\lambda)^N f=0$ for some $N\ge 1$. In one dimension the generalized eigenfunctions of $\Delta=\frac{d^2}{dx^2}$ were characterized in [3] and [5]. In $\mathbb{R}^n$ the result is

THEOREM 2. *If in Theorem 1 the sublinear growth condition (1.1) is replaced by the subexponential growth condition*

$$\lim_{k\to\infty}\frac{M_k}{(1+\epsilon)^k}=\lim_{k\to\infty}\frac{M_{-k}}{(1+\epsilon)^k}=0,$$

for all $\epsilon>0$, then f_0 is a generalized eigenfunction of Δ with eigenvalue $\lambda=-1$.

We shall extend this to all formally self-adjoint constant-coefficient differential operators in Section 2 (see Theorem 4). Although the following is well known to experts, it seems to be interesting enough to record here.

COROLLARY. *A smooth function of polynomial growth is a generalized eigenfunction of Δ with eigenfunction -1 if and only if the support of its Fourier transform is contained in the unit sphere $\{\xi : |\xi|=1\}$.*

Finally, we note that for $n\ge 2$ there are many eigenfunctions of Δ than have greater than polynomial growth. For example (when $n=2$)

$$f(x,y)=e^{2ix+\sqrt{3}y}$$

satisfies $\Delta f=-f$ and has exponential growth. However, it seems unlikely that it is possible to characterise eigenfunctions of Δ in the class of functions of exponential growth: Let ϕ be any continuous function on $[-\delta,\delta]$ and set

$$f_k(x)=\int_{-\delta}^{\delta}(\sin\theta+i\cos\theta)^{2k}\epsilon^{(\sin\theta+i\cos\theta)x}\phi(\theta)\,d\theta.$$

Then $\Delta f_k = \frac{d^2 f_k}{dx^2} = f_{k+1}$ and

$$|f_k(x)| \leq \int_{-\delta}^{\delta} e^{x \sin\theta} |\phi(\theta)|\, d\theta \leq M e^{(\sin\delta)|x|},$$

for all x. But f_0 is not an eigenfunction.

2. **The general results.** Let $x_1, \ldots, x_n$ be the usual coordinates in $\mathbb{R}^n$ and $i^2 = -1$. Set

$$D_j = \frac{1}{i}\frac{\partial}{\partial x_j}$$

The factor of $\frac{1}{i}$ is included to make the operator D_j formally self-adjoint. For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$, let $\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$ and $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$. Let

$$P(\xi) = \sum_\alpha a_\alpha \xi^\alpha$$

be a polynomial in ξ and let

$$L = P(D) = \sum_\alpha a_\alpha D^\alpha \tag{2.1}$$

be the corresponding constant-coeffcient linear partial differential operator. If P is real-valued then L will be formally self-adjoint. We now state our main result.

THEOREM 3. *Suppose $P(\xi) = \sum_\alpha a_\alpha \xi^\alpha$ is real-valued and $L = P(D)$. Let $\langle f_k \rangle_{-\infty}^{\infty}$ be a sequence of complex-valued functions on $\mathbb{R}^n$ so that*

$$L f_k = f_{k+1}$$

and

$$|f_k(x)| \leq M_k (1 + |x|)^a, \tag{2.2}$$

where $\langle M_k \rangle_{-\infty}^{\infty}$ satisfies the sublinear growth condition

$$\lim_{k \to \pm\infty} \frac{M_{|k|}}{k} = 0. \tag{2.3}$$

Then $f = f_+ + f_-$, where $L f_+ = f_+$ and $L f_- = -f_-$. If 1 (or -1) is not in the range of P then $f_+ = 0$ (or $f_- = 0$).

If $P(\xi) = -|\xi|^2$ then $L = \Delta$. Then $f_0 = f_-$ which yields Theorem 1. For operators such as the d'Alembertian $\Box = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial x_n^2}$, we see that both $f_+ \neq 0$ and $f_- \neq 0$ are possible. (cf. Theorem 3.1 of [6]).

The theorem applies to a class of operators more general than differential operators. In $\mathcal{S}(\mathbb{R}^n)$, the Fourier transform and its inverse are given by

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int e^{-i\xi\cdot x} f(x)\, dx$$

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and

$$\check{f}(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int e^{ix\cdot\xi} f(\xi)\, d\xi, \tag{2.4}$$

By duality these definitions extend to the space of tempered distributions; i.e., the dual space $\mathcal{S}(\mathbb{R}^n)$. Then, for the operator in (2.1), we have

$$\widehat{Lf}(\xi) = P(\xi)\hat{f}(\xi). \tag{2.5}$$

This can be used to define an operator L on the space of tempered distributions even when $P(\xi)$ is not a polynomial. Such operators are called *multiplier operators* or *translation-invariant pseudo-differential operators*. For example convolution operators $Lf = \phi * f$ are of this type. We note that for our result to hold it suffices that $P(\xi)$ be smooth and that (for each multi-index α) there be numbers C and N with

$$|D^\alpha P(\xi)| \leq C(1 + |\xi|)^N.$$

Then L (defined by (2.5)) is a linear operator on the space of tempered distributions. Although many such functions exist, for example $P(\xi) = Ce^{-|\xi|^2}$, the theorem is most interesting when L is a differential operator.

To prove the theorem we first show that the support of $\hat{f}_0$ is contained in the set $\{\xi : |P(\xi)| = 1\}$. Formally from (2.5) and the Fourier inversion formula (2.4) we get

$$f_k(x) = L^k f_0(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int e^{ix\cdot\xi} P(\xi)^k \hat{f}_0(\xi)\, d\xi.$$

If this is to stay bounded (as k varies), the support of $\hat{f}_0$ must be contained in the set $\{\xi : |P(\xi)| = 1\}$. More precisely, we have

PROPOSITION. *(A) If a function f satisfies, for $k = 0, 1, 2, \ldots$*

$$|L^k f(x)| \leq M_k(1 + |x|)^a,$$

where the constants M_k satisfy the subexponential growth condition

$$\lim_{k\to\infty} \frac{M_k}{(1+\epsilon)^k} = 0. \tag{2.6}$$

for all $\epsilon > 0$, then

$$\operatorname{spt}(\hat{f}) \subseteq \{\xi : |P(\xi)| \leq 1\}.$$

(B) If $\langle f_k \rangle_{k=-\infty}^{0}$ is a sequence of functions with $Lf_k = f_{k+1}$, for $k \leq -1$,

$$|f_k(x)| \leq M_k(1 + |x|)^a,$$

and

$$\lim_{k\to\infty} \frac{M_{|k|}}{(1+\epsilon)^k} = 0,$$

for all $\epsilon > 0$, then

$$\operatorname{spt}(\hat{f}_0) \subseteq \{\xi : |P(\xi)| \geq 1\}.$$

Unlike in Theorem 3 the function P can be complex valued and the proposition will still hold. This proposition is very closely related to the results of Section 3 of Garbardo's paper [3]. He shows this is related to the Paley-Wiener-Schwartz theorem.

LEMMA. *Let Φ and ϕ be C^∞-functions with ϕ compactly supported. Assume $|\Phi(\xi)| \leq r < 1$, for all $\xi \in$ spt(ϕ). For any sequence $\langle M_k \rangle_{k=0}^\infty$ of constants satisfying the subexponential growth condition (2.6) and, for any multi-index α,*

$$\lim_{k \to \infty} M_k \|D^\alpha(\phi \Phi^k)\|_{L_2} = 0.$$

PROOF. By the product rule there are constants $C(\beta, \gamma) \geq 0$ so that

$$\begin{aligned}\|D^\alpha(\phi\Phi^k)\|_{L_2} &= \Big\| \sum_{\beta+\gamma=\alpha} C(\beta,\gamma) D^\gamma \phi D^\beta \Phi^k \Big\|_{L_2} \\ &\leq \sum_{\beta+\gamma=\alpha} C(\beta,\gamma) \|D^\gamma \phi D^\beta \Phi^k\|_{L_2}\end{aligned}$$

Since $\operatorname{spt}(D^\alpha \phi) \subseteq \operatorname{spt}(\phi)$, we may assume $|\Phi(\xi)| \leq r < 1$ on the support of $\psi = D^\gamma \phi$. It is (thus) enough to show:

$$\lim_{k \to \infty} M_k \|\psi D^\beta(\Phi^k)\|_{L_2} = 0.$$

Assume $k > |\beta| := \beta_1 + \cdots + \beta_n$. Writing $\Phi^k = \Phi \cdots \Phi$ and using the product rule gives a sum with $k^{|\beta|}$ terms. Each term is a product of k factors, at least $k - |\beta|$ of which are Φ. The other factors are of the form $D^\gamma \Phi$, where $0 \leq |\gamma| \leq |\beta|$. Setting

$$\Psi_\beta(x) = \Big(\max_{|\gamma| \leq |\beta|} |D^\gamma \Phi(x)| \Big)^{|\beta|}$$

and using $|\Phi(x)| \leq r$ on spt(ψ), we get

$$\begin{aligned}|\psi D^\beta(\Phi^k)| &\leq k^{|\beta|} |\psi(x)| \, |\Phi(x)|^{k-|\beta|} \Psi_\beta(x) \\ &\leq k^{|\beta|} r^{k-|\beta|} |\psi(x)| \Psi_\beta(x).\end{aligned}$$

Thus

$$M_k \|\psi D^\beta(\Phi)^k\|_{L_2} \leq \|\psi \Psi_\beta\|_{L_2} k^{|\beta|} M_k r^{k-|\beta|}.$$

The growth condition (2.6) implies that the right-hand side goes to zero as $k \longrightarrow \infty$. ■

To prove part (A) of the proposition, it suffices to show $\langle \hat{f}, \phi \rangle = 0$ if $\phi \in C_0^\infty(\mathbb{R}^n)$ and $\operatorname{spt}(\phi) \cap \{\xi : |P(\xi)| \leq 1\} = \emptyset$. Since spt($\phi$) is compact, there is some $r < 1$ so that $\frac{1}{|P(\xi)|} \leq r$, for all $\xi \in$ spt(ϕ). Then

$$\begin{aligned}\langle \hat{f}, \phi \rangle &= \Big\langle P^k \hat{f}, \frac{\phi}{P^k} \Big\rangle \\ &= \Big\langle \widehat{L^k f}, \frac{\phi}{P^k} \Big\rangle \\ &= \Big\langle L^k f, \Big(\frac{\phi}{P^k}\Big)^\wedge \Big\rangle\end{aligned}$$

Choose an integer m with $2m \geq 2a + n + 1$. A calculation, using the hypothesis of the proposition and the Cauchy-Schwartz inequality, implies

$$
\begin{aligned}
|\langle \hat{f}, \phi \rangle| &\leq \int |L^k f(x)| \Big| \Big(\frac{\phi}{P^k}\Big)^\wedge \Big| \, dx \\
&\leq M_k \int \frac{(1+|x|)^a}{(1+|x|^2)^{\frac{m}{2}}} (1+|x|^2)^{\frac{m}{2}} \Big| \Big(\frac{\phi}{P^k}\Big)^\wedge \Big| \, dx \\
&\leq M_k \Big(\int \frac{(1+|x|)^{2a}}{(1+|x|^2)^m} \, dx \Big)^{\frac{1}{2}} \Big(\int (1+|x|^2)^m \Big| \Big(\frac{\phi}{P^k}\Big)^\wedge \Big|^2 \, dx \Big)^{\frac{1}{2}} \\
&= M_k C_1(a,m,n) \Big(\int (1+|x|^2)^m \Big| \Big(\frac{\phi}{P^k}\Big)^\wedge \Big|^2 \, dx \Big)^{\frac{1}{2}}.
\end{aligned}
$$

By a standard estimate (cf. [4], Chapter 1), there is a constant $C_2(m,n)$ with

$$\Big(\int (1+|x|^2)^m |\hat{f}(x)|^2 \, dx \Big)^{\frac{1}{2}} \leq C_2 \sum_{|\alpha| \leq m} \| D^\alpha f \|_{L_2}.$$

Using this in the above leads to

$$|\langle \hat{f}, \phi \rangle| \leq C_3(m,n,a) M_k \sum_{|\alpha| \leq m} \Big\| D^\alpha \Big(\frac{\phi}{P^k}\Big) \Big\|_{L_2}.$$

By the lemma the right-hand side of this goes to zero as $k \longrightarrow \infty$, and so $\langle \hat{f}, \phi \rangle = 0$. This completes the proof of part (A); part (B) is similar. Let $\phi \in C_0^\infty(\mathbb{R}^n)$ so that $\mathrm{spt}(\phi) \cap \{\xi : |P(\xi)| \geq 1\} = \emptyset$. We shall show that $\langle \hat{f}_0, \phi \rangle = 0$. Then, for some $r < 1$, the inequality $|P(\xi)| \leq r$ holds for all ξ in $\mathrm{spt}(\phi)$. Thus

$$
\begin{aligned}
\langle \hat{f}_0, \phi \rangle &= \langle \widehat{L^k f_{-k}}, \phi \rangle \\
&= \langle P^k \widehat{f_{-k}}, \phi \rangle \\
&= \langle f_{-k}, \widehat{P^k \phi} \rangle .
\end{aligned}
$$

The rest follows as in part (A).

We now prove Theorem 3. We first assume that -1 is not a value of $P(\xi)$, and show that $Lf_0 = f_0$. Let $S = \{\xi : P(\xi) = 1\}$. That $\mathrm{spt}(\hat{f}_0) \subseteq S$ follows from the growth conditions on the sequence $\langle f_k \rangle_{-\infty}^{\infty}$, the proposition, and the assumption that $P(\xi) \neq -1$.

The topology on the space $\mathcal{S}(\mathbb{R}^n)$ is defined by the seminorms

$$\|\phi\|_{N,m} = \sup_x \sum_{|\alpha| \leq N} (1+|x|)^m |D^\alpha \phi(x)|.$$

Therefore, since $\hat{f}_0$ is a continuous linear functional on $\mathcal{S}(\mathbb{R}^n)$, there is a constant C and integers m and N so that

$$|\langle \hat{f}_0, \phi \rangle| \leq C \|\phi\|_{N,m}, \tag{2.7}$$

for all $\phi \in S(\mathbb{R}^n)$. Therefore (as a distribution) $\hat{f}_0$ is of order $\leq N$.

CLAIM. *For this N,*

$$(P-1)^{N+1}\hat{f}_0 = 0. \tag{2.8}$$

To simplify notation set $h := (P-1)$. Then we need to show, for any compactly supported C^∞ function ϕ, that

$$\langle h^{N+1}\hat{f}_0, \phi \rangle := \langle \hat{f}_0, h^{N+1}\phi \rangle = 0.$$

Let $g\colon \mathbb{R} \longrightarrow [0,1]$ be a C^∞ function with $g = 1$ on $[-1/2, 1/2]$ and $g = 0$ outside $(-1,1)$. Set

$$g_r(t) := g\Big(\frac{t}{r}\Big).$$

Letting $B = \max\{ |g^{(k)}(t)| : t \in [-1,1], k \leq N\}$, we have

$$|g_r^{(k)}(t)| \leq \frac{B}{r^k} \leq \frac{B}{|t|^k}, \tag{2.9}$$

for all $k \leq N$. Set

$$H_r = g_r(h)h^{N+1}\phi.$$

Then $H_r = h^{N+1}\phi$ in a neighborhood of $\{\xi : h(\xi) = 0\} = \{\xi : P(\xi) = 1\} \supseteq \operatorname{spt}\hat{f}_0$. Thus by (2.7) we have

$$|\langle \hat{f}_0, h^{N+1}\phi \rangle| = |\langle \hat{f}_0, H_r \rangle| \leq C\|H_r\|_{N,m}.$$

To verify (2.8), it suffices to demonstrate $\|H_r\|_{N,m} \longrightarrow 0$ as $r \longrightarrow 0$. Write D^k for any D^α with $|\alpha| = k$ and $(Dh)^k$ for a product $D_{j_1}h \cdots D_{j_k}h$ of k first order partial derivatives of h. Then (ignoring factors of i) and assuming $k \leq N$

$$D^k(H_r) = D^k\big(g_r(h)h^{N+1}\phi\big)$$

is a sum of terms of the form

$$T_r(k_1, \ldots, k_{l+1}) = g_r^{(k_1)}(h)(Dh)^{k_1}(D^{k_2}h)\cdots(D^{k_l}h)h^{N+1-k_2-\cdots-k_l}D^{k_{l+1}}\phi$$

where $k_1 + \cdots + k_{l+1} = k$. Since the support of ϕ is compact, there is a bound K so that, for all $x \in \operatorname{spt}(\phi)$,

$$(1+|x|)^m|D^\alpha\phi(x)| \leq K \text{ and } |D^\alpha h(x)| \leq K,$$

whenever $|\alpha| \leq N$. Since $|h| \leq r$ on the support of H_r, the inequality (2.9) implies that on the support of $T_r(k_1, \ldots, k_{l+1})$,

$$\begin{aligned}(1+|x|)^m|T_r(k_1, \ldots, k_{l+1})| &\leq \frac{B}{|h|^{k_1}}|h|^{N+1-k_2-\cdots-k_l}KK^{k_{l-1}}K^{k_1} \\ &= B|h|^{N+1-k+k_{l+1}}K^{l+k_1} \\ &\leq BK^{l+k_1}r^{N+1-k+k_{l+1}}.\end{aligned}$$

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But $k \leq N$ so this goes to zero as $r \longrightarrow 0$. The sum defining $\|H_r\|_{N,m}$ is a finite sum of terms of this type and so $\|H_r\|_{N,m} \longrightarrow 0$ as $r \longrightarrow 0$. This completes the proof of the claim.

Inverting the Fourier transform (2.8) yields that

$$(L-1)^{N+1} f_0 = 0. \tag{2.10}$$

This equation implies

$$\operatorname{span}\{f_0, f_1, f_2, \ldots\} = \operatorname{span}\{f_0, Lf_0, L^2 f_0, \ldots\} = \operatorname{span}\{f_0, \ldots, L^N f_0\}.$$

We shall now show that we can take $N = 0$ in (2.10). If not then $(L-1)f_0 \neq 0$. Let K be the largest positive integer so that $(L-1)^K f \neq 0$. Clearly $K \leq N$. Thus

$$f := (L-1)^{K-1} f_0 \in \operatorname{span}\{f_0, \ldots, f_N\}$$

will satisfy

$$(L-1)^2 f = 0 \text{ and } (L-1)f \neq 0. \tag{2.11}$$

Write

$$f = a_0 f_0 + \cdots + a_N f_N,$$

for constants $a_0, \ldots, a_N$. Then

$$L^k f = a_0 f_k + \cdots + a_N f_{N+k}.$$

If $C_k = |a_0| M_k + \cdots + |a_N| M_{k+N}$, then this and (2.2) imply

$$|(L^k f)(x)| \leq C_k (1 + |x|)^a. \tag{2.12}$$

By (2.3) these satisfy the sublinear growth condition

$$\lim_{k \to \infty} \frac{C_k}{k} = 0. \tag{2.13}$$

An induction using (2.11) implies for $k \geq 2$ that

$$L^k f = kLf - (k-1)f = k(L-1)f + f.$$

$$\left|\big((L-1)f\big)(x)\right| \leq \frac{1}{k} |(L^k f)(x)| + \frac{|f(x)|}{k} \leq \frac{C_k}{k}(1 + |x|)^a + \frac{|f(x)|}{k}$$

Letting $k \longrightarrow \infty$ and using (2.13) implies $(L-1)f = 0$. But this contradicts (2.11). Consequently, $N = 0$ in (2.10). This completes the proof in the case that -1 is not in the range of P.

In the case that $+1$ is not in the range of P we apply the same argument to $-L$ to conclude $Lf_0 = -f_0$. In the general case, let $L_0 = L^2$. Then $\widehat{L_0 f}(\xi) = P(\xi)^2 \hat{f}(\xi)$. $L_0 f_{2k} = f_{2(k+1)}$ and $P(\xi)^2 \neq -1$. Thus we can (as before) conclude, for the sequence $\langle f_{2k} \rangle_{k=-\infty}^{\infty}$ that

$$L_0 f_0 = L^2 f_0 = f_0.$$

Set $f_+ = \frac{1}{2}(f_0 + Lf_0)$ and $f_- = \frac{1}{2}(f_0 - Lf_0)$. Then $f = f_+ + f_-$, $Lf_+ = f_+$, and $Lf_- = -f_-$. This completes the proof of Theorem 3.

THEOREM 4. *If, in Theorem 3, we replace (2.3) with*

$$\lim_{k\to\pm\infty} \frac{M_{|k|}}{(1+\epsilon)^k} = 0, \tag{2.14}$$

for all $k > 0$, then the span of $\langle f_k \rangle$ is finite dimensional. Moreover, $f_0 = f_+ + f_-$, where, for some integer N, $(L-1)^N f_+ = 0$ and $(L+1)^N f_- = 0$. Thus f_+ (or f_-) is a generalized eigenfunction of L with eigenvalue $+1$ (or -1).

The proof will be based on the following result from linear algebra.

LEMMA. *Let X be a finite dimensional complex vector space, and let $A\colon X \longrightarrow X$ be a linear map with eigenvalues $\lambda_1, \dots, \lambda_p$. Then $X = X_1 \oplus \cdots \oplus X_p$, where $X_j = \ker\big((A-\lambda_j)^N\big)$ and $N = \dim X$.*

This can be deduced from the Jordan normal form. (cf. [1], Chapter 10.)

We first prove Theorem 4 under the assumption that $P(\xi) \neq -1$. Using the growth condition (2.14) and the proposition, we may still conclude that $\operatorname{spt}(\hat{f}_0) \subseteq S = \{\xi : P(\xi) = 1\}$. But then, as before, we can conclude that (2.10) holds. But this is enough to complete the proof in this case. A similar argument shows that if $P(\xi) \neq 1$, then $(L+1)^N f_0 = 0$.

In the general case we again let $L_0 = L^2$ and $P_0 = P^2$. Then $P_0(\xi) \neq -1$ and the span of $\langle f_{2k} \rangle$ is finite dimensional. The map L takes the span of $\langle f_{2k} \rangle$ onto the span of $\langle f_{2k+1} \rangle$. Thus X is finite dimensional. Any $f \in X$ will have $\operatorname{spt}(f)$ inside the set defined by $P(\xi) = \pm 1$. From this it is not hard to show the only possible eigenvalues of L restricted to X are $+1$ and -1. The result now follows from the last lemma.

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REFERENCES

1. G. Birkoff and S. MacLane, *A Survey of Modern Algebra*, MacMillan, New York, 1965.
2. H. Burkhill, *Sequences characterizing the sine function*, Math. Proc. Camb. Phil. Soc. **89**(1981), 71–77.
3. Jean-Pierre Gabardo, *Tempered distributions with spectral gaps*, Math. Proc. Camb. Phil. Soc. **106**(1989), 143–162.
4. L. Hormander, *Linear Partial Differential Operators*, Springer-Verlag, New York, 1969.
5. R. Howard, *A note on Roe's characterization of the sine function*, Proc. Amer. Math. Soc. **105**(1989), 658–663.
6. J. Roe, *A characterization of the sine function*, Math. Proc. Camb. Phil. Soc. **87**(1980), 69–73.

7. W. Rudin, *Functional Analysis*, McGraw-Hill, New York, 1973.
8. R. S. Strichartz, *Characterization of eigenfunctions of the Laplacian by boundedness conditions*, Trans. Amer. Math. Soc. (to appear).

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