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A MARTINGALE CHARACTERIZATION OF PÓLYA–LUNDBERG PROCESSES

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Abstract

We study exponential families within the class of counting processes and show that a mixed Poisson process belongs to an exponential family if and only if it is either a Poisson process or has a gamma structure distribution. This property can be expressed via exponential martingales.

Keywords: Mixed Poisson process; Pólya–Lundberg process; exponential family

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1. Introduction

Since mixed Poisson processes were introduced as a generalization of homogeneous Poisson processes they have been intensively studied. A detailed survey of the theory developed and results obtained is given in the monograph *Mixed Poisson Processes* [2].

An important issue is how mixed Poisson processes can be characterized within more general classes of processes. A well-known result [8] in this context is the characterization of mixed Poisson processes within the class of general point processes via the conditional uniformity of its occurrence times. Some relevant recent articles are [4], [3], and [9]. The first article proves a characterization within the class of general point processes via normalized event occurrence times, and the latter two characterize mixed Poisson processes within the class of birth processes via martingales involving transition intensities.

The present article, however, does not deal with characterizations of mixed Poisson processes within more general classes of processes, but rather proves a characterization of Pólya–Lundberg processes within the class of mixed Poisson processes. Pólya–Lundberg processes, i.e. mixed Poisson processes whose structure distributions are gamma distributions, have been of special interest for as long as mixed Poisson processes have been studied. They not only seem to be the appropriate choice to model the number of occurrences of certain events in applications, but also are probably the easiest to treat analytically. The characterization given in this article underlines the special role of these processes. The characteristic property is an exponential martingale property which will be deduced from studies of exponential families of stochastic processes. An overview of this topic is given in the monograph *Exponential Families of Stochastic Processes* [6].

The article is organized as follows. In Section 2 we first concentrate on the definition of such exponential families. We employ a concept proposed in [5], where exponential families were introduced as equivalence classes containing at least two elements related by an equivalence relation defined on a set of probability measures on a filtered, measurable space. Then we

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study exponential families of mixed Poisson processes and determine all existing exponential families. It turns out that these are simply the family of Poisson processes and the families of Pólya–Lundberg processes. We explain the correspondence between this result and studies in [10] concerning birth-and-death processes and exponential families. Finally, in Section 3 we present our main theorem, which states that Pólya–Lundberg processes can be characterized within the class of all mixed Poisson processes using exponential martingales.

2. Exponential families of mixed Poisson processes

Consider the following canonical model. Let Ω be the space of all simple counting functions $\omega\colon T = [0,\infty) \to \mathbb{N}$ and $\mathcal{F}$ the σ-algebra generated by all cylindric sets. Furthermore, consider the canonical process X_T, with $X_t(\omega) = \omega(t)$ for $\omega \in \Omega$ and $t \in T$, and the natural filtration $\{\mathcal{F}_t\}_{t\in T}$ generated by X_T. By $\mathcal{P}$ we denote the set of all probability measures on $(\Omega, \mathcal{F})$. For a measure $\mathrm{P} \in \mathcal{P}$ we denote by P_t, $t \in T$, the restriction of P to $\mathcal{F}_t$.

Since we consider a canonical model, in the sequel we will sometimes refer to a measure $\mathrm{P} \in \mathcal{P}$ as the 'corresponding' process.

In order to define exponential families we use a concept proposed in [5]. The definition is based on the following relation on $\mathcal{P}$.

Definition 1. We say that two measures $\mathrm{P}, \mathrm{Q} \in \mathcal{P}$ are in relation, written $\mathrm{P} \sim \mathrm{Q}$, if, for every $t \in T$, the measure Q_t is absolutely continuous with respect to P_t and if there are functions $c, d\colon T \to [0,\infty)$ such that the Radon–Nikodým derivative $\mathrm{dQ}_t/\mathrm{dP}_t$ satisfies

$$\frac{\mathrm{dP}_t}{\mathrm{dQ}_t} = \mathrm{e}^{c(t)X_t + d(t)} \qquad \mathrm{P}_t\text{-almost surely.} \tag{1}$$

This relation is an equivalence relation with the help of which we now define exponential families.

Definition 2. An equivalence class of '$\sim$' with at least two elements is called an exponential family.

Overall, this nonparametric approach to exponential families, which are usually defined as parametric families of measures, has two advantages: its independence of any parametrization and the more general mathematical structure supporting it.

Now consider the set $\mathcal{M} \subset \mathcal{P}$ which consists of all mixed Poisson processes. Recall that under $\mathrm{P} \in \mathcal{M}$ the process X_T is called a mixed Poisson process if its distribution, P_{X_T}, satisfies

$$\mathrm{P}_{X_T}(A) = \int_0^\infty \mathrm{P}^\lambda(A)\,\mathrm{d}U(\lambda), \qquad A \in \mathcal{F},$$

where P^λ describes the distribution of a Poisson process with intensity λ and U is a distribution concentrated on $[0,\infty)$. The distribution U is called the structure distribution of the mixed Poisson process. If U is a gamma distribution with scale parameter $\varphi > 0$ and shape parameter γ, i.e. $u = \Gamma(\varphi,\gamma)$, then we call the corresponding mixed Poisson process a Pólya–Lundberg process. By $\hat{u}$ we denote the Laplace transform of U, i.e. $\hat{u}(t) = \int_0^\infty \mathrm{e}^{-\lambda t}\,\mathrm{d}U(\lambda)$.

In order to determine exponential families of mixed Poisson processes, we will deduce equivalent descriptions of the equivalence $\mathrm{P} \sim \mathrm{Q}$ for two measures $\mathrm{P}, \mathrm{Q} \in \mathcal{M}$ via the respective structure distributions, u_{P} and u_{Q}, and Laplace transforms, $\hat{u}_{\mathrm{P}}$ and $\hat{u}_{\mathrm{Q}}$. The connection between

P and the Laplace transform $\hat{u}_{\mathrm{P}}$ is described by

$$\mathrm{P}(X_t = k) = \frac{(-t)^k}{k!} \hat{u}_{\mathrm{P}}^{(k)}(t), \qquad t > 0,\ k \in \mathbb{N}_0,$$

where $\hat{u}_{\mathrm{P}}^{(k)}$ denotes the kth derivative of $\hat{u}_{\mathrm{P}}$.

Now we consider the following question: Can a measure $\mathrm{P} \in \mathcal{M}$ be equivalent to a measure, Q, which does not correspond to a mixed Poisson process $\mathrm{Q} \in \mathcal{P} \setminus \mathcal{M}$; that is, can we restrict '$\sim$' to $\mathcal{M}$ without reducing the equivalence classes? This question is answered in the following proposition.

Proposition 1. *Let* $\mathrm{P}, \mathrm{Q} \in \mathcal{P}$ *be two equivalent measures. Additionally, let* P *be a mixed Poisson process, i.e.* $\mathrm{P} \in \mathcal{M}$. *Then* Q *is also a mixed Poisson process, i.e.* $\mathrm{Q} \in \mathcal{M}$.

Proof. We will apply the fact that a mixed Poisson process is characterized by the conditional uniformity of its event occurrence times (see [8]), which can be expressed by stating that

$$\mathrm{P}(X_{t_1} = k_1, \ \ldots, \ X_{t_{n-1}} = k_{n-1} \mid X_{t_n} = k_n) = \frac{k_n! \, t_1^{k_1}}{t_n^{k_n} \, k_1!} \prod_{l=2}^{n} \frac{(t_l - t_{l-1})^{k_l - k_{l-1}}}{(k_l - k_{l-1})!} \tag{2}$$

holds for $n \in \mathbb{N}$, for $t_1, \ldots, t_n \in T$ with $0 \leq t_1 < \cdots < t_n$, and for $k_1, \ldots, k_n \in \mathbb{N}_0$ with $0 \leq k_1 \leq \cdots \leq k_n$.

Consider two measures, $\mathrm{P} \in \mathcal{M}$ and $\mathrm{Q} \in \mathcal{P}$, with $\mathrm{P} \sim \mathrm{Q}$. By condition (1) we have

$$\mathrm{Q}(X_{t_1} = k_1, \ \ldots, \ X_{t_n} = k_n) = \mathrm{e}^{c(t_n) X_{t_n} + d(t_n)} \, \mathrm{P}(X_{t_1} = k_1, \ \ldots, \ X_{t_n} = k_n)$$

for $n \in \mathbb{N}$, for $k_1, \ldots, k_n \in \mathbb{N}$, for $t_1, \ldots, t_n \in [0, t]$, $t_n \geq t_i$, $i = 1, \ldots, n$, and for some nonnegative functions c and d. Then, since

$$\begin{aligned}
\mathrm{Q}(X_{t_1} = k_1, \ \ldots, \ X_{t_{n-1}} = k_{n-1} \mid X_{t_n} = k_n) &= \frac{\mathrm{Q}(X_{t_1} = k_1, \ \ldots, \ X_{t_n} = k_n)}{\mathrm{Q}(X_{t_n} = k_n)} \\
&= \frac{\mathrm{e}^{c(t_n) k_n + d(t_n)} \, \mathrm{P}(X_{t_1} = k_1, \ \ldots, \ X_{t_n} = k_n)}{\mathrm{e}^{c(t_n) k_n + d(t_n)} \, \mathrm{P}(X_{t_n} = k_n)} \\
&= \frac{\mathrm{P}(X_{t_1} = k_1, \ \ldots, \ X_{t_n} = k_n)}{\mathrm{P}(X_{t_n} = k_n)} \\
&= \mathrm{P}(X_{t_1} = k_1, \ \ldots, \ X_{t_{n-1}} = k_{n-1} \mid X_{t_n} = k_n),
\end{aligned}$$

(2) holds under Q if and only if it holds under P.

Consequently, we can simply restrict '$\sim$' to $\mathcal{M}$. We are then able to establish the following characterization for the equivalence of two measures.

Proposition 2. *Let* P *and* Q *be two mixed Poisson processes,* $\mathrm{P}, \mathrm{Q} \in \mathcal{M}$, *and let* $c, d \colon T \to \mathbb{R}$ *be real functions. Then the following statements are equivalent.*

(i) *The measure* P *is equivalent to* Q, *i.e.* $\mathrm{P} \sim \mathrm{Q}$, *and the equivalence is determined by the functions* c *and* d.

(ii) *The equality* $\mathrm{Q}(X_t = k) = \mathrm{e}^{c(t)k + d(t)} \, \mathrm{P}(X_t = k)$ *holds for all* $t \in T$ *and* $k \in \mathbb{N}_0$.

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(iii) *For the Laplace transforms,* $\hat{u}_{\mathrm{P}}$ *and* $\hat{u}_{\mathrm{Q}}$*, of the structure distributions of* P *and* Q*, we have*

$$\begin{aligned}\hat{u}_{\mathrm{Q}}^{(k)}(t) &= \mathrm{e}^{c(t)k+d(t)}\hat{u}_{\mathrm{P}}^{(k)}(t), \qquad t>0,\ k\in\mathbb{N}_0,\\ d(0)&=0.\end{aligned}$$

Proof. First, (i) holds if and only if

$$\mathrm{Q}(X_{t_1}=k_1,\ \ldots,\ X_{t_n}=k_n) = \mathrm{e}^{c(t_n)k_n+d(t_n)}\,\mathrm{P}(X_{t_1}=k_1,\ \ldots,\ X_{t_n}=k_n) \tag{3}$$

holds for all $n\in\mathbb{N}$, all $t_1,\ldots,t_n$ with $0\le t_1<\cdots<t_n\le t$, and all $k_1,\ldots,k_n\in\mathbb{N}_0$. Without loss of generality, we assume that $k_1\le k_2\le\cdots\le k_n$.

Now choose $n\in\mathbb{N}$, $t_1,\ldots,t_n$, and $k_1,\ldots,k_n\in\mathbb{N}_0$ accordingly. By (2), we have

$$\mathrm{P}(X_{t_1}=k_1,\ \ldots,\ X_{t_n}=k_n) = \frac{k_n!\,t_1^{k_1}}{t_n^{k_n}\,k_1!}\prod_{l=2}^{n}\frac{(t_l-t_{l-1})^{k_l-k_{l-1}}}{(k_l-k_{l-1})!}\,\mathrm{P}(X_{t_n}=k_n).$$

Since this is similarly valid for Q, (3) reduces to

$$\mathrm{Q}(X_{t_n}=k_n) = \mathrm{e}^{c(t_n)k_n+d(t_n)}\,\mathrm{P}(X_{t_n}=k_n). \tag{4}$$

If additionally $t_n>0$, this is equivalent to

$$\hat{u}_{\mathrm{Q}}^{(k_n)}(t_n) = \mathrm{e}^{c(t_n)k_n+d(t_n)}\,\hat{u}_{\mathrm{P}}^{(k_n)}(t_n).$$

For $t_n=0$, the condition $d(0)=0$ is necessary and sufficient for (4) to hold for all $k_n\ge 0$.

The equivalence of (i), (ii), and (iii) follows from these considerations.

As an additional consequence of this proposition we find that, since Laplace transforms of distributions are continuous and infinitely often differentiable on $(0,\infty)$, so too are the functions c and d.

The following lemma is an essential result for finding exponential families. It supplies necessary and sufficient conditions for a measure $\mathrm{P}\in\mathcal{M}$ to belong to an exponential family depending only on the Laplace transform corresponding to P and not on any Laplace transform corresponding to an equivalent measure distinct from P.

Lemma 1. *For a measure* $\mathrm{P}\in\mathcal{M}$ *and functions* $c,d\colon T\to\mathbb{R}$*, the following statements are equivalent.*

(i) *There is a measure* $\mathrm{Q}\in\mathcal{M}$ *that is distinct from* P *yet equivalent to* P*. The equivalence* $\mathrm{P}\sim\mathrm{Q}$ *is determined by* c *and* d*.*

(ii) *The function* d *satisfies* $d\not\equiv 0$ *and* $d(0)=0$*. Furthermore,* c *and* d *are continuous and differentiable on* $(0,\infty)$ *and, for all* $t\in(0,\infty)$ *and all* $k\in\mathbb{N}$*, satisfy*

$$(c'(t)(k-1)+d'(t))\hat{u}_{\mathrm{P}}^{(k-1)}(t) = (\mathrm{e}^{c(t)}-1)\,\hat{u}_{\mathrm{P}}^{(k)}(t), \tag{5}$$

where a prime denotes differentiation.

Proof. First suppose (i) to be valid. By Proposition 2 we then have

$$\begin{aligned}\hat{u}_{\mathrm{Q}}^{(k)}(t) &= \mathrm{e}^{c(t)k+d(t)}\,\hat{u}_{\mathrm{P}}^{(k)}(t), \qquad t>0,\ k\in\mathbb{N}_0,\\ d(0)&=0,\end{aligned}$$

and the functions c and d are continuous and differentiable on $(0, \infty)$. Considering the first equation for $k = 0$, we see that $d \not\equiv 0$ since $\mathrm{P} \neq \mathrm{Q}$ and, consequently, $\hat{u}_{\mathrm{P}} \neq \hat{u}_{\mathrm{Q}}$. Thus, we have

$$\begin{aligned}
&\mathrm{e}^{c(t)k+d(t)}\, \hat{u}_{\mathrm{P}}^{(k)}(t) \\
&\quad = \hat{u}_{\mathrm{Q}}^{(k)}(t) \\
&\quad = \frac{\mathrm{d}}{\mathrm{d}t} \hat{u}_{\mathrm{Q}}^{(k-1)}(t) = \frac{\mathrm{d}}{\mathrm{d}t} \mathrm{e}^{c(t)(k-1)+d(t)}\, \hat{u}_{\mathrm{P}}^{(k-1)}(t) \\
&\quad = (c'(t)(k-1) + d'(t)) \mathrm{e}^{c(t)(k-1)+d(t)} \hat{u}_{\mathrm{P}}^{(k-1)}(t) + \mathrm{e}^{c(t)(k-1)+d(t)}\, \hat{u}_{\mathrm{P}}^{(k)}(t) \\
&\quad = [(c'(t)(k-1) + d'(t)) \hat{u}_{\mathrm{P}}^{(k-1)}(t) + \hat{u}_{\mathrm{P}}^{(k)}(t)] \mathrm{e}^{c(t)(k-1)+d(t)}, \qquad t > 0,\ k \in \mathbb{N}.
\end{aligned}$$

Finally, we obtain

$$(c'(t)(k-1) + d'(t))\, \hat{u}_{\mathrm{P}}^{(k-1)}(t) = (\mathrm{e}^{c(t)} - 1)\, \hat{u}_{\mathrm{P}}^{(k)}(t), \qquad t > 0,\ k \in \mathbb{N},$$

proving that (ii) follows from (i).

Now suppose (ii) to be valid. Define a function $\hat{u}_{\mathrm{Q}} \colon T \to \mathbb{R}$ by

$$\hat{u}_{\mathrm{Q}}(t) := \mathrm{e}^{d(t)}\, \hat{u}_{\mathrm{P}}(t), \qquad t \in T.$$

We will show that $\hat{u}_{\mathrm{Q}}$ is a Laplace transform and that the measure $\mathrm{Q} \in \mathcal{M}$ which corresponds to $\hat{u}_{\mathrm{Q}}$ is equivalent to P. First we prove the following, by induction on k:

$$\hat{u}_{\mathrm{Q}}^{(k)}(t) = \mathrm{e}^{c(t)k+d(t)}\, \hat{u}_{\mathrm{P}}^{(k)}(t), \qquad t > 0,\ k \in \mathbb{N}_0. \tag{6}$$

By definition, the equation holds for $k = 0$. Now suppose the upper equation to be valid for $k - 1$, $k \in \mathbb{N}$. Then it also holds for k:

$$\begin{aligned}
\hat{u}_{\mathrm{Q}}^{(k)}(t) &= \frac{\mathrm{d}}{\mathrm{d}t} \hat{u}_{\mathrm{Q}}^{(k-1)}(t) = \frac{\mathrm{d}}{\mathrm{d}t} (\mathrm{e}^{c(t)(k-1)+d(t)} \hat{u}_{\mathrm{P}}^{k}(t)) \\
&= (c'(t)(k-1) + d'(t)) \mathrm{e}^{c(t)(k-1)+d(t)} \hat{u}_{\mathrm{P}}^{(k-1)}(t) + \mathrm{e}^{c(t)(k-1)+d(t)} \hat{u}_{\mathrm{P}}^{(k)}(t) \\
&= \mathrm{e}^{c(t)(k-1)+d(t)} ((c'(t)(k-1) + d'(t)) \hat{u}_{\mathrm{P}}^{(k-1)}(t) + \hat{u}_{\mathrm{P}}^{(k)}(t)) \\
&\overset{(5)}{=} \mathrm{e}^{c(t)(k-1)+d(t)} ((\mathrm{e}^{c(t)} - 1) \hat{u}_{\mathrm{P}}^{(k)}(t) + \hat{u}_{\mathrm{P}}^{(k)}(t)) \\
&= \mathrm{e}^{c(t)k+d(t)} \hat{u}_{\mathrm{P}}^{(k)}(t), \qquad t > 0.
\end{aligned}$$

Because $\hat{u}_{\mathrm{Q}}$ is completely monotone on $(0, \infty)$ (see (6)) and $\hat{u}_{\mathrm{Q}}(0) = 1$, $\hat{u}_{\mathrm{Q}}$ is actually the Laplace transform of a probability distribution (see, for instance, [1, p. 439]). Let U_{Q} be this distribution and let $\mathrm{Q} \in \mathcal{M}$ be the corresponding mixed Poisson process. Because $d \not\equiv 0$, we have $\hat{u}_{\mathrm{Q}} \neq \hat{u}_{\mathrm{P}}$ and, hence, $\mathrm{Q} \neq \mathrm{P}$. Moreover, (6) implies that Q is equivalent to P. Thus, there is an exponential family which contains P.

Let us now consider two examples. Using Lemma 1, we can easily verify that each Poisson process is in an exponential family.

Under P, let X_T be a Poisson process with intensity $\lambda > 0$, i.e. a process whose structure distribution is a Dirac distribution, $U_{\mathrm{P}} = \delta_\lambda$, with $\hat{u}_{\mathrm{P}}(t) = \mathrm{e}^{-\lambda t}$, $t \in T$. Choose a $c_0 \in \mathbb{R} \setminus \{0\}$ and let c and d be

$$c(t) = c_0, \quad d(t) = -\lambda\, (\mathrm{e}^{c_0} - 1)\, t, \qquad t \in T. \tag{7}$$

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Then c and d are continuous and differentiable on $(0, \infty)$ and

$$(c'(t)(k-1) + d'(t))\hat{u}_{\mathrm{P}}^{(k-1)}(t) = (\mathrm{e}^{c(t)} - 1)\hat{u}_{\mathrm{P}}^{(k)}(t).$$

holds for $t > 0$ and $k \in \mathbb{N}$. Moreover, we have $d \not\equiv 0$ and $d(0) = 0$. Thus, owing to Lemma 1, the ordinary Poisson process belongs to an exponential family. Equally, we can show that Pólya–Lundberg processes are contained in exponential families.

Under P, let X_T be a Pólya–Lundberg process with structure distribution $U_{\mathrm{P}} = \Gamma(\varphi, \gamma)$, $\varphi, \gamma > 0$. Then

$$\hat{u}_{\mathrm{P}}(t) = \left(1 + \frac{t}{\varphi}\right)^{-\gamma}, \qquad t \in T,\ k \in \mathbb{N},$$

is the Laplace transform of U_{P}. Let c and d be

$$c(t) = \ln\left(\frac{t+\varphi}{t+\varphi\alpha}\right), \quad d(t) = \gamma\left(\ln\left(\frac{t+\varphi}{t+\varphi\alpha}\right) + \ln\alpha\right), \qquad t \in T, \tag{8}$$

where $\alpha \in \mathbb{R} \setminus \{1\}$. Then c and d are continuous and differentiable on $(0, \infty)$ and (5) holds. Additionally, as $d(0) = 0$ and $d \not\equiv 0$, P belongs to an exponential family.

The conditions of Lemma 1 for a Laplace transform $\hat{u}_{\mathrm{P}}$ to correspond to a measure P of an exponential family are actually very restrictive. The following proposition shows that the above examples are in fact the only ones possible.

Proposition 3. *For a measure* P $\in \mathcal{M}$, *the following statements are equivalent.*

(i) *The measure* P *belongs to an exponential family.*

(ii) *There exist* $\lambda > 0$ *and* $\varphi, \gamma > 0$ *such that the respective structure distributions,* U_{P}, *of* P *are* $U_{\mathrm{P}} = \delta_\lambda$ *and* $U_{\mathrm{P}} = \Gamma(\varphi, \gamma)$. *In other words,* X_T *is either a Poisson process or a Pólya–Lundberg process under* P.

The functions c *and* d *that determine the measures which are equivalent to* P *are as given in (7) and (8) for* $U_{\mathrm{P}} = \delta_\lambda$ *and, respectively,* $U_{\mathrm{P}} = \Gamma(\varphi, \gamma)$.

Before we can prove this proposition we need the following technical lemma.

Lemma 2. *Condition (ii) of Lemma 1 implies that*

$$(\mathrm{e}^{-c(t)} - 1)^2 v(t_0) = c'(t)\,\mathrm{e}^{-c(t)}, \qquad t > 0,$$

where

$$v(t_0) = \left(\frac{\hat{u}'_{\mathrm{P}}(t_0)}{\hat{u}_{\mathrm{P}}(t_0)} - \frac{\hat{u}''_{\mathrm{P}}(t_0)}{\hat{u}'_{\mathrm{P}}(t_0)}\right)\frac{1}{\mathrm{e}^{-c(t_0)} - 1}$$

and $t_0 > 0$ *is arbitrary.*

Proof. Let $t_0 > 0$. By (5), we have

$$\frac{\hat{u}_{\mathrm{P}}^{(k)}(t)}{\hat{u}_{\mathrm{P}}^{(k-1)}(t)} = \frac{c'(t)(k-1) + d'(t)}{\mathrm{e}^{c(t)} - 1}, \qquad t > 0,\ k \in \mathbb{N}.$$

The solution, $\hat{u}_{\mathrm{P}}^{(k-1)}$, to this ordinary differential equation satisfies

$$\hat{u}_{\mathrm{P}}^{(k-1)}(t) = \hat{u}_{\mathrm{P}}^{(k-1)}(t_0)\left(\frac{\mathrm{e}^{-c(t)} - 1}{\mathrm{e}^{-c(t_0)} - 1}\right)^{k-1}\mathrm{e}^{I(t)}, \qquad t > 0,\ k \in \mathbb{N}, \tag{9}$$

with $I(t) := \int_{t_0}^{t}[d'(s)/(\mathrm{e}^{c(s)} - 1)]\,\mathrm{d}s$.

We proceed by evaluating (9) for $k = 1, 2, 3$, which leads to a system of equations from which we can derive a differential equation for the function c. Evaluating (9) for $k = 1$ yields

$$\hat{u}_{\mathrm{P}}(t) = \hat{u}_{\mathrm{P}}(t_0)\, \mathrm{e}^{I(t)}, \qquad t > 0, \tag{10}$$

and, consequently,

$$\hat{u}'_{\mathrm{P}}(t) = \hat{u}_{\mathrm{P}}(t_0)\mathrm{e}^{I(t)} \frac{d'(t)}{\mathrm{e}^{c(t)} - 1}, \qquad t > 0. \tag{11}$$

Evaluating (9) for $k = 2$ yields

$$\hat{u}'_{\mathrm{P}}(t) = \hat{u}'_{\mathrm{P}}(t_0) \frac{\mathrm{e}^{-c(t)} - 1}{\mathrm{e}^{-c(t_0)} - 1} \mathrm{e}^{I(t)}, \qquad t > 0, \tag{12}$$

and combining (11) and (12) leads to

$$d'(t) = \frac{\hat{u}'_{\mathrm{P}}(t_0)}{\hat{u}_{\mathrm{P}}(t_0)} \frac{(\mathrm{e}^{-c(t)} - 1)(\mathrm{e}^{c(t)} - 1)}{\mathrm{e}^{-c(t_0)} - 1}, \qquad t > 0. \tag{13}$$

We also have

$$\hat{u}''_{\mathrm{P}}(t) = \hat{u}'_{\mathrm{P}}(t_0)\left(\frac{-c'(t)\mathrm{e}^{-c(t)}}{\mathrm{e}^{-c(t_0)} - 1} + \frac{\hat{u}'_{\mathrm{P}}(t_0)}{\hat{u}_{\mathrm{P}}(t_0)}\left(\frac{\mathrm{e}^{-c(t)} - 1}{\mathrm{e}^{-c(t_0)} - 1}\right)^2\right)\mathrm{e}^{I(t)}, \qquad t > 0, \tag{14}$$

which is obtained by differentiating (12) and substituting for $d'(t)$ using (13). Equation (9) for $k = 3$ is

$$\hat{u}''_{\mathrm{P}}(t) = \hat{u}''_{\mathrm{P}}(t_0)\left(\frac{\mathrm{e}^{-c(t)} - 1}{\mathrm{e}^{-c(t_0)} - 1}\right)^2 \mathrm{e}^{I(t)}, \qquad t > 0. \tag{15}$$

Combining (14) and (15) finally leads to the following differential equation for c:

$$(\mathrm{e}^{-c(t)} - 1)^2 \underbrace{\left(\frac{\hat{u}'_{\mathrm{P}}(t_0)}{\hat{u}_{\mathrm{P}}(t_0)} - \frac{\hat{u}''_{\mathrm{P}}(t_0)}{\hat{u}'_{\mathrm{P}}(t_0)}\right)\frac{1}{\mathrm{e}^{-c(t_0)} - 1}}_{v(t_0)} = c'(t)\mathrm{e}^{-c(t)}, \qquad t > 0.$$

Notice that the continuous function v, as a function of t_0, does not change sign over $(0, \infty)$, because having $v(t_0) = 0$ for some $t_0 > 0$ implies that U_{P} is a Dirac distribution and, therefore, that $v \equiv 0$.

Proof of Proposition 3. It remains to show that part (i) of Proposition 3 implies part (ii). Assume (i) to be valid, i.e. assume $\mathrm{P} \in \mathcal{M}$ to belong to an exponential family. Let $t_0 > 0$. Then Lemma 2 implies the following differential equation:

$$(\mathrm{e}^{-c(t)} - 1)^2 v(t_0) = c'(t)\mathrm{e}^{-c(t)}, \qquad t > 0. \tag{16}$$

In the sequel, we will solve this equation for c and deduce $\hat{u}_{\mathrm{P}}$. We distinguish the cases $v \equiv 0$ and $v(t_0) \neq 0$ for all $t_0 > 0$.

First assume that $v \equiv 0$, i.e. that U_{P} is a Dirac distribution. We are interested in the points at which U_{P} can be concentrated and in the corresponding functions c and d. Equation (16) now reads $0 = c'(t)\mathrm{e}^{-c(t)}$ and implies that $c(t) = c_0,\ t > 0$. Thus, by (13) we have

$$d'(t) = \frac{\hat{u}'_{\mathrm{P}}(t_0)}{\hat{u}_{\mathrm{P}}(t_0)} (\mathrm{e}^{c_0} - 1).$$

Consequently, taking into account the fact that $d(0) = 0$, we obtain

$$d(t) = \frac{\hat{u}'_{\mathrm{P}}(t_0)}{\hat{u}_{\mathrm{P}}(t_0)}(\mathrm{e}^{c_0} - 1)t, \qquad t > 0.$$

Since

$$I(t) = \int_{t_0}^{t} \frac{d'(s)}{\mathrm{e}^{c(s)} - 1}\,\mathrm{d}s = \frac{\hat{u}'_{\mathrm{P}}(t_0)}{\hat{u}_{\mathrm{P}}(t_0)}(t - t_0), \qquad t > 0,$$

(10) leads to

$$\hat{u}_{\mathrm{P}}(t) = \hat{u}_{\mathrm{P}}(t_0)\exp\biggl\{\frac{\hat{u}'_{\mathrm{P}}(t_0)}{\hat{u}_{\mathrm{P}}(t_0)}(t - t_0)\biggr\}, \qquad t > 0.$$

Taking the logarithmic derivative with respect to t, we have

$$\frac{\hat{u}'_{\mathrm{P}}(t)}{\hat{u}_{\mathrm{P}}(t)} = \frac{\hat{u}'_{\mathrm{P}}(t_0)}{\hat{u}_{\mathrm{P}}(t_0)}, \qquad t > 0.$$

This means that, for $t > 0$, the quotient $\hat{u}'_{\mathrm{P}}(t)/\hat{u}_{\mathrm{P}}(t)$ is independent of t. Additionally, this quotient is less than or equal to 0, with equality if and only if $U = \delta_0$. The latter case need no longer be considered, since $U = \delta_0$ implies that $d \equiv 0$, which contradicts condition (ii) of Lemma 1. Thus, with $\lambda := -\hat{u}'_{\mathrm{P}}(t_0)/\hat{u}_{\mathrm{P}}(t_0)$, we have the following representation for $\hat{u}_{\mathrm{P}}$:

$$\hat{u}_{\mathrm{P}}(t) = C\mathrm{e}^{-\lambda t}, \qquad t > 0.$$

Since a Laplace transform $\hat{u}_{\mathrm{P}}$ is continuous and satisfies $\hat{u}_{\mathrm{P}}(0) = 1$, the constant C has to be equal to 1. In summary, we have the following representations for $\hat{u}_{\mathrm{P}}$, c, and d:

$$\hat{u}_{\mathrm{P}}(t) = \mathrm{e}^{-\lambda t}, \quad c(t) = c_0, \quad d(t) = -\lambda(\mathrm{e}^{c_0} - 1)t, \qquad t > 0.$$

Because these functions are continuous, they can be extended onto the whole interval $T = [0, \infty)$. By inspection, $\hat{u}_{\mathrm{P}}$ is the Laplace transform of a Dirac distribution at $\lambda > 0$ and the measure $\mathrm{P} \in \mathcal{M}$ corresponds to an ordinary Poisson process with intensity $\lambda > 0$.

Now consider the case in which $v(t_0) \neq 0$. With $v := v(t_0)$ and $g(t) := \mathrm{e}^{-c(t)} - 1$, the differential equation

$$(\mathrm{e}^{-c(t)} - 1)^2 v(t_0) = c'(t)\,\mathrm{e}^{-c(t)}$$

(i.e. (18)) can be transformed into

$$g(t)^2 v = -g'(t) \quad \text{or, equivalently,} \quad v = -\frac{g'(t)}{g(t)^2}, \qquad t > 0,$$

and, thus,

$$g(t) = \frac{1}{vt + a}, \qquad t > 0.$$

Since $g(t) \in (-1, \infty)$ for $t > 0$, the integration constant a is restricted to satisfy

$$a \geq 0 \quad \text{for } v > 0 \qquad \text{and} \qquad a \leq -1 \quad \text{for } v < 0.$$

From g we obtain a representation for c,

$$c(t) = -\ln\biggl(1 + \frac{1}{vt + a}\biggr), \qquad t > 0.$$

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By (13), we have

$$d'(t) = \frac{\hat{u}'_{\mathrm{P}}(t_0)}{\hat{u}_{\mathrm{P}}(t_0)} \frac{vt_0 + a}{vt + a} \frac{-1}{vt + a + 1}, \qquad t > 0,$$

and, consequently, taking the fact that $d(0) = 0$ into account,

$$d(t) = \frac{\hat{u}'_{\mathrm{P}}(t_0)}{\hat{u}_{\mathrm{P}}(t_0)} \frac{vt_0 + a}{v} \left(\ln\left(\frac{vt + a + 1}{vt + a} \right) - \ln\left(\frac{a + 1}{a} \right) \right), \qquad t > 0.$$

For $I(t)$ we obtain

$$I(t) = \int_{t_0}^{t} \frac{d'(s)}{\mathrm{e}^{c(s)} - 1} \, \mathrm{d}s = \frac{\hat{u}'_{\mathrm{P}}(t_0)}{\hat{u}_{\mathrm{P}}(t_0)} \frac{vt_0 + a}{v} \ln\left(\frac{vt + a}{vt_0 + a} \right), \qquad t > 0.$$

Equation (10) leads to

$$\hat{u}_{\mathrm{P}}(t) = \hat{u}_{\mathrm{P}}(t_0) \, \mathrm{e}^{I(t)} = \hat{u}_{\mathrm{P}}(t_0) \left(\frac{vt + a}{vt_0 + a} \right)^{[\hat{u}'_{\mathrm{P}}(t_0)/\hat{u}_{\mathrm{P}}(t_0)][(vt_0 + a)/v]}, \qquad t > 0. \tag{17}$$

Additionally, as

$$\frac{vt_0 + a}{v} = \frac{g(t_0)^{-1}}{v} = \frac{1}{v(\mathrm{e}^{-c(t_0)} - 1)}$$

is valid for $t_0 > 0$, we obtain the following representation for $\hat{u}_{\mathrm{P}}$ from (17):

$$\hat{u}_{\mathrm{P}}(t) = \hat{u}_{\mathrm{P}}(t_0) \left(\frac{vt + a}{vt_0 + a} \right)^{[\hat{u}'_{\mathrm{P}}(t_0)/\hat{u}_{\mathrm{P}}(t_0)][1/v(\exp\{-c(t_0)\} - 1)]}, \qquad t > 0.$$

By taking the logarithmic derivative twice, we can show that neither

$$\frac{\hat{u}'_{\mathrm{P}}(t_0)}{\hat{u}_{\mathrm{P}}(t_0)} \frac{1}{v(\mathrm{e}^{-c(t_0)} - 1)}$$

nor a/v depends on the choice of t_0. Thus, setting

$$\gamma := -\frac{\hat{u}'_{\mathrm{P}}(t_0)}{\hat{u}_{\mathrm{P}}(t_0)} \frac{1}{v(t_0)(\mathrm{e}^{-c(t_0)} - 1)} > 0 \quad \text{and} \quad \varphi := \frac{a}{v(t_0)} > 0,$$

we can express $\hat{u}_{\mathrm{P}}$ as

$$\hat{u}_{\mathrm{P}}(t) = C \left(1 + \frac{t}{\varphi} \right)^{-\gamma}, \qquad t > 0.$$

Since $\hat{u}_{\mathrm{P}}$ is a Laplace transform and, thus, continuous at 0 with $\hat{u}_{\mathrm{P}}(0) = 1$, the constant C has to be equal to 1. By inspection, $\hat{u}_{\mathrm{P}}$ is the Laplace transform of a gamma distribution with parameters $\varphi, \gamma > 0$. With $\alpha := (a + 1)/a$, where $a > 0$ and $a < -1$ imply that $\alpha \in \mathbb{R} \setminus \{1\}$, we obtain

$$c(t) = \ln\left(\frac{t + \varphi}{t + \varphi\alpha} \right), \quad d(t) = \gamma \left(\ln\left(\frac{t + \varphi}{t + \varphi\alpha} \right) + \ln \alpha \right), \qquad t > 0.$$

The above representations for $\hat{u}_{\mathrm{P}}$, c, and d can be continuously extended to the point 0.

We now present an alternative way of proving the last proposition, using Liptser and Shiryayev's [7, Theorem 19.7] method of characterizing absolute continuity of two measures $\mathrm{P}, \mathrm{Q} \in \mathscr{P}$ by their compensators $\{A_t^{\mathrm{P}}\}_{t\geq 0}$ and $\{A_t^{\mathrm{Q}}\}_{t\geq 0}$. The measure Q is absolutely continuous with respect to P if and only if there exists a nonnegative process $\{\lambda_t^{\mathrm{P,Q}}\}_{t\geq 0}$ which is predictable with respect to $\{\mathcal{F}_t\}_{t\in T}$ and such that

$$A_t^{\mathrm{Q}}(\omega) = \int_0^t \lambda_s^{\mathrm{P,Q}}(\omega)\,\mathrm{d}A_s^{\mathrm{P}}(\omega), \qquad t < \infty,$$

and

$$\int_0^\infty \Big(1 - \sqrt{\lambda_s^{\mathrm{P,Q}}(\omega)}\Big)^2 \,\mathrm{d}A_s^{\mathrm{P}}(\omega) < \infty$$

hold for P-almost all $\omega \in \Omega$.

Now, for $\mathrm{P} \in \mathcal{M}$, and analogously for $\mathrm{Q} \in \mathcal{M}$, we have (see [2, p. 65])

$$A_t^{\mathrm{P}}(\omega) = \int_0^t \kappa_{X_s(\omega)}^{\mathrm{P}}(s)\,\mathrm{d}s \quad \text{P-almost everwhere (P-a.e.)},$$

where $\kappa_n^{\mathrm{P}}(t)$, $n \in \mathbb{N}$, $t > 0$, are the transition intensities of the mixed Poisson process (and, hence, birth process) P. So, for $\mathrm{P}, \mathrm{Q} \in \mathcal{M}$ we obtain

$$\lambda_t^{\mathrm{P,Q}}(\omega) = \frac{\kappa_{X_t(\omega)}^{\mathrm{Q}}(t)}{\kappa_{X_t(\omega)}^{\mathrm{P}}(t)} \quad \text{P-a.e.},$$

which is a predictable process if and only if the quotient on the right-hand side does not depend on ω, i.e. if and only if $\lambda_t^{\mathrm{P,Q}}(\omega) \equiv \lambda_t^{\mathrm{P,Q}}$ for $\omega \in \Omega$. Additionally, we have $\lambda_t^{\mathrm{P,Q}} \in (0,1)\cup(1,\infty)$ for $\mathrm{P} \neq \mathrm{Q}$.

Another part of [7, Theorem 19.7] states that the Radon–Nikodým derivatives $\mathrm{dQ}_t \,/\, \mathrm{dP}_t$ for $t \geq 0$ can be represented as

$$\frac{\mathrm{dQ}_t}{\mathrm{dP}_t}(\omega) = \exp\bigg\{\int_0^t \ln \lambda_s^{\mathrm{P,Q}}(\omega)\,\mathrm{d}X_s(\omega) - (A_t^{\mathrm{Q}}(\omega) - A_t^{\mathrm{P}}(\omega))\bigg\} \quad \mathrm{P}_t\text{-a.e.} \tag{18}$$

Now let us return to our initial question: Which measures $\mathrm{P}, \mathrm{Q} \in \mathcal{M}$ can be related to each other via '$\sim$'?

The definition of $\mathrm{P} \sim \mathrm{Q}$ requires, for $t \in [0,\infty)$, that

$$\frac{\mathrm{dQ}_t}{\mathrm{dP}_t}(\omega) = \mathrm{e}^{c(t)X_t(\omega)+d(t)} \quad \mathrm{P}_t\text{-a.e.},$$

which, together with (18), leads to

$$\int_0^t \ln \lambda_s^{\mathrm{P,Q}}\,\mathrm{d}X_s(\omega) - (A_t^{\mathrm{Q}}(\omega) - A_t^{\mathrm{P}}(\omega)) = c(t)X_t(\omega) + d(t) \quad \mathrm{P}_t\text{-a.e.}$$

Partial integration yields

$$\begin{aligned}\ln \lambda_t^{\mathrm{P,Q}} X_t(\omega) &- \int_0^t \frac{(\mathrm{d}/\mathrm{d}s)\lambda_s^{\mathrm{P,Q}}}{\lambda_s^{\mathrm{P,Q}}} X_s(\omega)\,\mathrm{d}s - \lambda_t^{\mathrm{P,Q}} A_t^{\mathrm{P}}(\omega) + \int_0^t \frac{\mathrm{d}}{\mathrm{d}s}\lambda_s^{\mathrm{P,Q}} A_s^{\mathrm{P}}(\omega)\,\mathrm{d}s + A_t^{\mathrm{P}}(\omega)\\ &= c(t)X_t(\omega) + d(t)\end{aligned}$$

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for P_t-almost all ω. If, for a fixed $\omega \in \Omega$, this equation is valid and if $t > 0$ is a continuity point of the path $X_T(\omega)$, then we can differentiate with respect to t and obtain

$$(1 - \lambda_t^{\mathrm{P,Q}}) \frac{\mathrm{d}}{\mathrm{d}t} A_t^{\mathrm{P}}(\omega) = c'(t) X_t(\omega) + d'(t),$$

which implies that

$$\kappa_{X_t(\omega)}^{\mathrm{P}}(t) = \frac{\mathrm{d}}{\mathrm{d}t} A_t^{\mathrm{P}}(\omega) = \frac{c'(t)}{1 - \lambda_t^{\mathrm{P,Q}}} X_t(\omega) + \frac{d'(t)}{1 - \lambda_t^{\mathrm{P,Q}}}.$$

Furthermore, for $t > 0$ and $n \in \mathbb{N}$, the set $\{\omega \in \Omega \colon X_t(\omega) = n,\ \lim_{s \to t-} X_s(\omega) = n\}$, i.e. the set of counting functions which at time t do not jump and are in state n, has positive P_t-measure. The transition intensities must thus satisfy

$$\kappa_n^{\mathrm{P}}(t) = \frac{c'(t)}{1 - \lambda_t^{\mathrm{P,Q}}} n + \frac{d'(t)}{1 - \lambda_t^{\mathrm{P,Q}}}, \qquad n \in \mathbb{N},\ t > 0.$$

However, the only processes $\mathrm{P} \in \mathcal{M}$ with transition intensities $\kappa_n^{\mathrm{P}}(t)$ that are linear in n for fixed t are Poisson processes and Pólya–Lundberg processes (see [2]).

To find the measures $\mathrm{Q} \in \mathcal{M}$ which can be equivalent to P consider the quotient

$$\kappa_{X_t(\omega)}^{\mathrm{Q}}(t) / \kappa_{X_t(\omega)}^{\mathrm{P}}(t).$$

Since it must not depend on ω if the two measures P and Q are to be in relation to each other, we find that a Poisson process can only be equivalent to a Poisson process and that a Pólya–Lundberg process with structure distribution $\Gamma(\varphi, \gamma)$, $\varphi, \gamma > 0$, and transition intensities $\kappa_n(t) = (\gamma + n)/(\varphi + t)$ can only be equivalent to Pólya–Lundberg processes with the same parameter γ.

An immediate consequence of Proposition 3 is the following corollary, which specifies exponential families in $\mathcal{M}$.

Corollary 1. *The only existing exponential families in $\mathcal{M}$ are the exponential family of homogeneous Poisson processes $\{\mathrm{P} \in \mathcal{M} \colon U_{\mathrm{P}} = \delta_\lambda,\ \lambda > 0\}$ and exponential families of Pólya–Lundberg processes $\{\mathrm{P} \in \mathcal{M} \colon U_{\mathrm{P}} = \Gamma(\varphi, \gamma),\ \varphi > 0\}$, where the shape parameter $\gamma > 0$ of the corresponding gamma structure distributions remains constant within each such exponential family (and serves to distinguish between them).*

Proof. Proposition 3 determines the only measures $\mathrm{P} \in \mathcal{M}$ belonging to exponential families and the only functions c and d leading to equivalent measures. Combining these, and therefore calculating for such a $\mathrm{P} \in \mathcal{M}$ and all possible appropriate functions c and d the equivalent measures $\tilde{\mathrm{P}} \in \mathcal{M}$ via $\hat{u}_{\tilde{P}}(t) = \mathrm{e}^{d(t)} \hat{u}_{\mathrm{P}}(t)$, we obtain the above-stated exponential families.

There is a correspondence between the above result and a theorem of Ycart [10]. He characterized continuous-time birth-and-death processes which are such that their one-dimensional distributions at any instant lie in a given exponential family, $\mathcal{F}$, generated by a nonnegative measure, μ, on $\mathbb{N}$, that is, $\mathcal{F} = \{\mathrm{P}_\theta \colon \theta \in \Theta\}$ with

$$\frac{\mathrm{dP}_\theta}{\mathrm{d}\mu}(n) = \frac{\mathrm{e}^{\theta n}}{\sum_{m=0}^{\infty} \mu(m) \mathrm{e}^{\theta m}}$$

and $\Theta = \{\theta \in \mathbb{R} \colon \sum_{m=0}^{\infty} \mu(m) \mathrm{e}^{\theta m} < \infty\}$.

Now assume $\mathrm{P} \in \mathcal{M}$ to be a mixed Poisson process with structure distribution U_{P} and the above property. Denote by $U_{P,t}$ the structure distribution of the distribution of $\mathrm{P} \circ X_t$ for $t > 0$. Let $\mathrm{Q} \in \mathcal{M}$ be a mixed Poisson process whose structure distribution is equal to $U_{\mathrm{P},t}$ for some $t > 0$ with $t \neq 1$. Then we have $\mathrm{P} \sim \mathrm{Q}$ with $\mathrm{P} \neq \mathrm{Q}$ and we find, from Corollary 1, that P and Q are either Poisson processes or Pólya–Lundberg processes. Hence, the characterization of mixed Poisson processes with one-dimensional distributions coming from a single exponential family is complete.

3. A martingale characterization of Pólya–Lundberg processes

In the preceding section we emphasized the special position of Pólya–Lundberg processes within the class of mixed Poisson processes. In the sequel, we will deduce a martingale characterization thereof.

First, consider the following proposition, which characterizes the process of densities $\{\mathrm{dQ}_t/\mathrm{dP}_t\}_{t\geq 0} = \{\mathrm{e}^{c(t)X_t+d(t)}\}_{t\geq 0}$ of two equivalent measures $\mathrm{P}, \mathrm{Q} \in \mathcal{M}$ as a martingale.

Proposition 4. *Given a measure* $\mathrm{P} \in \mathcal{M}$ *and functions* $c, d\colon T \to \mathbb{R}$*, the following statements are equivalent.*

(i) *There is a measure* $\mathrm{Q} \in \mathcal{M}$ *such that, for every* $t \in T$*, the measure* Q_t *is absolutely continuous with respect to* P_t *and the corresponding Radon–Nikodým derivative satisfies* $\mathrm{dQ}_t/\mathrm{dP}_t = \mathrm{e}^{c(t)X_t+d(t)}$ P_t*-a.e.*

(ii) *Under* P*, the process* $\{\mathrm{e}^{c(t)X_t+d(t)}\}_{t\in T}$ *is a martingale with respect to the natural filtration* $\{\mathcal{F}_t\}_{t\in T}$ *of* X_T *and the expectation of* $\mathrm{e}^{c(t)X_t+d(t)}$ *is equal to* 1 *for all* $t \in T$.

Proof. We first derive (ii) from (i). Since, for $t \geq 0$, $s \in [0, t]$, and $A \in \mathcal{F}_s$, we have

$$\int_A \frac{\mathrm{dQ}_t}{\mathrm{dP}_t}\,\mathrm{dP} = \int_A \frac{\mathrm{dQ}_s}{\mathrm{dP}_s}\,\mathrm{dP},$$

the process of Radon–Nikodým derivatives $\{\mathrm{e}^{c(t)X_t+d(t)}\}_{t\in T}$ is a martingale with respect to $\{\mathcal{F}_t\}_{t\in T}$.

We now derive (i) from (ii). It is evident that if we define a projective family of measures $\{\mathrm{Q}_t\}_{t\geq 0}$ by $\mathrm{dQ}_t/\mathrm{dP}_t := \mathrm{e}^{c(t)X_t+d(t)}$, then there exists a measure $\mathrm{Q} \in \mathcal{P}$ such that the measures Q_t are the restrictions of Q to $\mathcal{F}_t$. We have to show that Q necessarily lies in $\mathcal{M}$.

For $d \equiv 0$ the measure P itself satisfies (i). Notice that $d \equiv 0$ implies that $1 = \mathrm{E_P}(\mathrm{e}^{c(t)X_t})$ and, hence, that $c \equiv 0$ or $U_{\mathrm{P}} = \delta_0$. Now let $d \not\equiv 0$. We will apply Lemma 1, to show that $d(0) = 0$ and that c and d are continuous and differentiable on $(0, \infty)$ and such that

$$(c'(t)(k-1) + d'(t))\hat{u}_{\mathrm{P}}^{(k-1)}(t) = (\mathrm{e}^{c(t)} - 1)\hat{u}_{\mathrm{P}}^{(k)}(t) \tag{19}$$

holds for $t > 0$ and $k \in \mathbb{N}$.

First note that $d(0) = -\ln \mathrm{E_P}(\mathrm{e}^{c(0)X_0}) = 0$. Continuity and differentiability can be deduced from the martingale property, as follows. Let $t > 0$ be fixed. The martingale property corresponds to

$$\mathrm{E_P}(\mathrm{e}^{c(t)X_t+d(t)} \mid X_s) = \mathrm{e}^{c(s)X_s+d(s)}, \qquad s \in [0, t],$$

because X_T is a Markovian process under P. We can equivalently write

$$\sum_{k=0}^{\infty} \mathrm{e}^{c(t)k+d(t)}\,\mathrm{P}(X_t = k \mid X_s = k_s) = \mathrm{e}^{c(s)k_s+d(s)}, \qquad s \in [0, t],\ k_s \in \mathbb{N}_0.$$

Since the conditional probabilities $\mathrm{P}(X_t = k \mid X_s = k_s)$ are

$$P(X_t = k \mid X_s = k_s) = \begin{cases} (-1)^{k-k_s} \dfrac{(t-s)^{k-k_s}}{(k-k_s)!} \dfrac{\hat{u}_{\mathrm{P}}^{(k)}(t)}{\hat{u}_{\mathrm{P}}^{(k_s)}(s)} & \text{for } s \in [0,t] \text{ and } k \geq k_s, \\ 0 & \text{for } s \in [0,t] \text{ and } k < k_s, \end{cases}$$

we have

$$\mathrm{e}^{c(s)k_s + d(s)} \hat{u}_{\mathrm{P}}^{(k_s)}(s) = \sum_{k=k_s}^{\infty} \mathrm{e}^{c(t)k+d(t)} (-1)^{k-k_s} \frac{(t-s)^{k-k_s}}{(k-k_s)!} \hat{u}_{\mathrm{P}}^{(k)}(t), \qquad s \in [0,t],\ k_s \in \mathbb{N}_0.$$

Since, for $k_s \in \{0, 1\}$, the right-hand side of this equation is a continuous and differentiable function of s for $s \in [0, t]$, and since t can be chosen to be arbitrarily large, the functions c and d must be continuous and differentiable on $(0, \infty)$.

To prove (19) we differentiate the last equation with respect to s, obtaining

$$\begin{aligned} &(c'(s)k_s + d'(s))\mathrm{e}^{c(s)k_s+d(s)} \hat{u}_{\mathrm{P}}^{(k_s)}(s) + \mathrm{e}^{c(s)k_s+d(s)} \hat{u}_{\mathrm{P}}^{(k_s+1)}(s) \\ &\quad = \sum_{k=k_s+1}^{\infty} \mathrm{e}^{c(t)k+d(t)} (-1)^{k-k_s-1} \frac{(t-s)^{k-k_s-1}}{(k-k_s-1)!} \hat{u}_{\mathrm{P}}^{(k)}(t), \qquad s \in (0,t),\ k_s \in \mathbb{N}_0. \end{aligned}$$

Since power series are continuous in their convergence interval, taking the limit $s \uparrow t$ finally leads to

$$(c'(t)k_s + d'(t))\mathrm{e}^{c(t)k_s+d(t)} \hat{u}_{\mathrm{P}}^{(k_s)}(t) + \mathrm{e}^{c(t)k_s+d(t)} \hat{u}_{\mathrm{P}}^{(k_s+1)}(t) = \mathrm{e}^{c(t)(k_s+1)+d(t)} \hat{u}_{\mathrm{P}}^{(k_s+1)}(t)$$

and, thus,

$$(c'(t)k_s + d'(t))\hat{u}_{\mathrm{P}}^{(k_s)}(t) = (\mathrm{e}^{c(t)} - 1)\hat{u}_{\mathrm{P}}^{(k_s+1)}(t), \qquad k_s \in \mathbb{N}_0.$$

Such a representation exists for every $t > 0$. Hence, all the conditions of Lemma 1(ii) are satisfied and, consequently, there exists a measure $\mathrm{Q} \in \mathcal{M}$ that satisfies the conditions of Proposition 4(i) (cf. Lemma 1(i)).

An immediate consequence of Proposition 4 is the following martingale characterization of Pólya–Lundberg processes within the class of mixed Poisson processes.

Theorem 1. *The following statements are equivalent for a measure* $\mathrm{P} \in \mathcal{M}$.

(i) *There are functions* $c, d \colon T \to \mathbb{R}$, *with* d *nonconstant, such that* $\{\mathrm{e}^{c(t)X_t+d(t)}\}_{t\in T}$ *under* P *is a martingale with respect to* $\{\mathcal{F}_t\}_{t\in T}$.

(ii) *The measure* P *corresponds either to a Poisson process or to a Pólya–Lundberg process.*

Proof. Theorem 1 follows from Propositions 3 and 4.

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