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# AGEING PROPERTIES AND SERIES SYSTEMS

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**Abstract**

The comparison of lifetimes has been treated extensively during the last decade. A wide variety of mathematical objects have been defined, which, in reliability theory, are used to quantify ageing properties. In this work, using the equilibrium variable, we give a new viewpoint on ageing properties. Moreover, we give new bounds on the moments of series systems.

*Keywords:* Ageing property; stochastic order; series system; IFR; DMRL; HNBUE

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## 1. Introduction

Ageing was defined for positive random variables forty years ago. In the last decade, many developments have taken place. In order to take into account the numerous fields of application, various new approaches have been proposed. Usually an ageing property is based on a comparison (with respect to a stochastic order) between the lifetime and the residual time (or another lifetime).

### 1.1. The equilibrium variable

Let $X$ be a nonnegative continuous random variable with absolutely continuous distribution $F_X$. Let us denote the survival distribution by $\overline{F}_X = 1 - F_X$, the density function by $f_X$, the expectation, which is assumed to be finite, by $m = \mathrm{E}[X]$, and the second-order moment, which is also assumed to be finite, by $m_2 = \mathrm{E}[X^2]$.

Now, let $\tilde{X}$ be a random variable with density function $f_{\tilde{X}}$, defined from the tail of $X$ by

$$f_{\tilde{X}}(t) = \frac{\overline{F}_X(t)}{m} \quad \text{for any positive } t.$$

Its survival function is denoted by $\overline{F}_{\tilde{X}}(\cdot)$, and its mean $\tilde{m}$ is well known to be

$$\tilde{m} = \frac{m_2}{2m}.$$

We call $\tilde{X}$ the *equilibrium variable* of $X$. Its distribution is known as the *equilibrium distribution* of $X$ in renewal process theory. It is also called the *integrated tail function*.

If the variable $X$ has a finite moment of order $k$, $k \geq 2$, then the equilibrium variable of order $k$ is defined as

$$\tilde{X}^{(k)} = \widetilde{\tilde{X}^{(k-1)}}$$

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using the following density:

$$f_{\tilde{X}^{(k)}}(t) = \frac{\overline{F}_{\tilde{X}^{(k-1)}}(t)}{\mathrm{E}[\tilde{X}^{(k-1)}]},$$

where, for $k = 1$, $\tilde{X}^{(1)} = \tilde{X}$.

Let $l_X = \min\{u : \overline{F}_X(u) = 0\}$ and, for $t \leq l_X$, let us define $X_t$, the residual lifetime of $X$, using its survival distribution, as follows:

$$\overline{F}_{X_t}(x) = \frac{\overline{F}_X(t+x)}{\overline{F}_X(t)}.$$

Also, the hazard rate is defined by

$$h_X(t) = \frac{f_X(t)}{\overline{F}_X(t)}.$$

### 1.2. Usual stochastic orders

In order to compare different ways to define ageing, let us recall the usual definitions of the partial stochastic orders that we will use (see, for example, Shaked and Shanthikumar (1994)).

**Definition 1.** Let $X$ and $Y$ be nonnegative continuous random variables such that $l_X = \min\{u : \overline{F}_X(u) = 0\}$ and $l_Y = \min\{u : \overline{F}_Y(u) = 0\}$, respectively, and $l = \min\{l_X, l_Y\}$.

(a) The random variable $X$ is said to be smaller than $Y$ with respect to the usual stochastic order (written $X \leq_{\mathrm{st}} Y$) if, for any $u \geq 0$,

$$\overline{F}_X(u) \leq \overline{F}_Y(u).$$

(b) The random variable $X$ is said to be smaller than $Y$ in the hazard-rate order (written $X \leq_{\mathrm{hr}} Y$) if, for any $u \in [0, l]$,

$$h_X(u) \geq h_Y(u)$$

or, equivalently,

$$\frac{\overline{F}_X(u)}{\overline{F}_Y(u)} \quad \text{is decreasing.} \tag{1}$$

(c) The random variable $X$ is said to be smaller than $Y$ in the likelihood ratio (denoted by $X \leq_{\mathrm{lr}} Y$) if, for any $0 \leq u \leq v \leq l$,

$$f_X(u)f_Y(v) - f_X(v)f_Y(u) \geq 0.$$

These definitions come from different fields of reliability theory, and they are used in the limiting case, where $l = +\infty$.

For two nonnegative random variables $X$ and $Y$, the following chain of implications is well known:

$$X \leq_{\mathrm{lr}} Y \Rightarrow X \leq_{\mathrm{hr}} Y \Rightarrow X \leq_{\mathrm{st}} Y. \tag{2}$$

### 1.3. Ageing properties

The ageing property is introduced in reliability to quantify the fact that a system which is functioning at time $t$ has a survival lifetime that is less than the initial one. The comparison can be based upon various criteria depending on stochastic order. The most popular notions of ageing are based on the comparison of the residual lifetime to the initial lifetime with respect to the failure rate and the stochastic and mean orders (see, for example, Desphande *et al.* (1986), Klefsjö (1982), and Shaked and Shanthikumar (1994)).

We propose to define these ageing properties in terms of equilibrium variables.

**Definition 2.** A nonnegative continuous random variable $X$ can be defined as follows. (This list is not exhaustive.)

(a) The random variable $X$ is *increasing failure rate (IFR)* if, for any $t \in [0, l_X]$,

$$h_X(t) = \frac{f_X(t)}{\overline{F}_X(t)} \quad \text{is increasing,}$$

or, equivalently,

$$\tilde{X} \leq_{\mathrm{lr}} X.$$

(b) The random variable $X$ is *decreasing mean residual life (DMLR)* if, for any $t \in [0, l_X]$,

$$\mathrm{E}[X_t] = \frac{\int_t^\infty \overline{F}(u)\,\mathrm{d}u}{\overline{F}(t)} \quad \text{is decreasing,}$$

or, equivalently (see, for example, Belzunce *et al.* (1999)),

$$\tilde{X} \text{ is IFR,} \tag{3}$$

or, equivalently,

$$\tilde{X} \leq_{\mathrm{hr}} X. \tag{4}$$

(c) The random variable $X$ is *harmonic new better than used in expectation (HNBUE)* if, for any $x \geq 0$,

$$\int_x^\infty \overline{F}_X(u)\,\mathrm{d}u \leq m\mathrm{e}^{-x/m}.$$

This is equivalent to (see Klefsjö (1982))

$$\tilde{X} \leq_{st} \mathcal{E}_m,$$

where $\mathcal{E}_m$ is an exponential random variable with mean $m$.

The following implications are well known (see, for example, Shaked and Shanthikumar (1994)):

$$\text{IFR} \Rightarrow \text{DMRL} \Rightarrow \text{HNBUE}. \tag{5}$$

## 2. Ageing properties and series system

In this section, we are interested in the mean time of a series system. We recall the following result (see Shaked and Shanthikumar (1994)) about the preservation of the orders st and hr.

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### 2.1. Stability by series

**Proposition 1.** *Let $X_1, \dots, X_n$ and $Y_1, \dots, Y_n$ be independent random variables with survival functions $\overline{F}_{X_1}, \dots, \overline{F}_{X_n}$ and $\overline{F}_{Y_1}, \dots, \overline{F}_{Y_n}$, respectively.*

*If $X_i \leq_* Y_i$ for all $i = 1, \dots, n$, then*

$$X_1 \wedge \dots \wedge X_n \leq_* Y_1 \wedge \dots \wedge Y_n,$$

*where $\leq_*$ denotes st or hr order and $X \wedge Y = \min(X, Y)$.*

*For lr ordering this implication is true for independent, identically distributed variables.*

The stability-by-series system is quite natural in an ageing context because of the following relation:

$$(X \wedge Y)_t =_{\mathrm{st}} X_t \wedge Y_t \quad \text{for all } t \geq 0.$$

Therefore, it is sufficient to have stability of the minimum.

The stability of the ageing property DMRL, based upon the residual life of $k$-out-of-$n$ systems, is given by Li and Zuo (2002).

### 2.2. Bounds for the moment of a series system

Bounds exist for the mean time-to-failure of a series system in the case of ageing components. Here, we present bounds for the moments of order $n$, $n \geq 2$, under the DMRL assumption. First, recall the following results.

**Lemma 1.** (Klefsjö (1982, p. 333).) *Let $X_1, \dots, X_p$ be $p$ independent nonnegative random variables. If $X_i$ has the HNBUE property for all $i$, then*

$$\mathrm{E}[X_1 \wedge \dots \wedge X_p] \geq \frac{1}{\sum_{i=1}^{p} (1/\mathrm{E}[X_i])}. \tag{6}$$

**Proposition 2.** *Let $X$ be a nonnegative random variable and $\tilde{X}^{(k)}$ be the equilibrium variable of $X$ of order $k$. If $X$ has a finite moment of order $n$ (denoted by $m_n$), then*

$$m_n = n! \prod_{k=0}^{n-1} \tilde{m}^{(k)} \quad \text{for all } n \geq 2, \tag{7}$$

*where $\tilde{m}^{(k)} = \mathrm{E}[\tilde{X}^{(k)}]$ is the mean of the equilibrium variable $\tilde{X}^{(k)}$. Moreover, $\tilde{m}^{(0)} = m$ and $\tilde{m}^{(1)} = \tilde{m}$.*

*Proof.* We have seen that, for $n = 2$,

$$m_2 = 2m\tilde{m}.$$

Integrating by parts gives

$$m_n = n\tilde{m}^{(n-1)} m_{n-1},$$

which proves (7) by induction.

**Corollary 1.** *Let $X$ be a nonnegative random variable and $\tilde{X}^{(n)}$ be the equilibrium variable of $X$ of order $n$. If $X$ has a finite moment $m_n$ of order $n$, then the mean of $\tilde{X}^{(n-1)}$ is equal to*

$$\tilde{m}^{(n-1)} = \mathrm{E}[\tilde{X}^{(n-1)}] = \frac{m_n}{n m_{n-1}}.$$

The connection between the ageing properties of the variable and its equilibrium variable is presented in the following result.

**Lemma 2.** *Let $X$ be a nonnegative random variable with a finite moment of order $n$. If $X$ has the DMRL property, then the equilibrium variable $\tilde{X}^{(n)}$ of $X$ of order $n$ is IFR.*

*Proof.* From the DMRL property of $X$ and the property given by (3), $\tilde{X}$ is IFR. Using the relation (5) gives the DMRL property of $\tilde{X}$. Thus, $\tilde{X}^{(2)}$ is IFR and, clearly, $\tilde{X}^{(n)}$ has the IFR property.

**Theorem 1.** *Let $X$ and $Y$ be independent nonnegative random variables. Let $\tilde{X}^{(k)}$ and $\tilde{Y}^{(k)}$ be, respectively, the independent equilibrium variables of $X$ and $Y$ of order $k$. If $X$ and $Y$ have the DMRL property and if the $n$th moments of $X$ and $Y$ are finite, then*

$$\tilde{X}^{(n)} \wedge \tilde{Y}^{(n)} \leq_{\mathrm{lr}} \widetilde{X \wedge Y}^{(n)}, \tag{8}$$

*where $\widetilde{X \wedge Y}^{(n)}$ is the equilibrium variable of $X \wedge Y$ of order $n$.*

*Proof.* For $n = 1$, $\tilde{X}$ and $\tilde{Y}$ are independent and, thus, the density of $\tilde{X} \wedge \tilde{Y}$, for $t \in [0, l]$, can be written as follows:

$$f_{\tilde{X}\wedge\tilde{Y}}(t) = f_{\tilde{X}}(t)\overline{F}_{\tilde{Y}}(t) + f_{\tilde{Y}}(t)\overline{F}_{\tilde{X}}(t).$$

The density of $\widetilde{X \wedge Y}$ is

$$f_{\widetilde{X\wedge Y}}(t) = \frac{\overline{F}_X(t)\overline{F}_Y(t)}{\mathrm{E}[X \wedge Y]}.$$

Using the notation $m = \mathrm{E}[X]$, $r = \mathrm{E}[Y]$, $m(t) = \mathrm{E}[X_t]$, and $r(t) = \mathrm{E}[Y_t]$, the likelihood ratio can be written as

$$\frac{f_{\tilde{X}\wedge\tilde{Y}}}{f_{\widetilde{X\wedge Y}}}(t) = \frac{\mathrm{E}[X \wedge Y]}{mr}(m(t) + r(t)).$$

From the assumption that $X$ and $Y$ have the DMRL property, we find that $m(t)$ and $r(t)$ are decreasing. Thus, for any $0 \leq u \leq v \leq l$,

$$f_{\tilde{X}\wedge\tilde{Y}}(u) f_{\widetilde{X\wedge Y}}(v) - f_{\tilde{X}\wedge\tilde{Y}}(v) f_{\widetilde{X\wedge Y}}(u) \geq 0,$$

which leads to $\tilde{X} \wedge \tilde{Y} \leq_{\mathrm{lr}} \widetilde{X \wedge Y}$.

Now, by induction, (8) is assumed to be true for $n - 1$, that is,

$$\tilde{X}^{(n-1)} \wedge \tilde{Y}^{(n-1)} \leq_{\mathrm{lr}} \widetilde{X \wedge Y}^{(n-1)}.$$

The density of $\tilde{X}^{(n)} \wedge \tilde{Y}^{(n)}$ is

$$f_{\tilde{X}^{(n)}\wedge\tilde{Y}^{(n)}}(t) = f_{\tilde{X}^{(n)}}(t)\overline{F}_{\tilde{Y}^{(n)}}(t) + f_{\tilde{Y}^{(n)}}(t)\overline{F}_{\tilde{X}^{(n)}}(t)$$

and the density of $\widetilde{X \wedge Y}^{(n)}$ is

$$f_{\widetilde{X\wedge Y}^{(n)}}(t) = \frac{\overline{F}_{\widetilde{X\wedge Y}^{(n-1)}}(t)}{\mathrm{E}\big[\widetilde{X \wedge Y}^{(n-1)}\big]}.$$

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The likelihood ratio $f_{\tilde{X}^{(n)}\wedge\tilde{Y}^{(n)}}(t)/f_{\widetilde{X\wedge Y}^{(n)}}(t)$ is

$$\mathrm{E}\big[\widetilde{X\wedge Y}^{(n-1)}\big]\bigg(\frac{\overline{F}_{\tilde{X}^{(n-1)}}(t)\overline{F}_{\tilde{Y}^{(n)}}(t)}{\mathrm{E}[\tilde{X}^{(n-1)}]\overline{F}_{\widetilde{X\wedge Y}^{(n-1)}}(t)}+\frac{\overline{F}_{\tilde{Y}^{(n-1)}}(t)\overline{F}_{\tilde{X}^{(n)}}(t)}{\mathrm{E}[\tilde{Y}^{(n-1)}]\overline{F}_{\widetilde{X\wedge Y}^{(n-1)}}(t)}\bigg).$$

Since $X$ and $Y$ have the DMRL property, the equilibrium variables of order $k$, $\tilde{X}^{(k)}$ and $\tilde{Y}^{(k)}$, are IFR for all $k\geq 1$ (see Lemma 2).

The relation (5) implies that $\tilde{Y}^{(n-1)}$ is DMRL and, from (4,) we obtain

$$\tilde{Y}^{(n)}\leq_{\mathrm{hr}}\tilde{Y}^{(n-1)}.$$

The preservation of hazard-rate order in a series system (see Proposition 1) gives

$$\tilde{X}^{(n-1)}\wedge\tilde{Y}^{(n)}\leq_{\mathrm{hr}}\tilde{X}^{(n-1)}\wedge\tilde{Y}^{(n-1)}.$$

The induction assumption is

$$\tilde{X}^{(n-1)}\wedge\tilde{Y}^{(n-1)}\leq_{\mathrm{lr}}\widetilde{X\wedge Y}^{(n-1)},$$

and the implication (2) gives

$$\tilde{X}^{(n-1)}\wedge\tilde{Y}^{(n-1)}\leq_{\mathrm{hr}}\widetilde{X\wedge Y}^{(n-1)},$$

which leads to

$$\tilde{X}^{(n-1)}\wedge\tilde{Y}^{(n)}\leq_{\mathrm{hr}}\widetilde{X\wedge Y}^{(n-1)}.$$

From the relation (1), the ratio

$$\frac{\overline{F}_{\tilde{X}^{(n-1)}}(t)\overline{F}_{\tilde{Y}^{(n)}}(t)}{\mathrm{E}[\tilde{X}^{(n-1)}]\overline{F}_{\widetilde{X\wedge Y}^{(n-1)}}(t)}$$

and, symmetrically, the ratio

$$\frac{\overline{F}_{\tilde{Y}^{(n-1)}}(t)\overline{F}_{\tilde{X}^{(n)}}(t)}{\mathrm{E}[\tilde{Y}^{(n-1)}]\overline{F}_{\widetilde{X\wedge Y}^{(n-1)}}(t)}$$

are both decreasing, which completes the proof.

A consequence of this result is given in the following corollary.

**Corollary 2.** *Let $X$ and $Y$ be independent nonnegative random variables. If $X$ and $Y$ have the DMRL property, then, for any $x\geq 0$,*

$$\int_x^\infty\overline{F}_X(u)\overline{F}_Y(u)\,\mathrm{d}u\geq\frac{1}{m+r}\int_x^\infty\overline{F}_X(u)\,\mathrm{d}u\int_x^\infty\overline{F}_Y(u)\,\mathrm{d}u.$$

*Proof.* Since $X$ and $Y$ are DMRL, using Theorem 1 we obtain

$$\tilde{X}\wedge\tilde{Y}\leq_{\mathrm{lr}}\widetilde{X\wedge Y}$$

and, from the relation (2),

$$\tilde{X}\wedge\tilde{Y}\leq_{\mathrm{st}}\widetilde{X\wedge Y}.$$

This inequality can be written as

$$\int_x^\infty \overline{F}_X(u)\overline{F}_Y(u)\,\mathrm{d}u \geq \frac{\mathrm{E}[X\wedge Y]}{mr}\int_x^\infty \overline{F}_X(u)\,\mathrm{d}u \int_x^\infty \overline{F}_Y(u)\,\mathrm{d}u.$$

The relations (5) and (6) give

$$\mathrm{E}[X\wedge Y] \geq \frac{1}{1/m+1/r},$$

and the above two inequalities yield

$$\int_x^\infty \overline{F}_X(u)\overline{F}_Y(u)\,\mathrm{d}u \geq \frac{1}{m+r}\int_x^\infty \overline{F}_X(u)\,\mathrm{d}u \int_x^\infty \overline{F}_Y(u)\,\mathrm{d}u.$$

Theorem 1 can be generalized to the case of $p$ random variables.

**Theorem 2.** *Let $X_1,\dots,X_p$ be $p$ independent nonnegative random variables and $\tilde{X}_i^{(k)}$, $i=1,\dots,p$, their independent equilibrium variables of order $k$. If $X_i$ has the DMRL property and the moment of order $n$, $n\geq 1$, exists, then*

$$\tilde{X}_1^{(n)}\wedge\cdots\wedge\tilde{X}_p^{(n)} \leq_{\mathrm{lr}} (X_1\widetilde{\wedge\cdots\wedge} X_p)^{(n)}$$

*for all $p\geq 2$.*

Now we are ready to propose a lower bound on the $n$th moment of a series system.

**Theorem 3.** *Let $X$ and $Y$ be independent nonnegative random variables and let $m_n$ and $r_n$, $n\geq 2$, be their $n$th moments, respectively.*

*If $X$ and $Y$ have the DMRL property, then*

$$\mathrm{E}[(X\wedge Y)^n] \geq n!\prod_{k=0}^{n-1}\frac{1}{1/\tilde{m}^{(k)}+1/\tilde{r}^{(k)}},$$

*where*

$$\tilde{m}^{(k)} = \frac{m_{k+1}}{km_k} \quad \text{and} \quad \tilde{r}^{(k)} = \frac{r_{k+1}}{kr_k}$$

*are the means of $\tilde{X}^{(k)}$ and $\tilde{Y}^{(k)}$ respectively.*

*Proof.* The $n$th moment $\mu_n$ of the random variable $X\wedge Y$ can be written, using Proposition 2 (see (7)), as

$$\mu_n = n!\prod_{k=0}^{n-1}\mathrm{E}\big[\widetilde{X\wedge Y}^{(k)}\big].$$

However, Theorem 1 gives

$$\tilde{X}^{(k)}\wedge\tilde{Y}^{(k)} \leq_{\mathrm{lr}} \widetilde{X\wedge Y}^{(k)}.$$

Since $\tilde{X}^{(k)}$ and $\tilde{Y}^{(k)}$ have the IFR property, (6) gives

$$\mathrm{E}[\tilde{X}^{(k)}\wedge\tilde{Y}^{(k)}] \geq \frac{1}{1/\tilde{m}^{(k)}+1/\tilde{r}^{(k)}},$$

so we obtain

$$\mathrm{E}\big[\widetilde{X \wedge Y}^{(k)}\big] \geq \mathrm{E}[\tilde{X}^{(k)} \wedge \tilde{Y}^{(k)}] \geq \frac{1}{1/\tilde{m}^{(k)} + 1/\tilde{r}^{(k)}},$$

and the result follows.

It is worth noting that this inequality is an equality in the exponential case. In terms of reliability, the lower bound is particularly useful.

The result in Theorem 3 can also be generalized to the case of $p$ random variables.

**Theorem 4.** *Let $X_1, \dots, X_p$ be $p$ independent nonnegative random variables. If $X_i$ has the DMRL property and its nth moment $m_n$ exists (for all $i$), then the nth moment of the random variable $X_1 \wedge \dots \wedge X_p$ satisfies*

$$\mathrm{E}[(X_1 \wedge \dots \wedge X_p)^n] \geq n! \prod_{k=0}^{n-1} \frac{1}{\sum_{i=1}^{p} 1/\tilde{m}_i^{(k)}}.$$

**Corollary 3.** *Let $X_1, \dots, X_p$ be $p$ independent, identically distributed nonnegative random variables. If $X_i$ has the DMRL property and its nth moment exists (for all $i$), then the nth moment of $X_1 \wedge \dots \wedge X_p$ has the lower bound*

$$\mathrm{E}[(X_1 \wedge \dots \wedge X_p)^n] \geq \frac{\mathrm{E}[(X_1^n)]}{p^n}. \tag{9}$$

A similar corollary can be deduced for other families of ageing properties, not only DMRL. For example, $\mathrm{DRL}_{\mathrm{Lt}}$ is a class introduced by Belzunce *et al.* (1999), (2001) from the Laplace order, and our formula (9) is valid for $\mathrm{DRL}_{\mathrm{Lt}}$.

As in the previous case, the inequality turns out to be an equality in the case of an exponential distribution. Computing the lower bounds above requires the computation of certain moments of the equilibrium variables. There are many ways to approximate these moments when the distribution is not completely known. A special case, which is particularly interesting, is $n = 2$ when the mean time-to-failure must be estimated with few data points.

## References

Belzunce, F., Ortega, E. and Ruiz, J. M. (1999). The Laplace order and ordering of residual lives. *Statist. Prob. Lett.* **42,** 145–156.

Belzunce, F., Ortega, E. and Ruiz, J. M. (2001). A note on stochastic comparisons of excess lifetimes of renewal processes. *J. Appl. Prob.* **38,** 747–753.

Desphande, J. V., Kochar, S. C. and Singh, H. (1986). Aspects of positive ageing. *J. Appl. Prob.* **23,** 748–758.

Klefsjö, B. (1982). The HNBUE and HNWUE classes of life distributions. *Naval Res. Logistics Quart.* **29,** 331–344.

Li, X. and Zuo, J. (2002). On the behaviour of some new ageing properties based upon the residual life of $k$-out-of-$n$ systems. *J. Appl. Prob.* **39,** 426–433.

Shaked, M. and Shanthikumar, J. G. (1994). *Stochastic Orders and Their Applications.* Academic Press, New York.