COMPACTNESS IN Hom(G, H)

H. H. CORSON AND I. GLICKSBERG

Let G be a locally compact abelian group with Bohr compactification G^a. Then [**3**, Theorem 1.2] any subset F of G compact in G^a is necessarily compact in G; alternatively, any *closed non-compact subset* F of G has its closure F^- in $G^a \neq F$; hence $F^- \backslash F \neq \emptyset$. One of our aims in the present note is to give a result (Corollary 6) which asserts that $F^- \backslash F$ has no points which are G_δs, so that $F^- \backslash F$ is a perfect set. Another aim is to give an extension of a cited result of [**3**] in which commutativity and local compactness are essentially irrelevant, and to unify the proofs.

For any topological space G, let G_d be the discrete version of G. Also, for any topological groups G, H, let Hom(G, H) be the space formed from the continuous homomorphisms from G to H by using the compact open topology. Then we have a continuous injection

$$\text{Hom}(G, H) \subseteq \text{Hom}(G_d, H)$$

while the topology in the image is that of pointwise convergence on G; evidently, any compact subset F of Hom(G, H) is compact in Hom(G_d, H). Our first result is the following partial converse.

THEOREM 1. *Suppose that G and H are topological groups and every closed subgroup of G is a Baire space.*† *Then $F \subset$ Hom(G, H) and compact in* Hom(G_d, H) *is necessarily compact in* Hom(G, H).

If G is locally compact abelian and H is the circle group, then this is precisely the assertion of [**3**, Theorem 1.2] for the character group of G.

Our proof of Theorem 1 has two sides: compactness arguments (which yield sequential compactness of F in Hom(G_d, H), and centre mainly about a simple argument of Grothendieck [**4**] also used in [**3**]), and a category result, quite possibly known and reminiscent of results of Baire (see [**1**, § 5, Exercise 23]), with which we shall begin, after giving some notation and a remark.

For topological spaces X and Y, $C(X, Y)$ denotes the continuous functions from X to Y in the compact open topology; if $Y = R$, the reals, we write $C(X)$.

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†That is, the Baire category theorem holds for each open subset U: U is not a countable union of nowhere dense sets. Note that some such hypothesis is needed; for example, take G a Hilbert space in the weak topology and H the reals.

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Remark. We shall use two simple and well-known facts about equicontinuous functions: (1) Suppose that F is compact in $C(X_d, Y)$ and that $x_0 \in X$. Then F is equicontinuous at x_0 for one uniform structure on Y consistent with the topology of Y if and only if the same is true for any such uniformity. (2) Suppose that F is as above and that D is a dense subset of F which is contained in $C(X, Y)$. If the restriction of D to each compact subset L of X is equicontinuous, as a set of functions from L to Y, then the closure†† of D in the compact open topology is F, and F is compact in this topology.

Proof of (1) *and* (2). (1) Follows easily from observing that F is equicontinuous at x_0 if and only if the map $(x, f) \to f(x)$ is continuous at (x_0, f_0) for each $f_0 \in F$. (Use compactness of F for the "if" part.)

(2) The usual proof of Ascoli's Theorem actually proves this slight generalization.

LEMMA 2. *Let X be a Baire space, Y a separable metric space, and F a compact metric subspace of $C(X_d, Y)$. Assume that there is a dense subspace D of F contained in $C(X, Y)$. Then F is equicontinuous at each point of a dense G_δ in X.*

Proof. By hypothesis, Y is topologized by a countable collection of real functions $\{g_i\}$. Note that $\{g_i \circ f: f \in F\}$ is a subset of $C(X_d)$ satisfying our hypotheses on F. Thus if we can establish the theorem for $Y = R$, the more general version will follow by the above Remark, using $\{g_i\}$ to define a metric on Y.

Hence we assume that $F \subset C(X_d)$. This defines a map e from X into $C(F)$ by evaluation. Let $C'(F)$ denote $C(F)$, but under the topology of pointwise convergence on D. It is easy to check that the identity map i from $C'(F)$ to $C(F)$ has the property that the inverse image of each closed ball in $C(F)$ is closed in $C'(F)$.

Now let $B_1, B_2, \ldots$ be a cover of $C(F)$ by closed balls of radius less than $1/n$. Then $\{i^{-1}(B_s)\}$ forms a closed cover for $C'(F)$. However, from our hypotheses, e is continuous from X into $C'(F)$. Hence $\{(i \circ e)^{-1}(B_s)\}$ forms a closed cover for X. It follows that there is a dense open subset U_n of X such that each point of U_n has a neighbourhood which is mapped into some B_s by $i \circ e$. Consequently, $\bigcap U_n$ is a dense G_δ in X, and $i \circ e$ is continuous at each point of $\bigcap U_n$. Since F is compact, $C(F)$ has the topology of uniform convergence; hence this means that for each $x \in \bigcap U_n$ and $\epsilon > 0$ there is a neighbourhood V of x for which $|f(x) - f(y)| < \epsilon$, all $f \in F$, $y \in V$, i.e., that F is equicontinuous at x.

COROLLARY 3. *Let G be a topological group each of whose closed subgroups is a Baire space, and let H be any topological group. Suppose that F is a compact metric subset of* $\mathrm{Hom}(G_d, H)$ *with a dense subset D contained in* $\mathrm{Hom}(G, H)$. *Then F is compact in the compact open topology.*

††The closure is in the space of all maps from X to Y whose restrictions to compacta are continuous.

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Proof. By the above Remark (2), we need only show that $D|K$ is equicontinuous on any compact $K \subset G$. Since any neighbourhood of the identity in H is the inverse image of a corresponding neighbourhood in a metric quotient of H, it suffices to show equicontinuity of the corresponding set of mappings from X into the metric quotient. Let $\{f_n\}$ be a countable dense subset of D.

Let G_0 be the closed subgroup K generates in G. Since $f_n(K)$ is compact metric, $\bigcup_n f_n(K)$ is necessarily separable in H, and so the closed subgroup H_0 generated by $\bigcup_m f_m(K)$ is separable, and contains $D(G_0)$. If we give H_0 a metric consistent with its topology, then $D|G_0$, as a subset of $\mathrm{Hom}(G_0, H_0)$, is equicontinuous at some $x_0 \in G_0$ by Lemma 2. By the Remark (1) above, we see that $D|G_0$ is equicontinuous at x_0 using the group uniformity on H_0. Therefore, $D|G_0$ is equicontinuous at each point of G_0 by translation.

As was noted there, [**3**, Theorem 1.2] gave an analogue of the uniform boundedness principle for locally compact abelian groups, and the same can be said for Theorem 1. But here the Banach space result follows from the same derivation: specifically, Corollary 3 contains the Banach-Steinhaus theorem, which easily yields uniform boundedness.

Our other ingredient in the proof of Theorem 1 is given by the following lemma of Grothendieck [**4**] related to a result of Smulian [**2**] and its argument.

LEMMA 4. *Let K be compact and Y metric and let F be any compact subset of $C(K_d, Y)$. If F is contained in $C(K, Y)$, then any sequence $\{f_n\}$ in F has a compact metric closure in $C(K_d, Y)$.*

Proof. Let K_0 be the compact quotient space of K formed by identifying points not separated by elements of $\{f_n\}^-$ and K_1 that obtained using only $\{f_n\}$. Evidently, K_1 is a metric space and we have a continuous map of K_0 onto K_1. But since points separated by $\{f_n\}^-$ must in fact be separated by $\{f_n\}$ itself, the map is one-to-one and K_0 and K_1 are homeomorphic; thus K_0 is compact metric. Now let $\{x_i\}$ be a countable dense set in K_0. It follows as above that the topology of pointwise convergence in $\{f_n\}^-$ is the same as the topology of pointwise convergence on $\{x_i\}$. Hence $\{f_n\}^-$ is compact metric.

Proof of Theorem 1. By the Remark (2) above, it suffices to show that $F|K$ is equicontinuous on each compact $K \subset G$.

To see that $F|K$ is equicontinuous we can assume that H is metric (exactly as in the proof of Corollary 3, replacing H by a metric quotient H_0 and F by its image in $\mathrm{Hom}(G, H_0)$, evidently a compact subset of $\mathrm{Hom}(G_d, H_0)$). Now conditional compactness in the metric space $C(K, H)$ is equivalent to conditional countable compactness; thus, if $F|K$ is not conditionally compact in $C(K, H)$, i.e., not equicontinuous, we have a sequence $\{f_n|K\}$, $f_n \in F$, which has no cluster point in $C(K, H)$. But $F|K$ is compact in $C(K_p, H)$ since F is compact in $\mathrm{Hom}(G_p, H)$, and thus by Lemma 4 we have $\{f_n|K\}^-$ compact metric in the topology of pointwise convergence on K. If G_0 is the closed subgroup of G generated by K, then since points of F separated by

G_0 are already separated by K, we conclude, by exactly the argument used in the proof of Lemma 4, that the topology of pointwise convergence on G_0 and that of pointwise convergence on K coincide on F. Thus $\{f_n|G_0\}^-$ is compact metric under pointwise convergence on G_0, and hence by Corollary 3, $\{f_n|G_0\}^-$ is compact in $\text{Hom}(G_0, H)$. Consequently, $\{f_n|K\}$ has a cluster point in $C(K, H)$ after all, a contradiction showing that $F|K$ is conditionally compact in $C(K, H)$ and completing our proof of Theorem 1.

One immediate consequence of Theorem 1 is that if F is a closed but a non-compact subset of $\text{Hom}(G, H)$, its closure F^- in $\text{Hom}(G_d, H)$, if compact, is distinct from F, i.e. $F^-\backslash F \neq \emptyset$. But more can be said.

THEOREM 5. *Suppose that G and H are topological groups and every closed subgroup of G is a Baire space. Suppose that every element of* $\text{Hom}(G_d, H)$ *which is continuous on all compact subsets of G is continuous (as is the case if G is locally compact or metric). Then if F is a closed subset of* $\text{Hom}(G, H)$ *whose closure F^- in* $\text{Hom}(G_d, H)$ *is compact, the subspace $F^-\backslash F$ of* $\text{Hom}(G_d, H)$ *has no points which are G_δs (and hence, is a perfect set).*

Proof. Suppose that $f_0 \in F^-\backslash F$ is a G_δ in $F^-\backslash F$; thus

$$\bigcap_n U_n \cap (F^-\backslash F) = \{f_0\},$$

where $\{U_n\}$ is a decreasing sequence of compact neighbourhoods of f_0 in F^-. If $\bigcap_n U_n \cap F = \emptyset$, our proof is complete since by choosing $f_n \in U_n \cap F$ we obtain a sequence any of whose cluster points lie in

$$\bigcap_n U_n \cap F^- = \bigcap_n U_n \cap (F^-\backslash F) = \{f_0\};$$

hence $f_n \to f_0$ in $\text{Hom}(G_d, H)$ by compactness of F^-. By Corollary 3, $f_n \to f_0$ in††† $\text{Hom}(G, H)$, and $f_0 \in F$ since F is closed in $\text{Hom}(G, H)$, a contradiction.

Thus, we can assume that $E = \bigcap_n U_n \cap F \neq \emptyset$. If $E = E^-$, its closure in $\text{Hom}(G_d, H)$, then E is compact in F^-, $f_0 \notin E$, and we can replace U_n by a compact subneighbourhood V_n of f_0 missing E, and hence obtain $\bigcap_n V_n \cap F = \emptyset$, yielding a contradiction as before. Thus

$$E \neq E^- \subset \bigcap_n U_n \cap F^- = E \cup (\bigcap U_n \cap (F^-\backslash F)) = E \cup \{f_0\},$$

hence $E^- = E \cup \{f_0\} \neq E$.

Now any compact subset of the compact subspace E^- of $\text{Hom}(G_d, H)$ which lies in E is contained in $\text{Hom}(G, H)$, and hence is compact in the compact open topology by Theorem 1. Each $f \in E$ has a compact neighbourhood V in E^- avoiding f_0, hence compact in $\text{Hom}(G, H)$, and since $\text{Hom}(G, H) \to \text{Hom}(G_d, H)$ is continuous, V is a neighbourhood of f in the subspace E of $\text{Hom}(G, H)$. Thus E is a locally compact subspace of $\text{Hom}(G, H)$.

†††$f_n \to f_0$ in the compact open topology; thus f_0 is continuous on compacta, hence in $\text{Hom}(G, H)$ by hypothesis.

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Moreover, the obvious one-to-one map of the one-point compactification E_∞ of E into $E^- = E \cup \{f_0\}$ sending ∞ onto f_0 is continuous at ∞ since then if U is an open neighbourhood of f_0 in E^-, $E^-\backslash U = E\backslash U$ is compact in $\mathrm{Hom}(G_d, H)$ and lies in $\mathrm{Hom}(G, H)$, hence is compact there, i.e., $(E \cap U) \cup \{\infty\}$ is a neighbourhood of ∞. Continuity elsewhere is evident so we have a homeomorphism: E^- is the one-point compactification of E.

Now E is not compact in $\mathrm{Hom}(G, H)$ (otherwise E is compact in $\mathrm{Hom}(G_d, H)$, hence equals E^-). As a consequence, E cannot be countably compact in $\mathrm{Hom}(G, H)$: for then it would be equicontinuous on compacta as we saw in the proof of Theorem 1, so that by the Remark (2) above, f_0 would be in the closure of E in the compact open topology, whence f_0 is continuous on compacta and therefore in $\mathrm{Hom}(G, H)$ by our final hypothesis. But now by Theorem 1, since $E^- \subset \mathrm{Hom}(G, H)$, E^- is compact in $\mathrm{Hom}(G, H)$, as is its closed (in $\mathrm{Hom}(G, H)$) subset E, while E is not compact there, as we have seen.

Thus we have a sequence $\{f_n\}$ in E with no cluster point in $\mathrm{Hom}(G, H)$. Hence $f_n \to \infty$ in its locally compact subspace E, and therefore $f_n \to f_0$ in E^-, i.e., pointwise. By Corollary 3, we have $f_n \to f_0$ uniformly on compacta, hence $f_0 \in \mathrm{Hom}(G, H)$ as before, and now we arrive again at the conclusion that E is compact in $\mathrm{Hom}(G, H)$, a contradiction, completing our proof.

COROLLARY 6. *If G is a locally compact abelian group and F is a closed non-compact subset with closure F^- in G^a, then $F^-\backslash F$ has no points which are G_δs and is a non-void perfect set.*

We need apply Theorem 5 to F as a subset of $\mathrm{Hom}(G, H)$, H the circle group.

Some of the results of [**3**] also extend to the present context; we shall point out two specifically, which follow from Theorem 1.

COROLLARY 7. *Suppose that X is a locally compact or metric space and $f: X \to \mathrm{Hom}(G, H)$ when followed by $\mathrm{Hom}(G, H) \to \mathrm{Hom}(G_d, H)$ is continuous, where G satisfies the hypotheses of Theorem 1. Then f is continuous.*

Proof. If X is locally compact and V is a compact neighbourhood of $x_0 \in X$, then $f(V)$ is compact in $\mathrm{Hom}(G_d, H)$, hence in $\mathrm{Hom}(G, H)$ by Theorem 1. If $\{x_\delta\}$ is a net in V converging to x_0, then $f(x_\delta) \to f(x)$ in $\mathrm{Hom}(G_d, H)$, thus $\{f(x_\delta)\}$ has at most $f(x)$ as a cluster point in $\mathrm{Hom}(G, H)$, hence converges because of the compactness of $f(V)$. If X is metric (or any space in which convergent sequences define the topology) we can replace V by $\{x_n\}^-$, where $x_n \to x_0$.

COROLLARY 8. *Suppose that G, H, and K are topological groups with G and H locally compact or complete metric. Suppose that $[\cdot\,,\cdot]: G \times H \to K$ has all sections $g \to [g, h]$ and $h \to [g, h]$ continuous homomorphisms. Then $[\cdot\,,\cdot]$ is jointly continuous.*

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Proof. $[g, \cdot] \in \mathrm{Hom}(H, K)$, and $g \to [g, \cdot]$ is a continuous map into $\mathrm{Hom}(H_d, K)$, so that, by Corollary 7, $g \to [g, \cdot]$ is continuous into $\mathrm{Hom}(H, K)$. If H is locally compact and V is a compact neighbourhood of its identity h_0, and W is a neighbourhood of the identity k_0 of K, then we have a neighbourhood U of the identity g_0 in G for which $[U, V] \subset W$, and our proof is complete. Again if G is locally compact, our proof is complete since we may interchange G and H; thus we can assume that G and H are both metric, and we only have to see that $g_n \to g_0$, $h_n \to h_0$ imply $[g_n, h_n] \to k_0$. But with $\{h_n\}^-$ in place of V, we have $[U, \{h_n\}^-] \subset W$ for some U for any given W, which yields convergence.

Remark (*added in proof*). Corollary 6 shows that a closed non-compact set F in a locally compact abelian group G in a sense has a "large" closure F^- in G^a. This is reflected in a further corollary: *F^- does not lie in a countable collection of cosets of G in G^a.*

We argue by contradiction, reducing to the easy case of a metric σ-compact group. First, we have a sequence $\{f_n\}$ in F tending to ∞ in G, and by hypothesis, the closure of $\{f_n\}$ lies in countably many cosets of G. Trivially there is a metric quotient H of G in which the image of our sequence tends to ∞, and since $G \to H$ extends to map G^a onto H^a, with each coset of H in H^a the image of a coset of G in G^a, the image of $\{f_n\}$ in H is a closed non-compact subset of H with closure in H^a contained in a countable union of cosets of H. Thus it suffices to obtain a contradiction from the case in which G is metric and $F = \{f_n\}$. Moreover, since $\{f_n\}$ lies in an open σ-compact subgroup G_0 of G we can replace G by G_0 and take G σ-compact: for $\{f_n\}^- \subset G_0^-$ and

$$\emptyset \neq \{f_n\}^- \cap (x + G) = \{f_n\}^- \cap G_0^- \cap (x + G)$$

implies $x + G = x_0 + G$, $x_0 \in G_0{}^-$; thus

$$(x_0 + G) \cap G_0^- = x_0 + G \cap G_0^- = x_0 + G_0$$

(since $y \in G \backslash G_0$ implies that there is a $\hat{g}$ in $G^\wedge \equiv 1$ on G_0, $\neq 1$ at y, whence $y \notin G_0^-$); thus $\{f_n\}^-$ lies in a countable union of cosets of G_0 in G_0^-, while G_0^- is precisely $G_0{}^a$ (as is most easily seen by computing its character group, $G^{a\wedge}/G_0^{-\perp}$, or the discrete version of $G^\wedge$ modulo its subgroup orthogonal to G_0^-, i.e., to G_0, hence the discrete version of $G_0{}^\wedge = G^\wedge/G_0{}^\perp$).

Therefore we can assume that G is both metric and σ-compact. But now $\{f_n\}^- \cap (x + G)$ is σ-compact in the topology of G translated to $x + G$, hence in the topology of G^a. Consequently, for $x \in \{f_n\}^- \backslash \{f_n\}$, since $\{f_n\}^-$ lies in countably many cosets of G, we have a sequence of compact neighbourhoods U_n of x in G^a with $\bigcap U_n \cap \{f_n\}^- \subset x + G$; and being compact in G^a and contained in $x + G$, it is compact in the topology of G translated to $x + G$ by [**3**, Theorem 1.2], and both topologies coincide. Thus $\bigcap U_n \cap \{f_n\}^-$ is a metrizable subspace of G^a, which of course implies that x is a G_δ in $\{f_n\}^-$, hence in $\{f_n\}^- \backslash \{f_n\}$, contradicting Corollary 6.

At least when G is metric and σ-compact, there is a local form of our result: *for any* $x \in F^- \backslash F$, *each neighbourhood* U *of* x *in* F^- *meets uncountably many cosets of* G. (Otherwise, with U compact we can find U_n as before with $\bigcap U_n \cap U \cap F^- \subset x + G$, yielding x a G_δ in F^-.)

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University of Washington,
Seattle, Washington