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A Bernstein–Walsh Type Inequality and Applications

Tejinder Neelon

Abstract. A Bernstein–Walsh type inequality for C^∞ functions of several variables is derived, which then is applied to obtain analogs and generalizations of the following classical theorems: (1) Bochnak–Siciak theorem: a C^∞ function on $\mathbb{R}^n$ that is real analytic on every line is real analytic; (2) Zorn–Lelong theorem: if a double power series $F(x, y)$ converges on a set of lines of positive capacity then $F(x, y)$ is convergent; (3) Abhyankar–Moh–Sathaye theorem: the transfinite diameter of the convergence set of a divergent series is zero.

1 Introduction

Let $F(x, y)$ be a double power series with complex coefficients. Suppose that every substitution $x = p(t)$ and $y = q(t)$, where p and q are convergent power series in t, transforms F into a convergent series in t. S. Bochner (circa 1945) asked whether the double series F is necessarily convergent. M. A. Zorn (1947) showed that the answer is affirmative by considering only the set of the linear substitutions, *i.e.,* if $F(\xi t, \eta t)$ is convergent for every $(\xi, \eta) \in \mathbb{C}^2$ then F is convergent. Later, R. Ree extended the result by further reducing the substitution set $S = \mathbb{C}^2$ to the set of real linear substitutions, *i.e.,* $S = \mathbb{R}^2$. A natural question then is, how small can the substitution set S in Zorn's theorem be? The complete answer was given by P. Lelong [7]. Since $F(\xi t, \eta t)$ and $F(\xi, \eta)$ are simultaneously convergent or divergent for $t \neq 0$ and $F(0, 0)$ is a constant, by excluding $(0, 0)$ from consideration, the substitution set S can be considered as a subset of the complex projective space $\mathbb{P}^1$. Then S can then be identified, modulo the point $(1, 0)$, with the subset $S' = \{\xi/\eta : (\xi, \eta) \in S\} \subset \mathbb{C}$. Lelong's result says that Zorn's theorem holds if and only if S' is not contained in an F_σ set of logarithmic capacity zero. This result was later rediscovered by A. Sathaye[13] (see also [1, 8]).

It is easy construct a function $f \in C^r(\mathbb{R}^n), 0 \le r < \infty$, whose restriction to each line segment in $\mathbb{R}^n$ is analytic. A function which is analytic on every analytic arc need not be even continuous (see [2]). However, the analog of Zorn's theorem for the analyticity of C^∞ functions was obtained independently by J. Bochnak [3] and J. Siciak [14] (see Corollary 7).

Evidently, Lelong's result is not well known. It is not referenced in any of the papers [1, 3, 13, 14]. During the write-up of the present work, the author's attention was brought to Lelong's paper by Robert Molzon.

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In this note we study the phenomenon described by the results mentioned above in function classes and power series rings other than the analytic class and the ring of convergent series.

We will use the following multi-index notations.

$$\mathbb{Z}_+^n := \{\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) : 0 \le \alpha_i \in \mathbb{Z}, \forall i\}.$$

For $x = \big(x_1, x_2, \dots, x_n\big)$, and $\alpha \in \mathbb{Z}_+^n$, put $x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ and $|\alpha| := \alpha_1 + \alpha_2 + \cdots + \alpha_n$.

For a function $f\colon \mathbb{R}^n \to \mathbb{R}$ and a set F, put $\|f\|_F := \sup_{x\in F} |f(x)|$.

In what follows, unless mentioned otherwise, $\{M_k\}_{k=0}^\infty$ will denote an arbitrary sequence of positive numbers.

2 Ultradifferentiable Classes

Let E be a subset of $\mathbb{R}^n$. Let $C\{M_k\}(E)$(resp., $C(M_k)(E)$) be the class of all functions $f \in C^\infty(\mathbb{R}^n)$ satisfying the following condition.

$$\exists h > 0 \text{ (resp., } \forall h > 0),$$
$$\exists C > 0 \text{ such that } \sup_E |\partial^\alpha f(x)| \le C h^{|\alpha|} M_{|\alpha|},\ \forall \alpha \in \mathbb{Z}_+^n.$$

The classes $C\{M_k\}(E)$ and $C(M_k)(E)$ are normed spaces via the norms defined by the optimal value of C above. Let $C\{M_k\} := \bigcap_K C\{M_k\}(K)$ and $C(M_k) := \bigcap_K C(M_k)(K)$ where the intersections run over an exhausting sequence of compact sets in $\mathbb{R}^n$. The spaces $C\{M_k\}$ and $C(M_k)$ equipped with the injective limit and the projective limit topology, respectively, are called the ultradifferentiable classes. The class $C(M_k)$ contains all compactly supported elements of $C\{M_k\}$. It follows from the Cauchy inequalities and Taylor's theorem that $C\{k!\}$ is precisely the class of analytic functions. The classes $C\{(k!)^\nu\}, \nu > 1$, known as Gevrey classes, are especially important in partial differential equations and harmonic analysis.

Let $\mathfrak{F}_n\{M_k\}$ be the vector space of all power series $F(Z) = \sum_\alpha f_\alpha Z^\alpha$, $Z = (z_1, z_2, \dots, z_n)$, $f_\alpha \in \mathbb{C}$, for which there is a constant $R > 0$ such that $|f_\alpha| \le R^{|\alpha|} M_{|\alpha|}, \forall \alpha$. The space $\mathfrak{F}_n\{1\}, \{1\} := \{1, 1, \dots\}$, is precisely the ring of convergent power series in n variables with coefficients in $\mathbb{C}$. The spaces $\mathfrak{F}_n\{M_k\}$ have been of recent interest; see *e.g.*, [9, 11, 15].

In the study of ultradifferentiable classes $C\{M_k\}$, $C(M_k)$ and $\mathfrak{F}_n\{M_k\}$, it is often necessary to put certain conditions on the sequence $\{M_k\}$. For example, the condition of logconvexity ($M_k^2 \le M_{k-1}M_{k+1}$) and the condition of differentiability ($\exists A > 0, M_{k+1} \le A^k M_k$) make an ultradifferentiable class closed under the product and the differentiation of functions, respectively, and the nonquasianalyticity condition ($\sum_k M_{k-1}/M_k < \infty$) ensures the existence of compactly supported elements in $C\{M_k\}$. There are many other commonly used conditions as well; see [9, 11, 15], and references therein. The sequences $\{M_k\}$ considered here and in [10] are arbitrary. Therefore, Theorem 2 and Theorem 3 and their consequences can be interpreted as estimations on the mixed derivatives of functions of several variables in terms of the derivatives of their restrictions to a family of polynomial curves.

3 A Bernstein–Walsh Type Inequality

For a compact subset K of $\mathbb{C}$ or $\mathbb{R}^2$ and an integer $k \geq 2$, define the k-th diameter of K as

$$\delta_k(K) := \max\left\{ \prod_{1\leq i<j\leq k} |z_i - z_j|^{\frac{2}{k(k-1)}} : z_1, z_2, \dots, z_k \in K \right\}.$$

The limit $\delta(K) := \lim_{k\to\infty} \delta_k(K)$ exists and is called the transfinite diameter of K. In $\mathbb{C}$, the notion of transfinite diameter coincides with the notion of logarithmic capacity. This is no longer true for the corresponding notions in higher dimensions.

Let Ω be an open subset of $\mathbb{C}^N$. Let $u\colon \Omega \to [-\infty, \infty)$ be an upper semicontinuous function which is not identically $-\infty$ on any connected component of Ω. The function u is said to be plurisubharmonic if for each $w \in \Omega$ and each $\xi \in \mathbb{C}^N$, the function $z \to u(w + z\xi)$ is either subharmonic or identically $-\infty$ on every connected component of the set $\{z \in \mathbb{C} : w + z\xi \in \Omega\}$. For example, for a holomorphic function f, the function $\log |f|$ is plurisubharmonic.

A subset $E \subseteq \mathbb{C}^N$ is said to be locally pluripolar (or polar when $N = 1$) if for each point $a \in E$ there is a neighborhood V of a and a plurisubharmonic function $u\colon V \to [-\infty, \infty)$ such that $E \cap V \subseteq \{z \in V : u(z) = -\infty\}$.

By Josefson's theorem a locally pluripolar set $E \subseteq C^N$ is pluripolar in the sense that E is contained in the set $\{u = -\infty\}$ for some plurisubharmonic function u on $\mathbb{C}^N$. A countable union of pluripolar subsets of $\mathbb{C}^N$ is pluripolar.

In $\mathbb{C}^N$, the zero set of any nonzero holomorphic function is pluripolar. Thus, in $\mathbb{C}^2$, any complex plane is pluripolar while any totally real plane (*e.g.*, $\mathbb{R}^2$) is nonpluripolar. There exists a generalized Cantor subset C of $[0, 1]$ that is nonpolar but has zero Hausdorff dimension. Since the product $E_1 \times E_2 \times \cdots \times E_n$ of nonpolar sets $E_j \subset \mathbb{C}$ is easily seen to be nonpluripolar in $\mathbb{C}^N$, by taking $E = C \times C \times \cdots \times C$ we see that there are nonpluripolar subsets in $\mathbb{R}^N \subset \mathbb{C}^N$ of zero Hausdorff dimension. Pluripolar sets can be characterized as the sets of zero logarithmic capacity.

The Bernstein–Walsh inequality states that for any nonpluripolar compact set $E \subseteq \mathbb{C}^N$ and relatively compact neighborhood U of E there is a constant $A \geq 1$ such that the inequality $\|P\|_U \leq A^k \|P\|_E$ holds for all complex polynomials $P\colon \mathbb{C}^N \to \mathbb{C}$ of degree k.

Further details regarding plurisubharmonic functions, pluripolar sets, and the Bernstein–Walsh inequality can be found, for example, in [5]

Let n and d be positive integers. We will view an element $\mathbf{q} \in \mathbb{C}^{nd} \simeq (\mathbb{C}^n)^d$ as an $n \times d$ matrix with $\mathbf{q}_{\cdot\nu} = (q_{1\nu}, q_{2\nu}, \dots, q_{d\nu})$, $1 \leq \nu \leq n$, as its rows and $\mathbf{q}_{j\cdot} = (q_{j1}, q_{j2}, \dots, q_{jn}) \in \mathbb{C}^n$, $1 \leq j \leq d$, as its columns.

Each element $\mathbf{q} \in \mathbb{C}^{nd}$ is the jet at 0 of a unique holomorphic curve in $\mathbb{C}^n$ defined by the polynomial map $\mathbf{q}(w) = \sum_{j=1}^{d} \frac{1}{j!} \mathbf{q}_{j\cdot} w^j$, $|w| \leq 1$. If $\mathbf{q} \in \mathbb{R}^{nd} \simeq (\mathbb{R}^n)^d + i0$, the curve $t \to \mathbf{q}(t)$, $t \in I = [-1, 1]$, lies in $\mathbb{R}^n$.

For a C^∞ function $f\colon \mathbb{R}^n \to \mathbb{R}$ and $\mathbf{q} \in (\mathbb{R}^n)^d$, let $f_{x\mathbf{q}}(t) := f(x + \mathbf{q}(t))$, and let $f_{x\mathbf{q}}^{(k)}(0)$ be the k-th derivative of $f_{x\mathbf{q}}$ at $t = 0$.

Proposition 1 *Let Λ be a compact nonpluripolar subset of $\mathbb{C}^{nd}$.*

(1) *If $\Lambda \subseteq \mathbb{R}^{nd}$, then there exists a constant $C > 0$ such that*

$$\max_{|\alpha|=k} |\partial^\alpha f(x)| \leq C^k \sup_{\mathbf{q}\in\Lambda} \left| f^{(k)}_{x\mathbf{q}}(0) \right|, \quad \forall x \in \mathbb{R}^n, \forall f \in C^\infty(\mathbb{R}^n), \tag{3.1}$$

and for all power series $F(x) = \sum_\alpha f_\alpha x^\alpha$, $f_\alpha \in \mathbb{R}$,

$$\max_{|\alpha|=k} |f_\alpha| \leq C^k \sup_{\mathbf{q}\in\Lambda} \left| f^{(\mathbf{q})}_j \right| \tag{3.2}$$

where $f^{(\mathbf{q})}_j$, $j \in \mathbb{Z}_+$, denotes the coefficient of t^j in $F(\mathbf{q}(t)) := \sum_\alpha f_\alpha \left(\mathbf{q}(t)\right)^\alpha$ as a series in t.

(2) *There exists a constant $C > 0$ such that the inequality*

$$\max_{|\alpha|=k} |F_\alpha| \leq C^k \sup_{\mathbf{q}\in\Lambda} \left| F^{(\mathbf{q})}_j \right| \tag{3.3}$$

holds for all power series $F(Z) = \sum_\alpha F_\alpha Z^\alpha, F_\alpha \in \mathbb{C}$, where $F^{(\mathbf{q})}_j$, $j \in \mathbb{Z}_+$ denotes the coefficient of w^j in $F(\mathbf{q}(w)) = \sum_\alpha F_\alpha \left(\mathbf{q}(w)\right)^\alpha$ as a series in w.

Proof For a $\boldsymbol{\beta} \in \left(\mathbb{Z}^n_+\right)^d$, let $\varpi(\boldsymbol{\beta}) := \sum_{j=1}^d j \left|\beta_{j\cdot}\right|$ denote the weight of $\boldsymbol{\beta}$, where $\left|\beta_{j\cdot}\right| = \sum_{\nu=1}^n \beta_{j\nu}$, $1 \leq j \leq d$, is the length of the j-th column.

Let $f \colon \mathbb{R}^n \to \mathbb{R}$ be a C^∞ function. Fix $x \in \mathbb{R}^n$. For $\boldsymbol{\beta} \in \left(\mathbb{Z}^n_+\right)^d$ put

$$f^{(\boldsymbol{\beta})}(x) = \begin{cases} \partial_1^{|\boldsymbol{\beta}_{\cdot 1}|} \cdots \partial_n^{|\boldsymbol{\beta}_{\cdot n}|} f(x) & \text{if } \varpi(\boldsymbol{\beta}) = k, \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$P(\mathbf{z}) := \sum_{|\boldsymbol{\beta}|\leq k} k!\, f^{(\boldsymbol{\beta})}(x) \prod_{j=1}^d \frac{z_{j1}^{\beta_{j1}} z_{j2}^{\beta_{j2}} \cdots z_{jn}^{\beta_{jn}}}{\beta_{j1}!\, \beta_{j2}! \cdots \beta_{jn}!\, \left[j!\right]^{|\beta_{j\cdot}|}} \tag{3.4}$$

denote the multivariate Bell polynomial of degree k in the variables $z = \left\{z_{j\nu}\right\}_{j,\nu}$.

By the multivariate version of the Faà Di Bruno formula for derivatives of composite functions (see [4, Corollary 2.11]), we have

$$f^{(k)}_{x\mathbf{q}}(0) = P\left(\mathbf{q}\right), \ \forall \mathbf{q} \in \Lambda. \tag{3.5}$$

Let $r \geq 1$ be such that $\Lambda \subset \left\{z \in \mathbb{C}^{nd} : |z| < 2r\right\}$. Since Λ is nonpluripolar, by the Bernstein–Walsh inequality there exists a constant (independent of the polynomial P and hence k) $A \geq 1$ such that

$$\max_{|\mathbf{z}|\leq r} \left|P\left(\mathbf{z}\right)\right| \leq A^k \cdot \max_{\mathbf{q}\in\Lambda} \left|P\left(\mathbf{q}\right)\right|. \tag{3.6}$$

An application of Cauchy's inequalities to the polynomial $P(\mathbf{z})$ yields

$$\left|f^{(\boldsymbol{\beta})}(x)\right| \leq \frac{1}{k!} \prod_{j=1}^{d} \beta_{j1}!\,\beta_{j2}! \cdots \beta_{jn}!\, \left[j!\right]^{|\beta_{j\cdot}|} \cdot \max_{|\mathbf{z}|=r} |P(\mathbf{z})|\,.$$

Since $\sum_{1\leq j\leq d} |\beta_{j\cdot}| \leq k$, we have $\prod_{j=1}^{d} \beta_{j1}!\,\beta_{j2}! \cdots \beta_{jn}! \leq k!$ and $\prod_{j=1}^{d} j!^{\,|\beta_{j\cdot}|} \leq d!^{\,dk}$. Now with $C = \left(Ad!^{\,d}\right)$, the inequality 3.1 follows from 3.5 and 3.6.

Since any power series is a Taylor series of a C^∞ function (E. Borel's theorem) and the Taylor series of the composition of two functions is the composition of their Taylor series, it follows that the Faà Di Bruno formula is valid for power series as well (see also [4, §3]). Thus, the above argument also yields the inequalities 3.2 and 3.3. ■

In his comments on an earlier draft of this paper, Professor N. Levenberg brought to the author's attention a result of J. Korevaar and J. Wiegerinck (see [6] for the refined version stated below) that is similar to the linear case $d = 1$ of Proposition 1. By using generalized Green functions, they proved that for a subset $E \subset \mathbb{S}^{n-1} := \{x \in \mathbb{R}^n : x_1^2 + \cdots + x_n^2 = 1\}$ there is a constant $\beta(E) > 0$ satisfying the inequality

$$(3.7) \qquad \max_{|\alpha|=k} |\partial^\alpha f(x)| \leq \beta(E)^{-k} \sup_{\boldsymbol{\xi}\in E} \Big| \sum_{|\alpha|=k} \xi^\alpha \partial^\alpha f(x) \Big|, \ \forall f \in C^\infty(\mathbb{R}^n),\ \forall x \in \mathbb{R}^n,$$

if and only if the closure of the circular set $E_c := \{e^{it}x : x \in E, t \in \mathbb{R}\}$ is nonpluripolar in $\mathbb{C}^n$. Furthermore, the optimal value of the constant $\beta(E)$ is shown to be equal to the Siciak capacity of the closure of E_c.

It can be shown that if a set $E \subset \mathbb{S}^{n-1}$ is nonpluripolar as a subset of the complexified unit sphere $\{z \in \mathbb{C}^n : z_1^2 + \cdots + z_n^2 = 1, x_j = \operatorname{Re} z_j, 1 \leq j \leq n\}$, then E_c is nonpluripolar in $\mathbb{C}^n$. Since the set of directions E is contained in $\mathbb{S}^{n-1}$ and is not required to be nonpluripolar in $\mathbb{C}^n$, the hypothesis on E in 3.7 is weaker than the one on Λ in the case $d = 1$ of Proposition 1.

When $k = 1$, the suprema in 3.1 and 3.7 are over the set of first-order directional derivatives with directions in $\{\mathbf{q}_{1\cdot} : \mathbf{q} \in \Lambda\}$ and E, respectively. When $k > 1$, the supremum in 3.7 is over a set of k-th order directional derivatives, while the supremum in 3.1 involves the partial derivatives of not only order k but of lower orders as well. For example, if $n = 2$, $d = 2$, $k = 2$, and $\Lambda = \mathbb{R}^2 \times \mathbb{R}^2$, the set $\left\{ f''_{x\mathbf{q}}(0) : \mathbf{q} \in \Lambda \right\}$ is the set of all linear combinations of first and second order partial derivatives of f at x.

Remark In the planer case $n = 2$, one can obtain inequalities like 3.1 without using the Bernstein–Walsh inequality. For example, by fixing a set of $k + 1$ points $\mathbf{q}^{(l)} := (q_{11}^{(l)}, q_{12}^{(l)}) \in \Lambda, 0 \leq l \leq k$, 3.5 can be viewed as a system of linear equations in unknowns $\binom{k}{\beta_{11}} \partial_1^{\beta_{11}} \partial_2^{k-\beta_{11}} f(x)$, $0 \leq \beta_{11} \leq k$, and its determinant

$$\det\left[\left(q_{11}^{(l)}\right)^{k-j} \left(q_{12}^{(l)}\right)^{j} \right]_{0\leq j,l\leq k}$$

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can be computed explicitly via the Vandermonde formula. (The case $n > 2$ involves so called hyperdeterminants.) Now, if the $\mathbf{q}^{(l)}$'s are chosen to be well-spread, Cramer's rule will yield the desired inequality. In [10], this approach combined with induction on n was applied to the linear case $d = 1$ with $\Lambda = S^{n-1}$. A similar approach was also adopted in [12] to study the restrictions of C^∞ functions to families of C^∞ plane curves (*i.e.*, $n = 2, d = \infty$) whose slope sets $\{q_{12}/q_{11} : \mathbf{q} \in \Lambda\}$ have positive transfinite diameter.

4 Results

Theorem 2 *Let a compact set $\Lambda \subseteq \mathbb{R}^{nd}$ be nonpluripolar in $\mathbb{C}^{nd}$. Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a C^∞ function and $E \subseteq \mathbb{R}^n$ be a subset. If for some (resp., for all) $h > 0$ the condition*

$$\Phi_{h,E}(\mathbf{q}) := \sup_{\substack{x\in E\\ k\geq 1}} \frac{\left|f_{x\mathbf{q}}^{(k)}(0)\right|}{h^k M_k} < \infty,\ \forall \mathbf{q} \in \Lambda, \tag{4.1}$$

is satisfied, then $f \in C\{M_k\}(E)$ (resp., $C(M_k)(E)$). Hence, if for every compact set K, the condition $\|\Phi_{h,K}\|_\Lambda < \infty$ is satisfied for some (resp., for all) $h > 0$, then $f \in C\{M_k\}$ (resp., $C(M_k)$).

The case $d = 1$ of Theorem 2 was proved in [10] with $\Lambda = \mathbb{S}^{n-1}$.

As in the study of separate analyticity, a mild boundedness hypothesis of the type 4.1 is necessary when dealing with function classes containing nonanalytic functions.

Proof Let a C^∞ function $f\colon \mathbb{R}^n \to \mathbb{R}$, a subset $E \subseteq \mathbb{R}^n$, and a constant $h > 0$ be such that the condition 4.1 is satisfied. Since the function $\mathbf{q} \to \Phi_{h,E}(\mathbf{q})$ is everywhere finite and lower semicontinuous, the sets

$$\Lambda_m := \{\mathbf{q} \in \Lambda : |f_{x\mathbf{q}}^{(k)}(0)| \leq mh^k M_k, \forall k\}, m \in \mathbb{Z}_+,$$

are closed and $\Lambda := \bigcup_m \Lambda_m$. Since Λ is nonpluripolar, Λ_m is nonpluripolar for some $m \geq 1$. Now the inequality 3.1 with $\Lambda = \Lambda_m$ implies that there is a constant $C > 0$ such that $|\partial^\alpha f(x)| \leq m\,(Ch)^{|\alpha|}\, M_{|\alpha|},\ \forall \alpha, \forall x \in E$. ∎

The above theorem also has a natural algebraic analog. The hypothesis 4.1 is unnecessary here.

Now, because the substitutions $\mathbf{q}(w)$ are going to be polynomials with complex coefficients, the "jet set" Λ is going to be a subset of $(\mathbb{C}^n)^d \simeq \mathbb{R}^{2nd}$.

Theorem 3 *Let Λ be a nonpluripolar subset of $\mathbb{C}^{nd}$. Let $F(Z)$ be a formal power series in n variables. If $F(\mathbf{q}(w)) \in \mathfrak{F}_1\{M_k\}$ for all $\mathbf{q} \in \Lambda$, then $F \in \mathfrak{F}_n\{M_k\}$.*

Proof Suppose $F(Z)$ is a formal power series in n variables such that $F(\mathbf{q}(w)) := \sum_{j=0}^{\infty} F_j^{(\mathbf{q})} w^j \in \mathfrak{F}_1\,\{M_k\}$ for all $\mathbf{q} \in \Lambda$. Since by the Faà di Bruno formula the coefficients $F_j^{(\mathbf{q})}$ are polynomial functions of $q_{j\nu}$, $1 \leq j \leq d, 1 \leq \nu \leq n$, the function

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$\mathbf{q} \to \sup_{j \geq 1} \left| F_j^{(\mathbf{q})} M_j^{-1/j} \right|$ is lower semicontinuous. In particular the sets

$$\Lambda_m := \left\{ \mathbf{q} \in \Lambda : \left| F_j^{(\mathbf{q})} \right| \leq m^j M_j, \forall j \right\}, m \in \mathbb{Z}_+,$$

are closed and $\Lambda := \bigcup_m \Lambda_m$. Since Λ_m must be nonpluripolar for some $m \geq 1$, the proof is completed by using the inequality 3.3 with $\Lambda = \Lambda_m$. ■

The special case $M_k = 1, \forall k \geq 1$, yields the following n-dimensional version of the Zorn–Lelong theorem (for a more general result see [8, Corollary 4.2]).

Corollary 4 *Let $F(Z)$ be a power series in n variables. If the series $F(w\xi)$, in w, is convergent for each ξ in a nonpluripolar subset of $\mathbb{C}^n$, then $F(Z)$ is convergent.*

For a power series $F(Z)$, $Z = (z_1, z_2, \dots, z_n)$, let $S_{\{M_k\}}(F)$ be the set of all those $s \in \mathbb{C}$ for which $F(sz_2, z_2, \dots, z_n) \in \mathfrak{F}_{n-1}\{M_k\}$. Then $S_{\{1\}}(F)$ is precisely the convergence set of F in the sense of A. Sathaye[13]. The higher-dimensional convergence sets were developed in [8].

For a power series $F(Z)$ and $s \in \mathbb{C}$, put

$$G_s(w_1, w_2, \dots, w_n) = F(w_1 + sw_2, w_2, \dots, w_n).$$

As the ring $\mathfrak{F}_n\{M_k\}$ is invariant under a linear change of coordinates, $F \in \mathfrak{F}_n\{M_k\}$ if and only if $G_s \in \mathfrak{F}_n\{M_k\}$. Observe that $\zeta \in S_{\{M_k\}}(G_s)$ if and only if $s + \zeta \in S_{\{M_k\}}(F)$.

Theorem 5 *If $F(Z)$ is a formal power series in n variables, then $F \in \mathfrak{F}_n\{M_k\}$ if and only if $S_{\{M_k\}}(F)$ is nonpolar.*

Proof If $F \in \mathfrak{F}_n\{M_k\}$, then $S_{\{M_k\}}(F) = \mathbb{C}$. Now, let $F(Z)$ be a power series such that $S_{\{M_k\}}(F)$ is nonpolar. For $s \in \mathbb{C}$, we can write

$$F(sz_2, z_2, \dots, z_n) = \sum_{\alpha \in \mathbb{Z}_+^{n-1}} F_\alpha(s) \left(Z'\right)^\alpha, \quad Z' = (z_2, \dots, z_n),$$

where $F_\alpha(s)$ is a polynomial in s of degree at most $|\alpha|$. By arguing as in the proof of Theorem 3, there exists a constant $C > 0$ and a compact nonpolar subset $S \subseteq S_{\{M_k\}}(F)$ such that $|F_\alpha(s)| \leq C^{|\alpha|} M_{|\alpha|}$, $\forall \alpha, \forall s \in S$. Since S is not locally polar, there exists $s_0 \in S$ such that $B(s_0, r) \cap S$ is nonpolar for all $r > 0$, where $B(s_0, r)$ denotes the closed ball of radius r and centered at s_0. By replacing $F(Z)$ with $G_{s_0}(w_1, w_2, \dots, w_n)$, if necessary, we may assume that S contains 0 and that $s_0 = 0$. Let $r > 0$ be such that $S \subset B(0, r)$. By the Bernstein–Walsh inequality, there exists a constant $A \geq 1$ such that $|F_\alpha(s)| \leq A^{|\alpha|} M_{|\alpha|}$, $\forall \alpha, \forall |s| \leq r$. Now, as in the proof of Proposition 1, Cauchy's inequalities yield the required estimates on the coefficients of the polynomials $F_\alpha(s)$. The proof is complete because each coefficient of F is a coefficient of some $F_\alpha(s)$. ■

Corollary 6 (Abhyankar–Moh–Sathaye theorem) *The transfinite diameter of the convergence set of a divergent series is zero.*

Abhyankar and Moh [1] used the notion of one-dimensional Hausdorff measure in place of transfinite diameter. The stronger result, Corollary 6, was later obtained by Sathaye [13] who also showed that convergence sets of divergent power series are precisely the sets contained in an F_σ set of zero transfinite diameter, and thus, in particular, rediscovered Lelong's result. The results in [1, 13] hold for power series with coefficients from general valued fields. Because of the use of the Bernstein–Walsh inequality, our methods work only for power series with real or complex coefficients.

Corollary 7 (Bochnak–Siciak Theorem [3, 14]) *If a C^∞ function $f\colon \mathbb{R}^n \to \mathbb{R}$ is analytic on every line segment through a point x_0, then f is analytic in a neighborhood of x_0.*

Proof Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a C^∞ function. For any $\xi \in \mathbb{R}^n$, the Taylor series $F(x)$ of f about x_0 can be written as

$$F(x_0 + \xi t) = \sum_{k=0}^{\infty} \frac{f^{(k)}_{x_0\xi}(0)}{k!} t^k.$$

If $f_{x_0\xi} \in C\{k!\}$ then $F(x_0 + \xi t)$ has a positive radius of convergence. Since $\mathbb{R}^n$ is nonpluripolar in $\mathbb{R}^n$, by Theorem 3 there is $r > 0$ such that $F(x)$ converges for $|x - x_0| \leq r$. Since $f(x_0 + t\xi) = F(x_0 + t\xi)$, $\forall t \in (-r, r), \forall \xi \in \mathbb{R}^n$, by our hypothesis, $f(x) \equiv F(x)$, $\forall x, |x - x_0| < r$. ■

The hypothesis on the jet set Λ, in the results stated above, depends on the ambient complex structure. But the hypothesis in Corollary 6 and in the case $n = 2$ and $d = 1$ of Theorem 2 can be stated without invoking the ambient complex structure, *e.g.*, by requiring that the transfinite diameter of $\Lambda \subseteq \mathbb{R}^2$ be positive. When dealing with real functions and real power series it would be desirable to have purely "real" hypothesis on Λ.

Since the jet set of a family of C^∞ or analytic curves lives in an infinite dimensional space, the Bernstein–Walsh inequality or Proposition 1 is not available. In [12], the author obtained, by direct methods, some partial results in the case of C^∞ plane curves ($n = 2$).

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Department of Mathematics
California State University San Marcos
San Marcos, CA 92096-0001
U.S.A.
e-mail: neelon@csusm.edu

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