A MOMENT PROBLEM

LAJOS TAKÁCS *

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1. Introduction

Let ν be a discrete random variable taking on nonnegative integer values and set $\mathbf{P}\{\nu = k\} = P_k$, $k = 0, 1, \cdots$. Suppose that the binomial moments

$$B_r = \mathbf{E}\left\{\binom{\nu}{r}\right\} = \sum_{k=r}^{\infty} \binom{k}{r} P_k, \qquad r = 0, 1, \cdots, \tag{1}$$

are finite. Frequently the problem arises under what conditions the probabilities P_k, $k = 0, 1, \cdots$, can be determined uniquely by the sequence of moments B_r, $r = 0, 1, \cdots$, and how it can be done.

In what follows we shall show that if $\limsup_{r\to\infty} B_r^{1/r} < \infty$, then $\{P_k\}$ can be determined uniquely by $\{B_r\}$ and we shall give an explicit formula for P_k, $k = 0, 1, \cdots$. If $\limsup_{r\to\infty} B_r^{1/r} = \infty$, then, in general, $\{P_k\}$ cannot be determined uniquely by $\{B_r\}$.

2. An inversion formula

The probabilities P_k, $k = 0, 1, \cdots$, can be determined in several possible ways, but formula (2) seems to be the most convenient one.

THEOREM. *Let ν be a discrete random variable taking on nonnegative integer values and set $\mathbf{P}\{\nu = k\} = P_k$, $k = 0, 1, \cdots$. If the binomial moments $B_r = \mathbf{E}\{\binom{\nu}{r}\}$, $r = 0, 1, \cdots$, are finite and if $\rho = \limsup_{r\to\infty} B_r^{1/r} < \infty$, then*

$$P_k = \sum_{r=k}^{\infty} \frac{\binom{r}{k}}{(1+q)^{r+1}} \sum_{j=k}^{r} (-1)^{j-k} \binom{r-k}{j-k} q^{r-j} B_j \tag{2}$$

where q is nonnegative and greater than (ρ^2-1). If, in particular, $\rho = \limsup_{r\to\infty} B_r^{1/r} < 1$, then we can always choose $q = 0$ and (2) reduces to

$$P_k = \sum_{r=k}^{\infty} (-1)^{r-k} \binom{r}{k} B_r. \tag{3}$$

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PROOF. The generating function

$$P(z) = \sum_{j=0}^{\infty} P_j z^j \tag{4}$$

is uniformly convergent in the circle $|z| < 1$ and $P(z)$ is regular if $|z| < 1$. Hence

$$P^{(k)}(z) = \frac{d^k P(z)}{dz^k} = k! \sum_{j=k}^{\infty} \binom{j}{k} P_j z^{j-k} \tag{5}$$

for $k = 0, 1, \cdots$ and $|z| < 1$. If $z = 0$ in (5), then we get

$$P_k = \frac{1}{k!} P^{(k)}(0). \tag{6}$$

Thus the problem of finding P_k can be reduced to finding $P(z)$ in a neighborhood of $z = 0$.

If B_r is finite, then

$$B_r = \frac{1}{r!} \left(\frac{d^r P(z)}{dz^r}\right)_{z=1} \tag{7}$$

and for $|z-1| < 1/\rho$

$$P(z) = \sum_{r=0}^{\infty} B_r (z-1)^r. \tag{8}$$

The right hand side of (8) is uniformly convergent in the circle $|z-1| < 1/\rho$ and $P(z)$ is regular if $|z-1| < 1/\rho$. Hence

$$P^{(k)}(z) = \frac{d^k P(z)}{dz^k} = k! \sum_{r=k}^{\infty} \binom{r}{k} B_r (z-1)^{r-k} \tag{9}$$

for $k = 0, 1, \cdots$, and $|z-1| < 1/\rho$.

If $\rho < 1$, and we put $z = 0$ in (9), then by (6) we get (3). We note that (3) is an oscillating series which is convergent if and only if $\lim_{r\to\infty} r^k B_r = 0$.

If $\rho < \infty$, then (9) is a regular function of z in the circle $|z-1| < 1/\rho$. By analytical continuation we can extend the definition of (9) to the domain $|z| < 1$ and in this domain (9) agrees with (5). Now we shall show that the definition of (9) can easily be extended to a neighborhood of $z = 0$ by using Euler's transformation of series. (Cf. Hardy [2], Chapter VIII.) Let $q \geqq 0$ and form the E_q-transform of (9),

$$P_q^{(k)}(z) = k! \sum_{r=k}^{\infty} \frac{1}{(1+q)^{r+1}} \sum_{j=0}^{r} \binom{r}{j} \binom{j}{k} q^{r-j} (z-1)^{j-k} B_j. \tag{10}$$

For $|z-1| < 1/\rho$ we have $P_q^{(k)}(z) = P^{(k)}(z)$ given by (9) because Euler's transformation is consistent. Now by using a theorem of Knopp [6] we can establish the domain R_q in which (10) is convergent and represents a regular function of z. Suppose that $P_q^{(k)}(z)$ is analytically extended along every ray of origin $z = 1$ until we reach the first singular point (if any) of $P_q^{(k)}(z)$ on the ray. Denote by Γ the set of all singular points obtained in this way. Then R_q can be represented as the set of points common to all the circles $|z-1+q(\gamma-1)| < (1+q)|\gamma-1|$ for $\gamma \in \Gamma$. Evidently $|\gamma-1| \geqq 1/\rho$ and $|\gamma| \geqq 1$ for all $\gamma \in \Gamma$ and there exists a $\gamma \in \Gamma$ such that $|\gamma-1| = 1/\rho$. Hence it follows that R_q always contains the point $z = 0$ if $q > (\rho^2-1)$ and R_q never contains $z = 0$ if $q \leqq (\rho-1)/2$. For example, if $|\gamma| \geqq (1+\rho)/\rho$ for every $\gamma \in \Gamma$, then we can choose q as any nonnegative number greater than $(\rho-1)/2$, however, if Γ contains a γ for which $|\gamma-1| = 1/\rho$ and $|\gamma| = 1$, then q must be chosen greater than (ρ^2-1) in order that R_q contain $z = 0$. Accordingly if $q > (\rho^2-1)$, then in some neighborhood of $z = 0$ we have $P_q^{(k)}(z) = P^{(k)}(z)$ given by (5). Thus by (6) we have $P_k = P_q^{(k)}(0)/k!$, $k = 0, 1, \cdots$, which yields (2).

3. Examples

(i) Suppose that $B_r = \boldsymbol{E}\{\binom{\nu}{r}\} = a^r/r!$, $r = 0, 1, \cdots$, where a is a positive number. Then $\lim_{r\to\infty} B_r^{1/r} = 0$ and $\rho = 0$. By (3)

$$P_k = \sum_{r=k}^{\infty} (-1)^{r-k} \binom{r}{k} \frac{a^r}{r!} = e^{-a} \frac{a^k}{k!}, \qquad k = 0, 1, \cdots. \tag{11}$$

(ii) Suppose that $B_r = \boldsymbol{E}\{\binom{\nu}{r}\} = a^r$, $r = 0, 1, \cdots$, where a is a positive number. Then $\lim_{r\to\infty} B_r^{1/r} = a$ and $\rho = a$. If $a < 1$, then we can apply formula (3) to obtain

$$P_k = \sum_{r=k}^{\infty} (-1)^{r-k} \binom{r}{k} a^r = \frac{a^k}{(1+a)^{k+1}}, \qquad k = 0, 1, \cdots. \tag{12}$$

If $a < \infty$, then we obtain by (2) that

$$P_k = \sum_{r=k}^{\infty} \binom{r}{k} \frac{a^k(q-a)^{r-k}}{(1+q)^{r+1}} = \frac{a^k}{(1+a)^{k+1}}, \qquad k = 0, 1, \cdots, \tag{13}$$

where $q > (a-1)/2$. The best choice is $q = a$.

(iii) Let $A_1, A_2, \cdots, A_n, \cdots$ be an infinite sequence of events. Denote by ν the number of events occurring among $A_1, A_2, \cdots, A_n, \cdots$. It can easily be seen that

$$B_r = \boldsymbol{E}\left\{\binom{\nu}{r}\right\} = \sum_{1 \leqq i_1 < i_2 < \cdots < i_r < \infty} \boldsymbol{P}\{A_{i_1} A_{i_2} \cdots A_{i_r}\}. \tag{14}$$

If $\rho = \limsup_{r\to\infty} B_r^{1/r} < 1$, then the probability that exactly k events occur among $A_1, A_2, \cdots, A_n, \cdots$ is given by (3). Formula (3) was found first by Jordan [3], [4], [5], for the case when $A_{n+1} = A_{n+2} = \cdots = 0$, the impossible event. (Cf. also [8].)

If $\rho = \limsup_{r\to\infty} B_r^{1/r} < \infty$, then the probability that exactly k events occur among $A_1, A_2, \cdots, A_n, \cdots$ is given by (2) with $q > (\rho^2-1)$. In some particular cases we can choose $q > (\rho-1)/2$.

(iv) Consider the previous example. The probability that at least one event occurs among $A_1, A_2, \cdots, A_n, \cdots$ is given by $\boldsymbol{P}\{A_1+A_2+\cdots+A_n+\cdots\} = 1-P_0$. If $\rho < 1$, then by (3)

$$\boldsymbol{P}\{A_1+A_2+\cdots+A_n+\cdots\} = \sum_{r=1}^{\infty}(-1)^{r-1}B_r. \tag{15}$$

If $\rho < \infty$, then by (2)

$$\boldsymbol{P}\{A_1+A_2+\cdots+A_n+\cdots\} = 1-\sum_{r=0}^{\infty}\frac{1}{(1+q)^{r+1}}\sum_{j=0}^{r}(-1)^j\binom{r}{j}q^{r-j}B_j \tag{16}$$

where $q > (\rho^2-1)$. In some particular cases we can choose $q > (\rho-1)/2$.

Formula (15) was found first by Poincaré [7] for the case when $A_{n+1} = A_{n+2} = \cdots = 0$, the impossible event. Dvoretzky [1] proved that (15) holds if $\lim_{r\to\infty} B_r = 0$.

(v) It is interesting to mention also the following simple example. A balanced coin is tossed repeatedly. We say that event A_n occurs if head does not appear among the first n tossings. Denote by ν the number of events occurring among $A_1, A_2, \cdots, A_n, \cdots$. By (14) $B_r = \boldsymbol{E}\{\binom{\nu}{r}\} = 1$ for $r = 0, 1, \cdots$. In this case (3) is divergent, but by (2) with $q > 0$ we get that $P_k = \boldsymbol{P}\{\nu = k\} = 1/2^{k+1}$ for $k = 0, 1, \cdots$, in agreement with a direct calculation.

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Department of Mathematical Statistics
Columbia University