J. Austral. Math. Soc. 20 (Series A) (1975), 85–109.

RECTIFIABLY AMBIGUOUS POINTS OF PLANAR SETS

FREDERICK BAGEMIHL and PAUL D. HUMKE

(Received 10 August 1973)

Communicated by G. Szekeres

Denote by P the Euclidean plane with a rectangular Cartesian coordinate system where the x-axis is horizontal and the y-axis is vertical. An *arc* in P shall mean a simple continuous curve $\Lambda:\{t: 0 \leqq t < 1\} \to P$ having the properties that $\text{limit}_{t\to 1}\Lambda(t)$ exists and $\text{limit}_{t\to 1}\Lambda(t) \neq \Lambda(t_0)$ for $0 \leqq t_0 < 1$. An *arc at a point* ζ in P shall be an arc Λ where $\lim_{t\to 1}\Lambda(t) = \zeta$. If S is an arbitrary subset of the plane, ζ is termed an ambiguous point relative to S provided there are arcs Λ and Γ at ζ with $\Lambda \subseteq S$ and $\Gamma \subseteq P - S$; such arcs are referred to as *arcs of ambiguity* at ζ. If A is a set of arcs we say a point ζ in P is *accessible* via A provided there is an arc at ζ which is an element of A. If B is also a collection of arcs, then A and B are said to be *pointwise disjoint* if whenever $\alpha \in A$ and $\beta \in B$, $\alpha \cap \beta = \varnothing$. The collections A and B are said to be *terminally arcwise disjoint* if whenever $\alpha \in A$ and $\beta \in B$ and both α and β are arcs at a point ζ in P, then $\alpha \cap \beta$ contains no arc at ζ. If S is a planar set, we let $\mathscr{A}(S)$ denote the set of all arcs contained in S. Note that if $S \cap T = \varnothing$ then $\mathscr{A}(S)$ and $\mathscr{A}(T)$ are pointwise disjoint collections of arcs.

In this paper we deal with accessibility of points via sets of rectifiable arcs and sets of totally nonrectifiable arcs, and related questions in ambiguous point theory. (An arc α is totally nonrectifiable if $\alpha/[t_1, t_2]$ is nonrectifiable for $0 \leqq t_1 < t_2 \leqq 1$.) Let $\mathscr{R}$ denote the set of all planar rectifiable arcs, and let $\mathscr{N}$ denote the set of all planar totally nonrectifiable arcs. Bagemihl (1966) showed that there is a set S_1 such that every point of the plane is an ambiguous point relative to S_1 and the arcs of ambiguity may be chosen to be rectifiable. In the first part of this paper we strengthen this result by showing that both

1. $\mathscr{A}(S_1) \subset \mathscr{R}$,
2. $\mathscr{A}(P - S_1) \subset \mathscr{R}$.

Secondly, we use S_1 to define a set $S_2 \subseteq P$ such that every point of the plane is an ambiguous point relative to S_2 and both

1. $\mathscr{A}(S_2) \subset \mathscr{N}$,
2. $\mathscr{A}(P - S_2) \subset \mathscr{N}$.

An example of a set S_3 is then presented such that every point of the plane is an ambiguous point relative to S_3 and yet

1. $\mathscr{A}(S_3) \subset \mathscr{R}$,
2. $\mathscr{A}(P - S_3) \subset \mathscr{N}$.

The final portion of the paper is devoted to proving a general theorem which shows that these three examples are, in a sense, extreme cases.

1. The Set S_1

The first part of this paper is devoted to the investigation of the set S_1 which was presented by Bagemihl (1966). We state this result as Theorem B below, and describe the construction of S_1 for completeness. (We also take this occasion to point out that Figures 5 and 6 in Bagemihl (1966) should be rotated through 90°.)

THEOREM B. *There exists a set $S_1 \subset P$ such that every point of P is a rectifiably ambiguous point relative to S_1.*

We shall introduce only the construction of S_1; for verification that S_1 has the stated properties, see Bagemihl (1966).

We construct S_1 and its complement $P - S_1 = T_1$ in the following manner. We first construct what we call a maze M. This consists of a certain number of horizontal and vertical rectilinear segments, some of which we put into S_1, the rest into T_1. The remaining points of P are then put into S_1 or T_1 in any way whatsoever, whereupon S_1 becomes completely defined. The maze itself is constructed in enumerably many stages: we first construct a submaze M_1, then add certain segments to M_1 to obtain a submaze M_2, and so on; and finally we set $M = \bigcup_{n=1}^{\infty} M_n$. Each submaze M_n in turn is constructed in four steps in a certain order. The procedure for constructing M_1 is different from that for the remaining submazes: we describe M_1 first, then give the procedure for constructing M_2 from M_1, this procedure is then repeated with M_2 to obtain M_3, and so on. Thus, from the second stage on, the procedure is essentially the same.

To construct M_1:

(a_1) put the vertical lines $x = 2n$ ($n = 0, \pm 1, \pm 2, \cdots$) into S_1,

(b_1) put the vertical lines $x = 2n + 1$ ($n = 0, \pm 1, \pm 2, \cdots$) into T_1,

(c_1) put those points of the horizontal lines $y = 2n$ ($n = 0, \pm 1, \pm 2, \cdots$) that have not already been accounted for into S_1,

(d_1) put those points of the horizontal lines $y = 2n + 1$ ($n = 0, \pm 1, \pm 2, \cdots$) that have not already been accounted for into T_1.

The resulting configuration of enumerably many vertical and horizontal straight lines constitutes the submaze M_1. Each point on these lines has been assigned unambiguously to one of the sets S_1, T_1. A portion of M_1 is illustrated in Figure 1. Here the heavy lines belong to S_1, the light lines to T_1. The point of intersection of a heavy line and a light line is marked with a black or a white dot according as this point belongs to S_1, or to T_1. The point of intersection of a heavy horizontal line and a heavy vertical line will be called an S_1-node, of a light horizontal line and a light vertical line a T_1-node. Observe that M_1 divides the plane into enumerably many squares of side length one, which will be called the squares of the first stage. For each of one of these squares, one vertex is an S_1-node and the opposite vertex is a T_1-node. This is the procedure for constructing M_2:

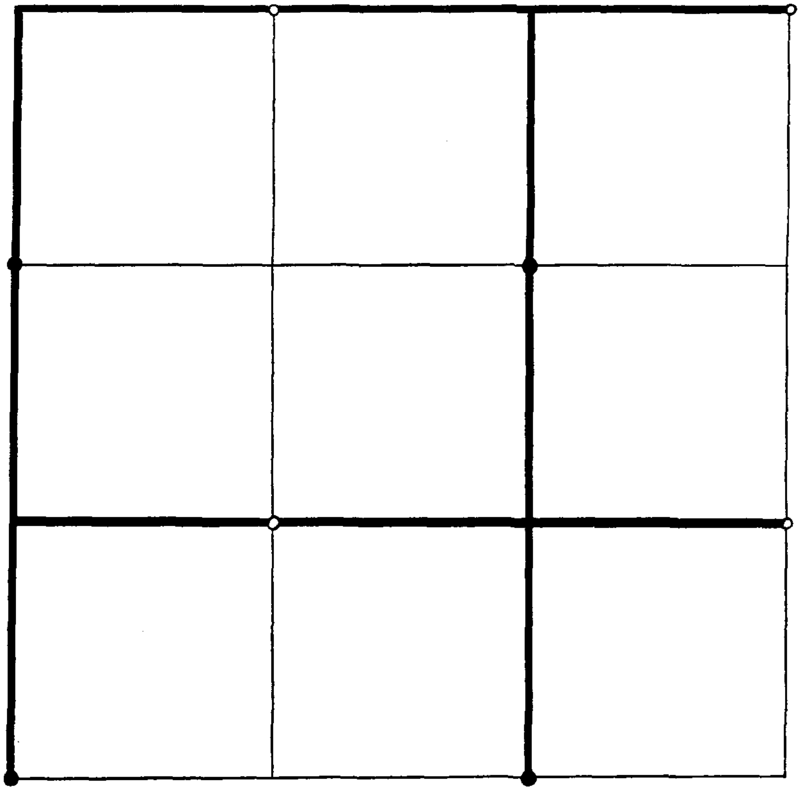

Figure 1.

(a_2) from every S_1-node of M_1, proceed in either direction horizontally a distance of 2/3, and at each of the two points reached erect an open vertical segment of length 2 with said point as midpoint; put these vertical segments into S_1, making the aforementioned two points new S_1-nodes;

(b_2) from every T_1-node of M_1 proceed as in (a_2), except put the resulting vertical segments into T_1, thus creating two new T_1-nodes;

(c_2) from every S_1-node of M_1 as well as those newly created by (a_2), proceed in either direction vertically a distance of 2/3, and at each of the two points

https://doi.org/10.1017/S1446788700023971 Published online by Cambridge University Press

reached erect an open horizontal segment of length 2/3 with the said point as midpoint; put these horizontal segments into S_1, making the aforementioned two points new S_1-nodes;

(d_2) from every T_1-node of M_1 as well as those newly created by (b_2), proceed as in (c_2), except put the resulting horizontal segments into T_1, thus creating two new T_1-nodes.

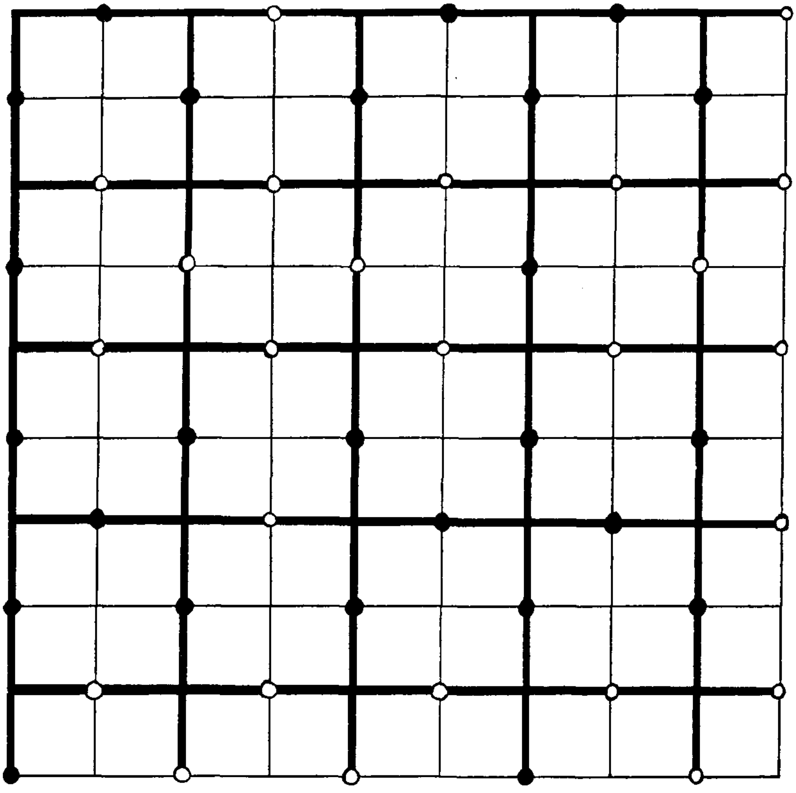

Figure 2.

The resulting configuration of M_1 and the newly added vertical and horizontal segments constitutes the submaze M_2. Each point on the enumerably many vertical and horizontal straight lines contained in M_2 has been assigned unambiguously to one of the two sets S_1, T_1. The portion of M_2 that arises from the portion of M_1 illustrated in Figure 1 is shown in Figure 2. Observe that M_2 divides the plane into enumerably many squares of side length 1/3, called the squares of the second stage. And again, for each one of these squares, one vertex is an S_1-node and the opposite vertex is a T_1-node.

Now to construct M_3, proceed as in the construction of M_2, except that in (a_3) and (b_3) the distance is 2/9 instead of 2/3 and the length is 2/3 instead of 2; and in (c_3) and (d_3) the distance is 2/9 instead of 2/3, and the length is also 2/9 instead of 2/3.

Proceeding successively in this fashion, we construct the submaze M_n for every natural number n. It divides the plane into enumerably many squares of side length $1/3^{n-1}$.

Finally, define M and S_1 as was indicated at the beginning.

As was noticed in Bagemihl (1966) at the conclusion of the proof of this Theorem B, if $\zeta \in P$ there are arcs at ζ of arbitrarily large diameter which are contained in S_1 and, likewise, there are arcs of arbitrarily large diameter which are contained in T_1. From this and the fact that $S_1 \cap T_1 = \varnothing$ we conclude that neither S_1 nor T_1 contains a loop. Let $\Lambda \in \mathscr{A}(S_1)$. Through a series of lemmas we shall show that Λ is a rectifiable arc.

LEMMA 1. *Suppose $\zeta \in S_1 \cup (P - M)$ and ζ is an interior point of a square Q_n of the nth stage. Suppose further that α and β are arcs at ζ such that*

1. *$\alpha(0)$ and $\beta(0)$ are in $P - \mathrm{Int}(Q_n)$,* [Int ≡ interior]
2. *$\alpha(t)$ and $\beta(t)$ are in S_1 for $0 \leqq t < 1$.*

Then $\alpha(t_1) = \beta(t_2)$ where

$$t_1 = \sup\{t : \alpha(t) \in \mathrm{Bd}(Q_n)\}, \quad [\mathrm{Bd} \equiv \text{boundary}]$$

$$t_2 = \sup\{t : \beta(t) \in \mathrm{Bd}(Q_n)\}.$$

PROOF. The case when $n = 1$ is typical, and we consider this case. In particular we let Q_1 be the square whose vertices are $A(0,0)$, $B(0,1)$, $C(1,1)$, and $D(1,0)$ where A is the S_1-node of Q_1 and C is the T_1-node of Q_1. Suppose that $\alpha(t_1) \neq \beta(t_2)$. Then as Q_1 is a square of the first stage and both $\alpha(t_1)$ and $\beta(t_2)$ lie on the boundary of Q_1, there is an arc Γ contained in $\mathrm{Bd}(Q_1) \cap S_1$ such that $\Gamma(0) = \alpha(t_1)$ and $\Gamma(1) = \beta(t_2)$. Hence, if there existed t_3 and t_4 such that

1. $t_1 < t_3 < 1$ and $t_2 < t_4 < 1$,
2. $\alpha(t_3) = \beta(t_4)$

then the arcs α, β, Γ would determine a loop, and as each of these arcs is in S_1 a contradiction would arise. The remainder of the proof is devoted to verifying the existence of t_3 and t_4.

If ζ_1 and ζ_2 are in P, for notational convenience we denote the closed line segment between ζ_1 and ζ_2 by $[\zeta_1, \zeta_2]$, and the open line segment between ζ_1 and ζ_2 by (ζ_1, ζ_2). At stage two of the construction the following points of Q_1 are assigned to either S_1 or T_1. Refer to Figure 3.

(a_2) The open segment $((2/3, 0), (2/3, 1))$ is assigned to S_1.

(b_2) The open segment $((1/3, 0), (1/3, 1))$ is assigned to T_1.

(c_2) The open horizontal segments $((0, 2/3), (1/3, 2/3))$ and $((1/3, 2/3), (1, 2/3))$ are placed into S_1.

https://doi.org/10.1017/S1446788700023971 Published online by Cambridge University Press

(d$_2$) The open horizontal segments $((0, 1/3), (2/3, 1/3))$ and $((2/3, 1/3), (1, 1/3))$ are placed into T_1.

Let R_1^* be that subsquare of Q_1 whose vertices are $(1/3, 1/3)$, $(1/3, 1)$, $(1, 1/3)$, and $(1, 1)$. The boundary of R_1^* is contained in T_1 except for the point $z(2/3, 1/3)$ which resides in S_1. Hence, if ζ is in the interior of R_1^*, then both α and β contain z, but this would imply the existence of t_3 and t_4 such that $t_1 < t_3 < 1$, $t_2 < t_4 < 1$, and $\alpha(t_3) = \beta(t_4) = z$, and that is impossible. Thus, if $\zeta \in R_1^*$ the lemma is valid.

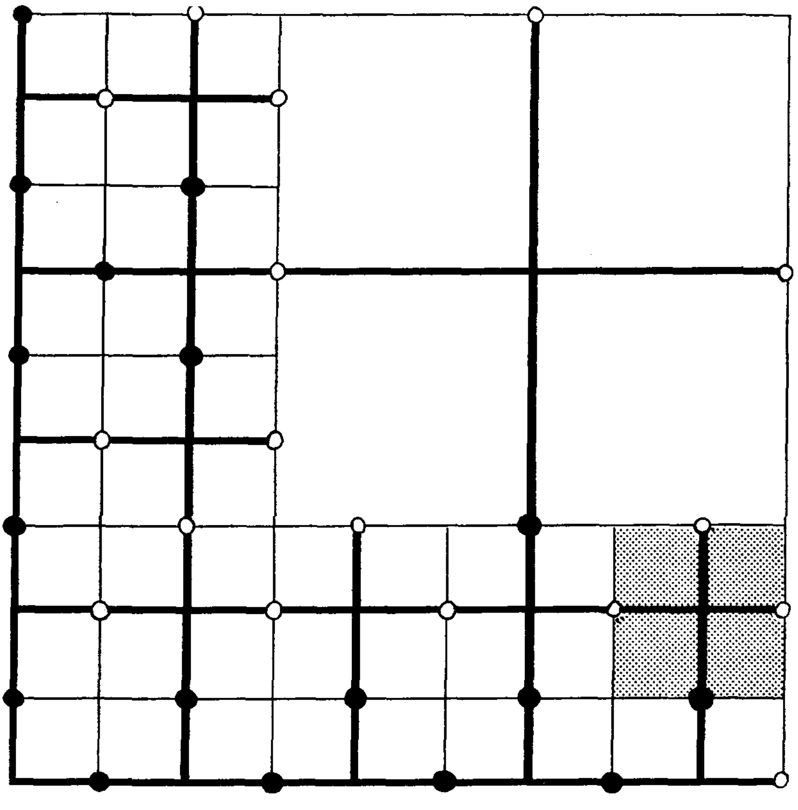

Figure 3.

Secondly, we show that if ζ is an interior point of the square region R_2^*, having vertices $(1/9, 1/9)$, $(1, 1/9)$, $(1/9, 1)$, and $(1, 1)$ the lemma is also valid. An inductive argument then provides that if ζ is in the interior of the square region R_n^*, whose vertices are $(1/3^n, 1/3^n)$, $(1, 1/3^n)$, $(1/3^n, 1)$, and $(1, 1)$ then the lemma is true. But, as ζ is an interior point of Q_1, $\zeta \in \bigcup_{n=1}^{\infty} \operatorname{Int} R_n^*$ and hence, the result follows. We exhibit the third stage of the construction within $Q_1 - R_1^*$ and consider the seven closed square subregions of R_2^* which border R_1^*. See Figure 3 where R_2^1 is shaded.

https://doi.org/10.1017/S1446788700023971 Published online by Cambridge University Press

1. R_2^1 having vertices (7/9, 1/9), (7/9, 1/3), (1, 1/3), and (1,1/9).
2. R_2^2 having vertices (7/9, 1/9), (7/9, 1/3), (5/9, 1/9), and (5/9, 1/3).
3. R_2^3 with vertices (5/9, 1/9), (5/9, 1/3), (1/3, 1/9), and (1/3, 1/3).
4. R_2^4 with vertices (1/3, 1/9), (1/3, 1/3), (1/9, 1/9), and (1/9,1/3).
5. R_2^5 with vertices (1/9, 1/3), (1/3, 1/3), (1/9, 5/9), and (1/3, 5/9).
6. R_2^6 having vertices (1/9, 5/9), (1/3, 5/9), (1/9, 7/9), and (1/3, 7/9).
7. R_2^7 having vertices (1/9, 7/9), (1/3, 7/9), (1/9, 1), and (1/3, 1).

Squares R_2^1, R_2^3, and R_2^4 have exactly one point of S_1 on their respective boundaries, and hence if both α and β intersect the interior of one of these three squares then both α and β must contain that point. That is, there is but one S_1-entrance to each one of these squares. It follows that if ζ is interior to one of R_2^1, R_2^3, or R_2^4 the lemma obtains. The remaining square which lies below R_1^* is R_2^2. Now, $R_2^2 \cup R_1^*$ has but one point of its boundary in S_1, and again, if both α and β intersect the interior of $R_2^2 \cup R_1^*$ then both α and β contain that point, and the lemma is valid.

The remaining squares are those to the left of R_1^*, and their union $R_2^5 \cup R_2^6 \cup R_2^7$ once more has exactly one point of its boundary in S_1. Hence, as before, if ζ is an interior point of $R_2^5 \cup R_2^6 \cup R_2^7$ the lemma is true. But,

$$\operatorname{Int} R_2^* - [(\operatorname{Int} R_2^1) \bigcup (\operatorname{Int} R_2^3) \bigcup \operatorname{Int}(R_2^4) \bigcup \operatorname{Int}(R_2^2 \bigcup R_1^*) \bigcup \operatorname{Int}(R_2^5 \bigcup R_2^6 \bigcup R_2^7)] \subset T_1 \bigcap M.$$

Consequently, if $\zeta \in \operatorname{Int} R_2^*$ then ζ is an interior point of one of the sets $R_2^1, R_2^3, R_2^4, R_2^2 \cup R_1^*$, or $R_2^5 \cup R_2^6 \cup R_2^7$, as ζ is in $S_1 \cup (P-M)$, and thus the lemma obtains. An inductive argument now completes the proof.

Lemma 2. *If α is an arc such that $\alpha(t) \in S_1$ for $0 \leqq t < 1$, then $\alpha(t) \in S_1 \cap M$ for $0 < t < 1$.*

Proof. Suppose, to the contrary, that there exists a number s^*, $0 < s^* < 1$, such that $\alpha(s^*) \notin M$. Consider the following two arcs at $\alpha(s^*)$:

1. $\alpha_1(t) = \alpha(s^* t)$ for $0 \leqq t < 1$,
2. $\alpha_2(t) = \alpha\left(\frac{s^* - 1}{1} t + \frac{s^* + 1}{2}\right)$ for $0 \leqq t < 1$.

As $\alpha(s^*) \notin M$ there exists a nested sequence of squares $\{Q_n : n = 1, 2, \cdots\}$, where Q_n is a square of the nth stage, such that $\bigcap_{n=1}^{\infty} Q_n = \alpha(s^*)$, and $\alpha(s^*)$ is an interior point of each Q_n; $n = 1, 2, \cdots$. Consequently, there is a natural number $N > 0$ such that both $\alpha_1(0)$ and $\alpha_2(0)$ are exterior to Q_N. As α_1 and α_2 are arcs at $\alpha(s^*)$ we may apply Lemma 1 to obtain numbers t_1 and t_2 such that

https://doi.org/10.1017/S1446788700023971 Published online by Cambridge University Press

$\alpha_1(t_1) = \alpha_2(t_2)$. It follows that $\alpha(s^*t_1) = \alpha\left(\frac{s^*-1}{2}t_2 + \frac{s^*+1}{2}\right)$ where

$$s^*t_1 < s^* < \frac{s^*-1}{2}t_2 + \frac{s^*+1}{2} < 1.$$

This, however, is impossible as α is an arc. One can easily verify that there are arcs in S_1 such that the initial points of those arcs do not lie on M (if $S_1 \not\subset M$). Hence, in this sense, Lemma 2 is a best possible result.

We will now need to refer to points of S_1 which were admitted to S_1 at a particular stage of the construction. For this reason we define

$S_1(n) = \{\zeta \in S_1 : \zeta$ was placed into S_1 during the nth stage of the construction and not before$\}$.

LEMMA 3. *Let α be an arc in S_1 with $\alpha(0) \in S_1(N)$ and $\alpha(t) \in \bigcup_{n=N}^{\infty} S_1(n)$ for $0 < t < 1$. Let $0 \leqq t_1 < t_2 < 1$ be such that $\alpha(t_1) \in S_1(m)$ and $\alpha(t_2) \in S_1(k)$. Then $m \leqq k$.*

PROOF. Suppose that $m > k$ and denote by Q_k a particular square of the kth stage containing $\alpha(t_1)$. We note that as $m > k$ and $k \geqq N$, $m > N$ and consequently $t_1 > 0$. The boundary of Q_k is part of the maze, M_k, of the kth stage of the construction and as such does not contain $\alpha(t_1) \in S_1(m)$. It follows then that $\alpha(t_1)$ is an interior point of Q_k. Define

1. $\gamma(t) = \alpha(t_1 t)$ for $0 \leqq t < 1$,
2. $\beta(t) = \alpha([t_1 - t_2]t + t_2)$ for $0 \leqq t < 1$.

Now, both γ and β are arcs at $\alpha(t_1)$, and $\alpha(t_1)$ is an interior point of Q_k. Further, $\alpha(0) = \gamma(0)$ and $\alpha(t_2) = \beta(0)$, and as $\alpha(0) \in S_1(N) \subset M_k$ and $\alpha(t_2) \in S_1(k) \subset M_k$, each of $\gamma(0)$ and $\beta(0)$ is a noninterior point of Q_k. We may therefore apply Lemma 1 to obtain numbers s_1 and s_2 such that $\gamma(s_1) = \beta(s_2)$. It follows that

$$\alpha(t_1 s_1) = \alpha([t_1 - t_2]s_2 + t_2)$$

where

$$0 \leqq t_1 s_1 < t_1 < [t_1 - t_2]s_2 + t_2 \leqq t_2.$$

This, however, contradicts the fact that α is an arc.

A consequence of this lemma is that if α is an arc satisfying the hypothesis of Lemma 3 and if t_1 and t_2 are numbers such that $0 \leqq t_1 < t_2 < 1$, with both $\alpha(t_1)$ and $\alpha(t_2)$ in $S_1(m)$ for some $m \geqq N$, then $\alpha(t) \in S_1(m)$ for $t_1 \leqq t \leqq t_2$.

LEMMA 4. *Let α be an arc in S_1 such that $\alpha(0) \in S_1(N)$ for some $N > 0$ and $\alpha(t) \in \bigcup_{n=N}^{\infty} S_1(n)$ for $0 < t < 1$. Then α is rectifiable.*

PROOF. Define $I_n = \{t \in [0,1) : \alpha(t) \in S_1(n)\}$. As N is the least number such that $\alpha \cup S_1(N) \neq \varnothing$, $I_k = \varnothing$ for $k < N$. Further, as $\alpha(0) \in S_1(N)$ and $\alpha(t) \in \bigcup_{n=N}^{\infty}$

$S_1(n)$, the consequence we mentioned of Lemma 3 above guarantees that I_n is either an interval, a point, or empty, for $n \geqq N$. Also, Lemma 3 insures that if $m < k$ then I_m lies to the left of I_k (i.e., if $x \in I_m$ and $y \in I_k$ then $x < y$). Hence in order to prove that α is rectifiable, it is sufficient to prove that α/I_n is rectifiable with length say L_n for $n = 1, 2, \cdots$, and in addition, that $\sum_{n=1}^{\infty} L_n < \infty$.

Evidently α/I_1 is of finite length. It is possible, however, to define arcs α in such a manner that α/I_1 is as long as any predetermined length. This is a unique property, though, not had by α/I_n for $n > 1$; in fact, in general, α/I_n has length less than $2/3^{n-1}$. To show that this is the case, we first notice that if Q_n is a square of the nth stage, then a side of Q_n has length $1/3^{n-1}$ for $n = 1, 2, \cdots$. Now, let $t \in I_n$ for $n > 1$ (if $I_n = \varnothing$ then the length of $\alpha/I_n = 0$) and let Q_{n-1} be a square of the $n-1$st stage which contains $\alpha(t)$. The boundary of Q_{n-1} lies in $T_1 \cup (\bigcup_{k=1}^{n-1} S_1(k))$ and consequently does not meet $S_1(n)$. As I_n is an interval and α/I_n is connected, it follows that α/I_n does not meet the exterior of Q_{n-1}. But $\alpha(t) \in Q_{n-1}$, and hence Q_{n-1} contains α/I_n. The maximum length of an arc in $S_1(n) \cap Q_{n-1}$ is $1/3^{n-2}$.

Consequently, α is a rectifiable arc and the length of α does not exceed $|\alpha/I_1| + \sum_{i=2}^{\infty} 1/3^{i-2} = |\alpha/I_1| + 3/2$.

The following lemma completes our work concerning S_1.

Lemma 5. *If $\Lambda \in \mathscr{A}(S_1)$, then Λ is rectifiable.*

Proof. As $\Lambda(t) \in S_1$ for $0 \leqq t < 1$, we may apply Lemma 2 to obtain that $\Lambda(t) \in S_1 \cup M$ for $0 < t < 1$. It follows that $\Lambda(t) \in \bigcup_{n=1}^{\infty} S_1(n)$ for $0 < t < 1$. Denote by N the smallest integer n such $\Lambda \cap S_1(n) \neq \varnothing$, and let t^* be such that $\Lambda(t^*) \in S_1(N)$. Define

1. $\Lambda_1(t) = \Lambda([1-t^*]t + t^*)$ for $0 \leqq t < 1$,
2. $\Lambda_2(t) = \Lambda(-t^*t + t^*)$ for $0 \leqq t < 1$.

It is evident that a necessary and sufficient condition for Λ to be rectifiable is that both Λ_1 and Λ_2 be rectifiable. But each of Λ_1 and Λ_2 satisfies the hypothesis of Lemma 4, and as such is rectifiable. This completes the proof of Lemma 5.

As S_1 and T_1 were constructed in similar fashion, we can verify the analogues of Lemmas 1 through 4 for T_1, and hence can establish the following result.

Lemma 5*. *If $\Gamma \in \mathscr{A}(T_1)$, then Γ is rectifiable.*

Thus, we have shown that not only is every point of the plane a rectifiably ambiguous point relative to S_1, but the only arcs contained wholly in either S_1 or in T_1 are rectifiable arcs. The results of this section are collected in Theorem 6.

Theorem 6. *There exists a set $S_1 \subset P$ such that every point of P is an ambiguous point relative to S_1, and both $\mathscr{A}(S_1) \subset \mathscr{R}$ and $\mathscr{A}(P - S_1) \subset \mathscr{R}$.*

2. The Sets S_2 and S_3

In this section we construct two other sets in P having the property that every point of P is an ambiguous point relative to that set. The first set we construct, S_2, has the additional property that both $\mathscr{A}(S_2) \subset \mathscr{N}$ and $\mathscr{A}(P-S_2) \subset \mathscr{N}$. The second construction provides a set S_3 having the property that $\mathscr{A}(S_3) \subset \mathscr{R}$ while $\mathscr{A}(P-S_3) \subset \mathscr{N}$.

The set S_2 is constructed as the image of S_1 under a suitable homeomorphism from P onto itself. Let Ψ be a continuous function of a real variable which is of bounded variation in no subinterval of real numbers. For the existence of such a function, sa function, see Carathéodory (1948; page 190). Then the graph of Ψ over any interval is nonrectifiable. Further, if f is a function of bounded variation on an interval $[a, b]$ then $\Psi + f$ is not of bounded variation on $[a, b]$ and its graph $\{(x, \Psi(x) + f(x)) \colon x \in [a, b]\}$, is also nonrectifiable.

We obtain S_2 from S_1 in two steps.

1. First rotate the set S_1 of Bagemihl's construction 45° in the clockwise direction about the origin to obtain the set S_2'.

2. Now, let Ψ be a function of a real variable which is continuous but of bounded variation in no interval of real numbers. Define $\Phi(x, y) = (x, y + \Psi(x))$. Then Φ is a homeomorphism from the plane onto itself, and we let S_2 be the image of S_2' under the mapping Φ. Denote by Γ the rotation about the origin through 45° followed by the mapping Φ. Then Γ is a homeomorphism from the plane onto itself such that

$$S_2 = \{\Gamma((x, y)) \colon (x, y) \in S_1\},$$

and we let

$$T_2 = \{\Gamma((x, y)) \colon (x, y) \in T_1\}.$$

If $\zeta \in P$, then $\zeta' = \Gamma^{-1}(\zeta)$ is an ambiguous point relative to S_1. Hence, there are arcs $\Lambda_1(\zeta')$ and $\Lambda_2(\zeta')$ at ζ' where $\Lambda_1(\zeta') \subset S_1$ and $\Lambda_2(\zeta') \subset T_1$. As Γ is a homeomorphism, $\Gamma \circ \Lambda_1(\zeta')$ and $\Gamma \circ \Lambda_2(\zeta')$ are arcs at ζ such that $\Gamma \circ \Lambda_1(\zeta') \subset S_2$ and $\Gamma \circ \Lambda_2(\zeta') \subset T_2$, and consequently ζ is an ambiguous point relative to S_2.

In order to verify that $\mathscr{A}(S_2) \subset \mathscr{N}$ it is sufficient to show that if Λ is an arc contained in S_2 then Λ is nonrectifiable. However, as Λ is contained in S_2 it follows that $\Gamma^{-1} \circ \Lambda$ is an arc contained in S_1, and by Lemma 2 we obtain that $\Gamma^{-1} \circ \Lambda(t) \in M$ except possibly when $t = 0$.

Let N denote the least integer such that $(\Gamma^{-1} \circ \Lambda) \cap S_1(N) \neq \varnothing$, and let t_1 be such that $\Gamma^{-1} \circ \Lambda(t_1) \in S_1(N)$. Define

$$\lambda(t) = \Gamma^{-1} \circ \Lambda([1 - t_1]t + t_1) \text{ for } 0 \leqq t < 1$$

and

$$I_n = \{t \in [0, 1); \lambda(t) \in S_1(n)\} \text{ for } n = N, N+1, \cdots.$$

https://doi.org/10.1017/S1446788700023971 Published online by Cambridge University Press

In the course of the proof of Lemma 4 we showed that I_n was an interval (possibly degenerate) for $n = N, N+1, \cdots$ and Lemma 2 insures that $\bigcup_{n=N}^{\infty} I_n = [0,1)$. It follows that there is an index $m \geqq N$ such that I_m is a nondegenerate interval, and as $\lambda / I_m \subset S_1(m)$, λ / I_m contains either a vertical or a horizontal line segment. As λ is a subarc of $\Gamma^{-1} \circ \Lambda$, $\Gamma^{-1} \circ \Lambda$ contains that same line segment, and consequently Λ is a nonrectifiable arc. Hence $\mathscr{A}(S_2) \subset \mathscr{N}$.

In a wholly analogous manner one can easily verify that $\mathscr{A}(T_2) \subset \mathscr{N}$. Our results concerning S_2 are contained in Theorem 7.

THEOREM 7. *There exists a set $S_2 \subset P$ such that every point of P is an ambiguous point relative to S_2, and both $\mathscr{A}(S_2) \subset \mathscr{N}$ and $\mathscr{A}(P - S_2) \subset \mathscr{N}$.*

The second set we construct in this section is a set S_3 having the following properties:

1. every point of P is ambiguous relative to S_3,
2. $\mathscr{A}(S_3) \subset \mathscr{R}$,
3. $\mathscr{A}(P - S_3) \subset \mathscr{N}$.

In order to construct S_3 we resort to a construction technique similar to that which Bagemihl used to define the set S_1. One preliminary construction is required.

Insertion of a Graph into an Arc

Let L_1 be a line in the plane, and let α be an arc in the plane such that $\alpha \cap L_1 = \varnothing$ and each line which is perpendicular to L_1 meets α in at most one point. Denote by L_2 a particular line which is perpendicular to L_1 and assume that $L_2 \cap \alpha \neq \varnothing$. Let $A = L_1 \cap L_2$ and $B = L_2 \cap \alpha$. Suppose further that $\varepsilon > 0$ is given, and that $g(x)$ is a continuous function defined for $0 \leqq x \leqq 1$ such that $g(0) = g(1) = 0$ and $-1 < g(x) < 1$ for $0 \leqq x \leqq 1$. We shall define what it means to ε-insert g into α along $[A, B]$, where $[A,B]$ denotes the closed line segment extending from A to B.

The general case is analogous to that where L_1 is the x-axis, $A = (1/2, 0)$, α is the graph of a continuous function $f(x)$ defined for $0 \leqq x \leqq 1$, and $f(1/2) > 0$. In this instance, $B = (1/2, f(1/2))$. Further assume that $0 < \varepsilon < 1/2$, and define the fluctuation of a function $h(x)$ defined on a closed interval $[a, b]$ as

$$\max\{h(x) : x \in [a, b]\} - \min\{h(x) : x \in [a, b]\}.$$

We define a "pyramid" consisting of an infinite sequence of closed rectangular regions, each of which is symmetric about the line segment $[A, B]$, has edges which are parallel to the coordinate axes, and lies between the graph of f and the x-axis, in the following manner.

i. Let $1/10 > \delta_1 > 0$ be such that both $\delta_1 < \varepsilon$ and the fluctuation of $f(x)$ on the closed interval $[1/2 - \delta_1, 1/2 + \delta_1]$ is less than $[1/10^2] f(1/2)$. Denote by R_1 the closed rectangular region having vertices

https://doi.org/10.1017/S1446788700023971 Published online by Cambridge University Press

$(1/2 - \delta_1, 0)$, $(1/2 + \delta_1, 0)$, $(1/2 - \delta_1, [9/10]f(1/2))$, and $(1/2 + \delta_1, [9/10]f(1/2))$.

The number δ_1 will be referred to as the *width of the insertion* and the choice of δ_1 precludes the possibility of the graph of f intersecting R_1.

ii. In general, let $1/10^n > \delta_n > 0$ be such that $\delta_n < \varepsilon$ and the fluctuation of $f(x)$ on the closed interval $[1/2 - \delta_n, 1/2 + \delta_n]$ is less than $[1/10^{n+1}]f(1/2)$. Denote by R_n the closed rectangular region having vertices

$$(1/2 - \delta_n, ([10^{n-1} - 1]/[10^{n-1}])f(1/2)),\ (1/2 + \delta_n, ([10^{n-1} - 1]/[10^{n-1}])f(1/2)),$$
$$(1/2 - \delta_n, ([10^n - 1]/10^n)f(1/2)),\ \text{and}\ (1/2 - \delta_n, ([10^n - 1]/10^n)f(1/2)).$$

We now place a copy of the graph of g into each of these rectangular regions, using the segment $[A, B]$ as an axis. The restriction that $g(0) = g(1) = 0$ insures that the inserted copies link in such a fashion that their union is an arc at B. Specifically, we define the graph of g placed into R_n $(n = 1, 2, \cdots)$ to be

$$G_n = \{(1/2 + \delta_n g[(10^n/9)(y/f(1/2)) - 10^n/9 + 10/9], y):$$
$$([10^{n-1} - 1]/10^{n-1})f(1/2) \leqq y \leqq ([10^n - 1]/10^n)f(1/2)\}.$$

The ε-insertion of g into f along $[A, B]$ is then $\bigcup_{n=1}^{\infty} G_n$. The insertion itself is the graph of a continuous function $g^*(y) = x$ defined for $0 < y < f(1/2)$. Suppose $0 < y_1 < f(1/2)$, and n is such that $([10^{n-1} - 1]/10^{n-1})f(1/2) \leqq y_1$. Then the construction provides that the fluctuation of g^* on $[y_1, f(1/2)]$ is at most $2\delta_n$. This completes our preliminary construction, and we are now able to proceed to the first stage of the construction of the set S_3.

We construct the set S_3, and its complementary set T_3, in a manner quite analogous to the way Bagemihl constructed the set S_1. Again a maze is constructed in an inductive fashion, and again this maze, M, will carry every arc of ambiguity. The difference is that $T_3 \cap M$ consists not of vertical and horizontal line segments as does $T_1 \cap M$, but rather of arcs which are totally nonrectifiable. These arcs are, however, graphs of functions inserted along either vertical or horizontal line segments. In particular, let $f(x)$ be a continuous function defined for $0 \leqq x \leqq 1$ such that $f(x)$ has the following properties:

1. f is of bounded variation in no subinterval of $[0, 1]$,
2. $-1/10 < f(x) < 1/10$ for $0 \leqq x \leqq 1$,
3. $f(0) = f(1) = 0$.

As f is of bounded variation in no subinterval of $[0, 1]$, its graph, F, is totally nonrectifiable. Stage 1 of the construction for M occurs in four parts.

(a_1) Put the vertical lines $x = 2n$ $(n = 0, \pm 1, \pm 2, \cdots)$ into S_3.

(b_1) Define $f^*(x) = f(x - [[x]])$ where $[[x]]$ is the greatest integer less than or equal to x. Denote the graph of f^* by F^*. Now, rotate F^* about the origin using $\pi/2$ as the angle of rotation, and translate the rotated set $2n + 1$ units horizontally to obtain the set F^*_{2n+1} where $n = 0, \pm 1, \pm 2, \cdots$. Place the sets F^*_{2n+1} into T_3. Each set F^*_{2n+1} is termed a vertical T_3-set of stage 1 and is said to have the line $x = 2n + 1$ as an axis.

(c_1) The plane has now been divided into enumerably many unbounded vertical "columns", and in this part we subdivide each column into bounded regions by introducing horizontal T_3-sets. As the construction in this third part is carried out similarly within each column, we shall restrict our attention to the column bounded by the y-axis and the vertical T_3-set F^*_1. From each of the points $(0, 2n + 1)$ $(n = 0, \pm 1, \pm 2, \cdots)$ 1/10-insert the function f into F^*_1 along the horizontal line segment extending from $(0, 2n + 1)$ to the set F^*_1, and place the points of these insertions, with the exception of their initial points on the y-axis which already belong to S_3, into T_3. Denote the width of this insertion by δ_1. These inserted sets are termed horizontal T_3-sets of stage 1, and their axes, which are the horizontal line segments along with the insertions occur, are at odd integer heights.

(d_1) The points of the horizontal lines $y = 2n$ $(n = 0, \pm 1, \pm 2, \cdots)$ which have not as yet been assigned, are now assigned to S_3.

This completes stage 1 of the construction of M. See Figure 4.

Stage 1 of the construction divides the plane into enumerably many regions which are called "grid squares" of the first stage. The intersection of a vertical T_3-arc with a horizontal T_3-arc is called a T_3-node, while an S_3-node is the intersection of a horizontal line segment in S_3 with a vertical line segment in S_3. Every grid square contains exactly one S_3-node and one T_3-node.

Stage 2 of the construction of M is typical of the construction at future stages, and occurs within the grid squares of stage 1. As the construction at this stage is carried out analogously within each grid square, we restrict our attention to the one having vertices (0,0), (1,0), (0,1), and (1, 1). The S_3-node of this grid square is (0,0), and the horizontal T_3-set has been inserted into the vertical T_3-set. The horizontal T_3-set bounding this grid square is the graph of a continuous function, $h(x)$, defined for $0 < x < 1$. We proceed as follows:

(a_2) Partition the interval $[0, 9/10]$ into an even number of subintervals $[x_0, x_1] = [0, x_1], [x_1, x_2], \cdots, [x_{2n-1}, x_{2n}] = [x_{2n-1}, 9/10]$ such that

1. $|x_k - x_{k-1}| < 1/10$ for $k = 1, 2, \cdots, 2n$.
2. All the partitioning intervals are of the same length, denoted by d.
3. The fluctuation of $h(x)$ on $[x_{k-1}, x_k]$ for $k = 1, 2, \cdots, 2n$ is less than $1/10^2$.

Erect a vertical line segment from the point $(x_{2k}, 0)$ to the point $(x_{2k}, h(x_{2k}))$ for $k = 1, 2, \cdots, n$, and place the points of these open segments into S_3. For notational convenience we denote the interval $[9/10, 1]$ by $[x_{2n}, x_{2n+1}]$.

(b_2) From the line $y = 0$, and along the vertical segments $[(x_{2k-1}, 0), (x_{2k-1}, h(x_{2k-1}))]$ for $k = 1, 2, \cdots, n$, ε_2-insert the function $f(x)$ into the graph of $h(x)$, where

$$\varepsilon_2 = \min\{1/10^2, d/10\}.$$

https://doi.org/10.1017/S1446788700023971 Published online by Cambridge University Press

Place the points of these insertions not already assigned into T_3. These newly inserted sets are called vertical T_3-sets of the second stage.

(c_2) Denote the vertical T_3-set inserted along the line segment $[(x_{2k-1},0), (x_{2k-1},h(x_{2k-1}))]$ by V_{2k-1} where $k = 1,2,\cdots,n$. Denote by V_{2n+1} that portion of the vertical T_3-set of stage 1 which has the line $x = x_{2n+1} = 1$ as an axis, and lies on the boundary of the grid square under consideration. The original grid square of stage 1 can now be considered as having been divided into columns

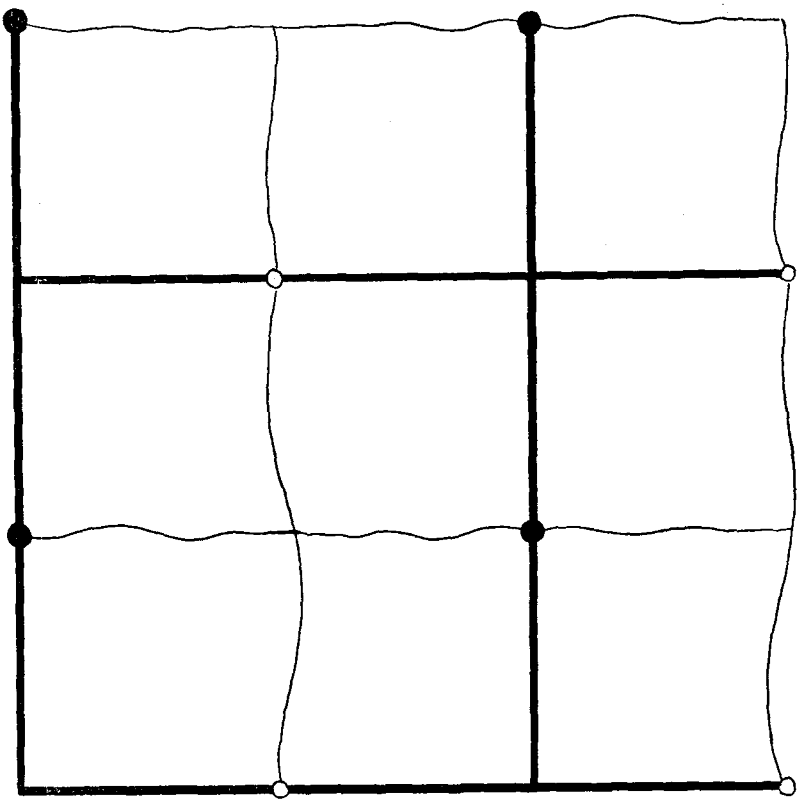

Figure 4.

determined by the original boundary of the grid square and by the newly inserted vertical T_3-sets. All save one of these columns are bounded vertically by an adjacent pair of vertical T_3-sets, while the other column is bounded on the right by a vertical T_3-set V_1 and on the left by the vertical S_3-set consisting of the segment $[(0,0),(0,1)]$. The construction continues within these columns. Each column of the former type has a vertical line segment separating the vertical T_3-sets which border it. If V_{2k-3} and V_{2k-1} $(k = 2,3,\cdots,n+1)$ form the vertical borders of this column, then the central segment (which is a vertical S_3-set) for

https://doi.org/10.1017/S1446788700023971 Published online by Cambridge University Press

that column is $[(x_{2k-2},0),(x_{2k-2},h(x_{2k-2}))]$. Denote $h(x_{2k-2})-\delta_1$ by C where δ_1 is the width of the horizontal insertion of (c_1). Also, V_{2k-3} and V_{2k-1} are graphs of continuous functions defined on the open intervals $(0,h(x_{2k-3}))$ and $(0,h(x_{2k-1}))$, and we denote those functions by $g_{2k-3}(y)=x$ and $g_{2k-1}(y)=x$, respectively. Partition the interval $[0,C]$ into an even number of subintervals, $[y_0,y_1]=[0,y_1],[y_1,y_2],\cdots,[y_{2m-1},y_{2m}]=[y_{2m-1},C]$ such that

1. $|y_q-y_{q-1}|<1/10$ for $q=1,2,\cdots,2m$.
2. All of the partitioning intervals are of the same length, denoted by d_1.
3. The fluctuation of $g_{2k-3}(y)$ and of $g_{2k-1}(y)$ on $[y_{q-1},y_q]$ is less than $1/10^2$ for $q=1,2,\cdots,2m$.

Now, along the horizontal segments $[(x_{2k-2},y_{2q-1}),\ (g_{2k-3}(y_{2q-1}),y_{2q-1})]$ and $[(x_{2k-2},y_{2q-1}),\ (g_{2k-1}(y_{2q-1}),y_{2q-1})]$ and from the point $(x_{2k-2},\ y_{2q-1})$ ε_3-insert the function $f(x)$ into V_{2k-3} and V_{2k-1}, respectively, where $\varepsilon_3=\min\{d_1/10,1/10^2\}$. The points of these insertions not as yet assigned are now placed into T_3. The column of the remaining type is handled similarly; however, horizontal insertions are into V_1 only, and hence in only one direction.

(d_2) For columns of the initial type, horizontal segments are now constructed which extend from V_{2k-3} to V_{2k-1} and which pass through the points (x_{2k-2},y_{2q}) for $q=1,2,\cdots,m$, and these horizontal segments are placed into S_3. For the remaining column, horizontal segments spanning the gap between $[(0,0),(0,1)]$ and V_1 are constructed and placed into S_3.

The maze M_2, then, consists of the maze M_1 described in the first stage of the construction together with the newly added points. See Figure 5. The plane has once again been subdivided into regions, which are termed grid squares of the second stage. Each grid square consists of its interior, which does not meet M_2, two line segments, one horizontal and one vertical meeting at a common endpoint (the S_3-node of this second-stage grid square), and the graphs of two continuous functions, each of which is of bounded variation in no subinterval on which it is defined. One of these graphs has been inserted into the other. Further, one graph has a horizontal axis and the other has a vertical axis. If we assume that for a particular grid square of stage two the horizontal T_3-arc has been inserted into the vertical T_3-arc, then the fluctuation of the horizontal T_3-arc is less than $2/10^2$, while the fluctuation of the vertical T_3-arc is less than $1/10^2$. The construction of M_3 is carried out within the grid squares of stage two and is analogous to that completed for M_2.

Proceeding inductively we obtain a maze M_n for each $n=1,2,\cdots$. Define $M=\bigcup_{n=1}^{\infty}M_n$. Finally, let S_3 consist exactly of those points which have been entered into S_3 during the course of the construction of M, and let $T_3=P-S_3$. This completes the construction, and we now proceed to verify that S_3 has the properties we initially claimed it would have.

First we must show that every point of the plane is an ambiguous point relative to S_3. To this end we let $\zeta \in P$, and define an arc at ζ in the following way. It is evident that there is a nested sequence of grid squares, $\{Q_n: n = 1, 2, \cdots\}$, such that Q_n is a grid square of the nth stage and $\bigcap_{n=1}^{\infty} Q_n = \{\zeta\}$. Let σ_n be the S_3-node of Q_n. The construction of M_{n+1} from M_n provides that if $\sigma_n \neq \sigma_{n+1}$ then there is an arc Γ_n lying in $S_3 \cap M_{n+1}$ such that $\Gamma_n(0) = \sigma_n$ and $\Gamma_n(1) = \sigma_{n+1}$. We define $\Gamma_n = \sigma_n$ if $\sigma_n = \sigma_{n+1}$. Letting $\Lambda_1^* = \bigcup_{n=1}^{\infty} \Gamma_n$ we find that Λ_1^*

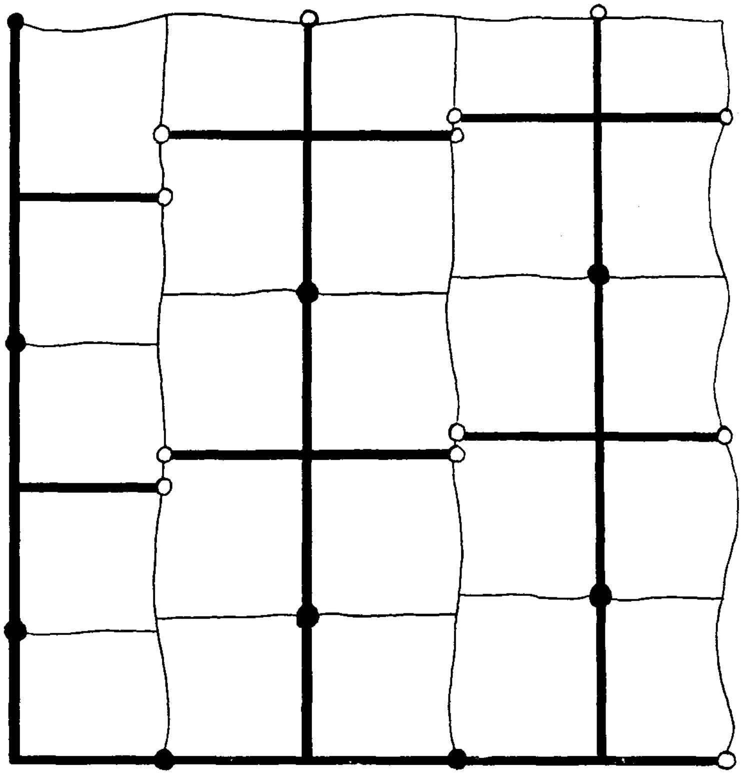

Figure 5.

provides a path (possibly not an arc) from σ_1 to ζ which lies entirely in S_3. It follows then that there is an arc $\Lambda_1 \subseteq \Lambda_1^*$ at ζ which lies entirely within S_3. Furthermore, due to the fact that σ_1 lies on a vertical straight line contained in S_3 (see stage one of the construction of S_3), it is possible to obtain such arcs at ζ of arbitrarily large diameter. In an analogous manner, arcs at ζ of arbitrarily large diameter which are contained in T_3 can be exhibited.

The existence of these arcs at ζ emanating from distant points allows us to conclude that neither S_3 nor T_3 contains a loop. The fact that neither S_3 nor

T_3 contains a loop, together with the similarity of construction between S_3 and S_1, allows us to prove the analogue of Lemma 1 for each of the sets S_3 and T_3. These results are listed below as Lemma 8a and Lemma 8b. The proof of each of these lemmas follows the proof of Lemma 1 closely and therefore is not given.

LEMMA 8a. *Suppose $\zeta \in S_3 \cup (P-M)$ and ζ is an interior point of a grid square Q_n of the nth stage. Suppose further that α and β are arcs at ζ such that*

1. *$\alpha(0)$ and $\beta(0)$ are in $P - \mathrm{Int}(Q_n)$,*
2. *$\alpha(t)$ and $\beta(t)$ are in S_3 for $0 \leqq t < 1$.*

Then $\alpha(t_1) = \beta(t_2)$ where

$$t_1 = \sup\{t : \alpha(t) \in \mathrm{Bd}(Q_n)\},$$

$$t_2 = \sup\{t : \beta(t) \in \mathrm{Bd}(Q_n)\}.$$

LEMMA 8b. *Suppose $\zeta \in T_3 \cup (P-M)$ and ζ is an interior point of a grid square Q_n of the nth stage. Suppose further that α and β are arcs at ζ such that*

1. *$\alpha(0)$ and $\beta(0)$ are in $P - \mathrm{Int}(Q_n)$,*
2. *$\alpha(t)$ and $\beta(t)$ are in T_3 for $0 \leqq t < 1$.*

Then $\alpha(t_1) = \beta(t_2)$ where

$$t_1 = \sup\{t : \alpha(t) \in \mathrm{Bd}(Q_n)\},$$

$$t_2 = \sup\{t : \beta(t) \in \mathrm{Bd}(Q_n)\}.$$

We are now able to use Lemmas 8a and 8b to prove the analogues of Lemmas 2 and 3 for this new construction. Only the following two analogues are needed, however, and we list them without further verification.

LEMMA 9. *If α is an arc such that $\alpha(t) \in T_3$ for $0 \leqq t < 1$, then $\alpha(t) \in T_3 \cap M$ for $0 < t < 1$.*

Define $S_3(n) = \{\zeta \in S_3 : \zeta$ was entered into S_3 during the nth stage of the construction and not before$\}$.

LEMMA 10. *Let α be an arc in S_3 with $\alpha(0) \in S_3(N)$ and $\alpha(t) \in \bigcup_{n=N}^{\infty} S_3(n)$ for $0 < t < 1$. Let $0 \leqq t_1 < t_2 < 1$ be such that $\alpha(t_1) \in S_3(m)$ and $\alpha(t_2) \in S_3(k)$. Then $m \leqq k$.*

Lemma 9 guarantees that if α is an arc such that $\alpha(t) \in T_3$ for $0 \leqq t < 1$ then α is nonrectifiable, for it is clear that if α contains a subarc which is imbedded in the maze M, then that subarc is totally nonrectifiable, and consequently α is nonrectifiable. Lemma 10 is important, for it enables us to prove the rectifiability of arcs that are subsets of S_3. We prove this in the spirit of Lemmas 4 and 5.

LEMMA 11. *Let α be an arc in S_3 such that $\alpha(0) \in S_3(N)$ for some $N > 0$ and $\alpha(t) \in \bigcup_{n=N}^{\infty} S_3(n)$ for $0 < t < 1$. Then α is rectifiable.*

PROOF. Again as in Lemma 4, define

$$I_n = \{t \in [0,1]: \alpha(t) \in S_3(n)\}.$$

As N is the least integer such that $\alpha \cap S_3(n) \neq \emptyset$ we have $I_k = \emptyset$ for $k < N$. Further, as $\alpha(0) \in S_3(N)$ and $\alpha(t) \in \bigcup_{n=N}^{\infty} S_3(n)$, Lemma 10 guarantees that I_n is either an interval, a point, or $\emptyset$ for $n \geqq N$. Lemma 10 also entails that if $x \in I_m$ and $y \in I_k$ and $m > k$ then $x > y$. Thus, in order to prove that α is rectifiable, it is sufficient to prove that both

1. α/I_n is rectifiable for $n = 1, 2, \cdots$

and

2. $\sum_{n=N}^{\infty} |\alpha/I_n| < \infty$.

As in Lemma 4, the case where $n = 1$ does not fit the pattern of the other cases. However, α/I_1 is of finite length. In general (i.e., for $n = 2, 3, \cdots$) we find the length of α/I_n to be less than $11/10^{n-1}$. It follows, then, that α is rectifiable and that the length of α does not exceed $|\alpha/I_1| + 11/9$.

LEMMA 12. *If $\alpha \in \mathscr{A}(S_3)$, then α is rectifiable.*

PROOF. The proof of Lemma 12 is identical with the proof of Lemma 5.

We collect the results of the previous lemmas concerning S_3 in the following theorem.

THEOREM 13. *There exists a set $S_3 \subset P$ such that every point of P is an ambiguous point relative to S_3, and $\mathscr{A}(S_3) \subset \mathscr{R}$ but $\mathscr{A}(P - S_3) \subset \mathscr{N}$.*

In view of the previous theorems one might conjecture that it is possible to define a set $S_4 \subset P$ such that every point of P is both rectifiably ambiguous relative to S_4, and nonrectifiably ambiguous relative to S_4. Indeed this is the case, and an example is constructed by letting S_4 be the image of S_1 under the function $\Psi: P \to P$ where

$$\Psi((x,y)) = (x + \psi(y), y)$$

and

$$\psi(y) = \begin{cases} y \sin(1/y) & \text{for } 0 < y, \\ 0 & \text{for } y \leqq 0. \end{cases}$$

If $\zeta \in P$ and ζ is not on the x-axis, then an arc α at ζ which lies in S_4 may be extended or shortened so as to include or exclude a nonrectifiable portion, and hence may be chosen to be either rectifiable or nonrectifiable. The same is true for an arc at ζ which lies in $P - S_4$. For every point ζ on the x-axis there is a nonrectifiable arc at ζ which is contained in the upper half-plane intersected with S_4, and a rectifiable arc at ζ contained in the lower half-plane intersected with S_4. Similar arcs lying in $P - S_4$ can also be found.

The question of whether terminally different arcs of approach can exist in both a set S and its complement for a large set of points is answered in the next, and concluding, section.

3. A General Theorem

This section is devoted to proving a general theorem which entails that if S is a planar set, the set of points which are both rectifiably ambiguous relative to S and totally nonrectifiably ambiguous relative to S is of first Baire category.

THEOREM 14. *Suppose that A_1, A_2, B_1, and B_2 are sets of planar arcs, and let $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$. Further, assume that*

1. *A_1 and A_2 are terminally arcwise disjoint,*
2. *B_1 and B_2 are terminally arcwise disjoint,*
3. *A and B are pointwise disjoint.*

Then the set of points which are accessible via each of the sets A_1, A_2, B_1, and B_2 is of first Baire category.

PROOF. Suppose to the contrary that the set of points of P which are accessible via each of the sets A_1, A_2, B_1, and B_2 is a set of second Baire category, Q. That is, if $\zeta \in Q$ there are arcs $\alpha'_1(\zeta)$, $\alpha'_2(\zeta)$, $\beta'_1(\zeta)$, and $\beta'_2(\zeta)$ at ζ where $\alpha'_1(\zeta) \in A_1$, $\alpha'_2(\zeta) \in A_2$, $\beta'_1(\zeta) \in B_1$, and $\beta'_2(\zeta) \in B_2$. We shall assign an ordered sextuple of rational numbers to ζ in the following manner using a technique developed by Bagemihl (1966).

1. Let $\Delta(\zeta)$ be a rational disc (i.e., a planar disc with a rational center and radius) which contains ζ and is such that the four arcs of accessibility meet the boundary of $\Delta(\zeta)$. Assign $\Delta(\zeta)$ to ζ and let

a. $t_1^* = \max\{t : \alpha'_1(\zeta; t) \in \mathrm{Bd}(\Delta(\zeta))\}$,

b. $t_2^* = \max\{t : \alpha'_2(\zeta; t) \in \mathrm{Bd}(\Delta(\zeta))\}$,

c. $t_3^* = \max\{t ; \beta'_1(\zeta; t) \in \mathrm{Bd}(\Delta(\zeta))\}$,

d. $t_4^* = \max\{t : \beta'_2(\zeta; t) \in \mathrm{Bd}(\Delta(\zeta))\}$.

Then define

a. $\alpha_1(\zeta; t) = \alpha'_1(\zeta; [1 - t_1^*]t + t_1^*)$; $0 \leqq t < 1$,

b. $\alpha_2(\zeta; t) = \alpha'_2(\zeta; [1 - t_2^*]t + t_2^*)$; $0 \leqq t < 1$,

c. $\beta_1(\zeta; t) = \beta'_1(\zeta; [1 - t_3^*]t + t_3^*)$; $0 \leqq t < 1$,

d. $\beta_2(\zeta; t) = \beta'_2(\zeta; [1 - t_4^*]t + t_4^*)$; $0 \leqq t < 1$.

2. If ζ_1 and ζ_2 are in P we let $[\zeta_1, \zeta_2]$ denote the closed line segment extending from ζ_1 to ζ_2. Let $\varepsilon(\zeta)$ be a rational number satisfying

$$0 < \varepsilon(\zeta) < 1/2 \min\{|[\alpha_1(\zeta;0), \beta_1(\zeta;0)]|, |[\alpha_1(\zeta;0), \beta_2(\zeta;0)]|, |[\alpha_2(\zeta;0), \beta_1(\zeta;0)]|, |[\alpha_2(\zeta;0), \beta_2(\zeta;0)]|\}$$

and assign $\varepsilon(\zeta)$ to ζ.

3. We now choose rational directions which approximate the directions of the rays emanating from the center of $\Delta(\zeta)$ to the initial points of the shortened arcs of accessibility at ζ. For notational convenience let $r(\theta)$ denote the ray whose initial point is the center of $\Delta(\zeta)$ and whose direction is θ. We define these approximating directions as follows:

a. Let $\theta_1(\zeta)$ be a rational direction such that $r(\theta_1(\zeta)) \cap \mathrm{Bd}(\Delta(\zeta))$ is within $(1/4)[\varepsilon(\zeta)]$ of $\alpha_1(\zeta;0)$.

b. Let $\theta_2(\zeta)$ be a rational direction such that $r(\theta_2(\zeta)) \cap \mathrm{Bd}(\Delta(\zeta))$ is within $(1/4)[\varepsilon(\zeta)]$ of $\alpha_2(\zeta;0)$.

c. Let $\phi_1(\zeta)$ be a rational direction such that $r(\phi_1(\zeta)) \cap \mathrm{Bd}(\Delta(\zeta))$ is within $(1/4)[\varepsilon(\zeta)]$ of $\beta_1(\zeta;0)$.

d. Finally, let $\phi_2(\zeta)$ be a rational direction such that $r(\phi_2(\zeta)) \cap \mathrm{Bd}(\Delta(\zeta))$ is within $(1/4)[\varepsilon(\zeta)]$ of $\beta_2(\zeta;0)$.

Assign these four directions to ζ.

The assigning is now completed and we define $Q(\Delta, \varepsilon, \theta_1, \theta_2, \phi_1, \phi_2)$ to be the set of all points in Q to which the ordered sextuple $(\Delta, \varepsilon, \theta_1, \theta_2, \phi_1, \phi_2)$ has been assigned. Evidently then $Q = \cup\, Q(\Delta, \varepsilon, \theta_1, \theta_2, \phi_1, \phi_2)$ where the union is taken over all admissible sextuples. As the set of indices over which the union is taken is an enumerable set, and as Q is of second Baire category, there is at least one index $(\Delta^*, \varepsilon^*, \theta_1^*, \theta_2^*, \phi_1^*, \phi_2^*)$ and a disc Δ_0 such that $Q^* = Q(\Delta^*, \varepsilon^*, \theta_1^*, \theta_2^*, \phi_1^*, \phi_2^*)$ is dense in Δ_0. It is apparent that $\Delta_0 \subset \Delta^*$. Once again, let $r(\theta)$ denote the ray whose initial point is at the center of Δ^* and whose direction is θ, and let

a. $\xi_1 = r(\theta_1^*) \cap \mathrm{Bd}(\Delta^*)$,

b. $\xi_2 = r(\theta_2^*) \cap \mathrm{Bd}(\Delta^*)$,

c. $\xi_1' = r(\phi_1^*) \cap \mathrm{Bd}(\Delta^*)$,

d. $\xi_2' = r(\phi_2^*) \cap \mathrm{Bd}(\Delta^*)$.

See Figure 6.

The disc Δ^* can now be classified according to the positions of the points ξ_1 and ξ_2 relative to the points ξ_1' and ξ_2'. In particular, we say Δ^* is of *type* 1 if the point pair $\{\xi_1, \xi_2\}$ does not separate the pair $\{\xi_1', \xi_2'\}$ on the boundary of Δ^* or if either $\xi_1 = \xi_2$ or $\xi_1' = \xi_2'$. We refer to Δ^* as of *type* 2 if the pair $\{\xi_1, \xi_2\}$ does separate the pair $\{\xi_1', \xi_2'\}$ on the boundary of Δ^*. Consequently, we have two cases to consider depending on the type of Δ^*. Before entering into a discussion of these particular cases, however, we prove two results. The first

https://doi.org/10.1017/S1446788700023971 Published online by Cambridge University Press

deals with the arcs $\alpha_1(\zeta)$ and $\alpha_2(\zeta)$, the second with $\beta_1(\zeta)$ and $\beta_2(\zeta)$, for a point $\zeta \in Q^* \cap \Delta_0$:

1. If $t^* = \sup\{t: \alpha_1(\zeta;t) \in \alpha_2(\zeta)\}$, then $t^* \neq 1$.
2. If $t^{**} = \sup\{t: \beta_1(\zeta;t) \in \beta_2(\zeta)\}$, then $t^{**} \neq 1$.

As the proof of 2. is analogous to the proof of 1., we prove only 1. Suppose that $\sup\{t: \alpha_1(\zeta;t) \in \alpha_2(\zeta)\} = 1$. There exists a t_1 such that $0 < t_1 < 1$ and $\alpha_1(\zeta;t) \in \Delta_0$ for $t_1 < t < 1$. Let

$$G = \{t: t_1 < t < 1 \text{ and } \alpha_1(\zeta;t) \notin \alpha_2(\zeta)\} \text{ and}$$

$$F = \{t: t_1 < t < 1 \text{ and } \alpha_1(\zeta;t) \in \alpha_2(\zeta)\}.$$

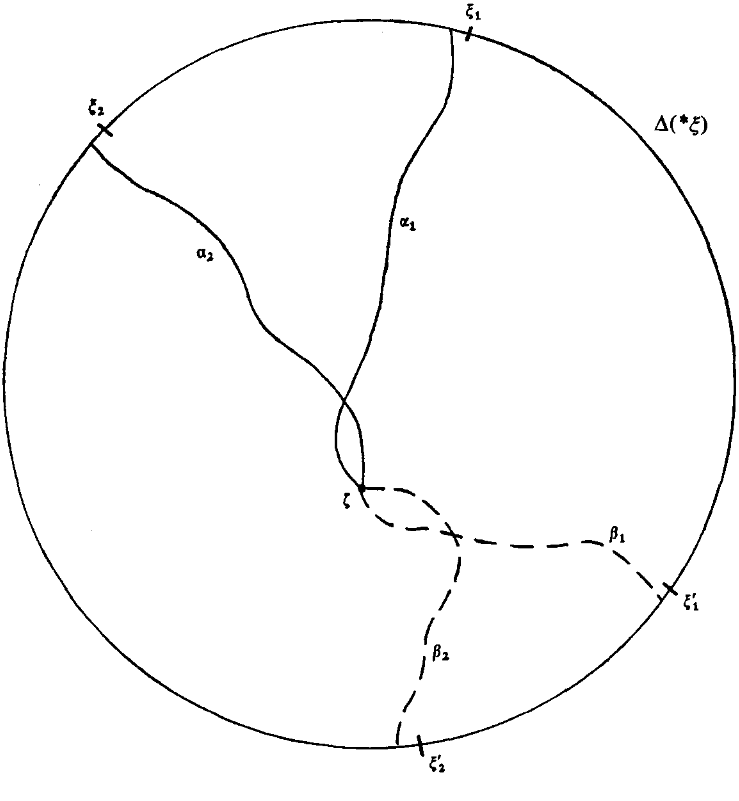

Figure 6.

As $\alpha_1'(\zeta) \in A_1$ and $\alpha_2'(\zeta) \in A_2$ and both $\alpha_1(\zeta)$ and $\alpha_2(\zeta)$ are arcs at ζ, it follows that $\alpha_1(\zeta) \cap \alpha_2(\zeta)$ contains no arc at ζ. We conclude that there exist two numbers t_2 and t_3 in F such that $t_1 < t_2 < t_3 < 1$ and $\{t: t_2 < t < t_3\} \subset G$. That is, $\alpha_1(\zeta;t_2) \in \alpha_2(\zeta)$ and $\alpha_1(\zeta;t_3) \in \alpha_2(\zeta)$ but $\alpha_1(\zeta;t) \notin \alpha_2(\zeta)$ for $t_2 < t < t_3$. As $\alpha_1(\zeta;t_2) \in \alpha_2(\zeta)$, there is a t_2' such that $\alpha_1(\zeta;t_2) = \alpha_2(\zeta;t_2')$; and as $\alpha_1(\zeta;t_3) \in \alpha_2(\zeta)$, there is a t_3' such that $\alpha_1(\zeta; t_3) = \alpha_2(\zeta; t_3')$. Let R denote the region bounded by the arcs $\alpha_1(\zeta)/[t_2, t_3]$ and $\alpha_2(\zeta)/[t_2', t_3']$. (We have made the tacit assumption that $t_2' < t_3'$, which may not be true. If $t_3' < t_2'$, an interchange of these two numbers in the definition of R is needed for notational correctness.) As $R \cap \Delta_0 \neq \varnothing$, there exists a $\zeta^* \in R \cap \Delta_0 \cap Q^*$. But $\beta_1(\zeta^*; 0)$ is on the boundary of Δ^*, and hence exterior to R. Further,

$$\operatorname*{limit}_{t \to 1} \beta_1(\zeta^*; t) = \zeta^* \in R.$$

It follows that $\beta_1(\zeta^*)$ meets the boundary of R. This, however, is impossible, as the boundary of R consists of subarcs of arcs in A, and A and B are pointwise disjoint. The proof of 2. is similar.

We now proceed to discuss the two cases mentioned previously.

CASE 1. Suppose that Δ^* is of type 1. See Figure 7. We show that if $\zeta \in \Delta_0 \cap Q^*$, then ζ lies on an arc contained in A, and also on an arc contained in B, thus contradicting the hypothesis that A and B are pointwise disjoint collections.

Let $\zeta \in \Delta_0 \cap Q^*$. In order to show ζ lies on an arc in B, we consider two subcases which depend on whether or not $\alpha_1(\zeta) \cap \alpha_2(\zeta) = \varnothing$.

a. $\alpha_1(\zeta) \cap \alpha_2(\zeta) \neq \varnothing$.

Let $t^* = \sup\{t : \alpha_1(\zeta; t) \in \alpha_2(\zeta)\}$; then as was shown earlier, $t^* < 1$. Let s^* be such that $\alpha_2(\zeta; s^*) = \alpha_1(\zeta; t^*)$, and denote by R_1 the region bounded by the arcs $\alpha_1(\zeta)/[t^*, 1)$ and $\alpha_2(\zeta)/[s^*, 1)$, and by the point ζ. As $\zeta \in \Delta_0$, we conclude that $R_1 \cap \Delta_0 \neq \varnothing$, and hence there is a $\zeta^* \in Q^* \cap R_1 \cap \Delta_0$. But $\beta_1(\zeta^*; 0)$ is on $\mathrm{Bd}(\Delta^*)$, while $\operatorname{limit}_{t \to 1} \beta_1(\zeta^*; t) = \zeta^*$, and consequently $\beta_1(\zeta^*)$ must intersect the boundary of R_1. As the sets of arcs A and B are pointwise disjoint, $\beta_1(\zeta^*) \cap \alpha_1(\zeta)/[t^*, 1) = \varnothing$ and $\beta_1(\zeta^*) \cap \alpha_2(\zeta)/[s^*, 1) = \varnothing$. It follows then that $\zeta \in \beta_1(\zeta^*)$, and hence $\zeta \in \beta_1'(\zeta^*)$.

b. $\alpha_1(\zeta) \cap \alpha_2(\zeta) = \varnothing$.

As Δ^* is of type 1, there is a path Γ on the boundary of Δ^* such that $\Gamma(0) = \xi_1$, $\Gamma(1) = \xi_2$, and neither ξ_1' nor ξ_2' is on Γ. Further, ξ_1 is within $(1/4)\varepsilon^*$ of $\alpha_1(\zeta; 0)$, ξ_2 is within $(1/4)\varepsilon^*$ of $\alpha_2(\zeta; 0)$, ξ_1' is within $(1/4)\varepsilon^*$ of $\beta_1(\zeta; 0)$, and ξ_2' is within $(1/4)\varepsilon^*$ of $\beta_2(\zeta; 0)$. Consequently, there is a path Γ^* on the boundary of Δ^* satisfying

i. $\Gamma^*(0) = \alpha_1(\zeta; 0)$,
ii. $\Gamma^*(1) = \alpha_2(\zeta; 0)$,
iii. $\beta_1(\zeta; 0) \notin \Gamma^*$ and $\beta_2(\zeta; 0) \notin \Gamma^*$.

Denote by R_2 the region bounded by the path Γ^*, the arcs $\alpha_1(\zeta)$ and $\alpha_2(\zeta)$, and the point ζ. Again $R_2 \cap \Delta_0 \neq \varnothing$, and we let $\zeta^* \in R_2 \cap \Delta_0 \cap Q^*$. Then ζ^* is an interior point of R_2 while $\beta_1(\zeta^*; 0)$ lies exterior to R_2, and hence $\beta_1(\zeta^*)$ must

Δ^* of Type 1

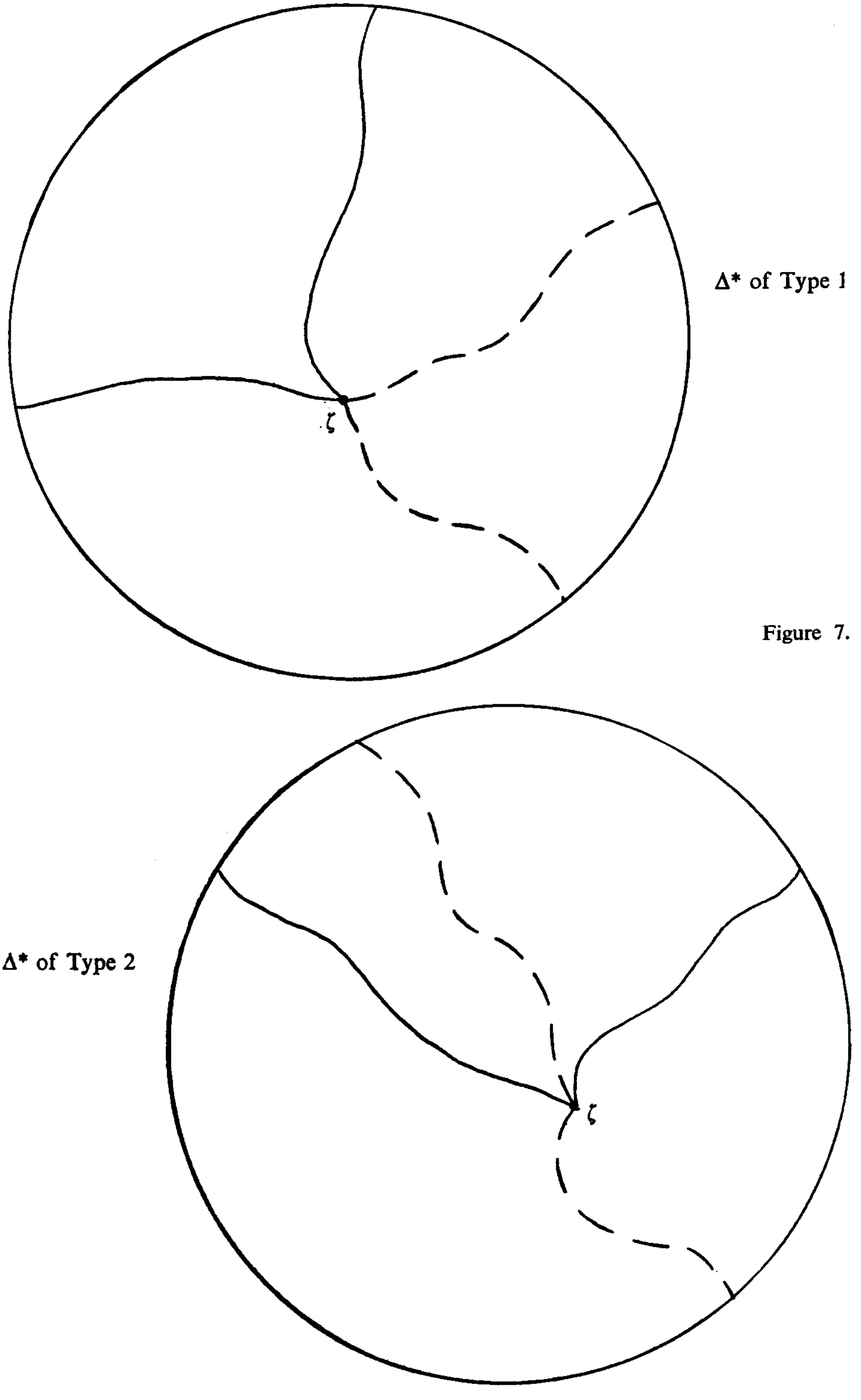

Figure 7.

Δ^* of Type 2

https://doi.org/10.1017/S1446788700023971 Published online by Cambridge University Press

meet the boundary of R_2. But $\beta_1(\zeta^*;0)$ is the only point of $\beta_1(\zeta^*)$ lying on the boundary of Δ^*, and $\beta_1(\zeta^*;0) \notin \Gamma^*$; hence $\beta_1(\zeta^*) \cap \Gamma^* = \emptyset$. Further, $\beta_1(\zeta^*) \cap \alpha_1(\zeta) = \emptyset$ and $\beta_1(\zeta^*) \cap \alpha_2(\zeta) = \emptyset$, as the sets A and B are pointwise disjoint. We again conclude that $\zeta \in \beta_1(\zeta^*)$, and thus that $\zeta \in \beta_1'(\zeta^*)$.

In either subcase, then, we find that ζ lies on an arc which is contained in the set B. In an analogous manner it can be shown that ζ also lies on an arc contained in A, and hence we reach the contradiction that the sets A and B are not pointwise disjoint. It must be, then, that Δ^* is of type 2.

CASE 2. Suppose Δ^* is of type 2. See Figure 7. We again show that if $\zeta \in \Delta_0 \cap Q^*$, then ζ lies both on an arc contained in B and on an arc contained in A, thus contradicting the hypothesis that A and B are pointwise disjoint. As before, we only show that ζ is on an arc contained in B, as this proof is wholly analogous to showing that ζ lies on an arc in A. We have the same two subcases to consider. Let $\zeta \in \Delta_0 \cap Q^*$.

a. $\alpha_1(\zeta) \cap \alpha_2(\zeta) \neq \emptyset$.

This subcase cannot occur, as the arcs $\alpha_1(\zeta)$ and $\alpha_2(\zeta)$ are separated by the union of the arcs $\beta_1(\zeta)$, $\beta_2(\zeta)$, and the point ζ.

b. $\alpha_1(\zeta) \cap \alpha_2(\zeta) = \emptyset$.

As Δ^* is of type 2, there exists an arc Γ on the boundary of Δ^* having the properties that $\Gamma(0) = \xi_1$, $\Gamma(1) = \xi_2$, and exactly one of ξ_1' or ξ_2' resides on Γ. Now $\alpha_1(\zeta;0)$, $\alpha_2(\zeta;0)$, $\beta_1(\zeta;0)$, and $\beta_2(\zeta;0)$ were chosen sufficiently close to (within $\varepsilon^*/4$ of) ξ_1, ξ_2, ξ_1', and ξ_2', respectively, for there to exist an arc Γ^* on the boundary of Δ^* with the following properties:

i. $\Gamma^*(0) = \alpha_1(\zeta;0)$,
ii. $\Gamma^*(1) = \alpha_2(\zeta;0)$,
iii. Γ^* contains exactly one of $\beta_1(\zeta;0)$ or $\beta_2(\zeta;0)$.

Suppose, for the sake of definiteness, that $\beta_1(\zeta;0) \in \Gamma^*$, and let R_2 denote the region bounded by Γ^*, the arcs $\alpha_1(\zeta)$ and $\alpha_2(\zeta)$, and the point ζ. Once again we find that $R_2 \cap \Delta_0 \neq \emptyset$, and consequently there is a $\zeta^* \in R_2 \cap \Delta_0 \cap Q^*$. But $\beta_2(\zeta^*;0)$ is within $\varepsilon^*/4$ of ξ_2', and hence is not on Γ^*. It follows then, that $\beta_2(\zeta^*;0)$ lies exterior to R_2. However, ζ^* is an interior point of R_2, and hence $\beta_2(\zeta^*)$ must intersect the boundary of R_2. As A and B are pointwise disjoint collections, we infer that both $\alpha_1(\zeta)$ and $\alpha_2(\zeta)$ miss $\beta_2(\zeta^*)$. Also, $\beta_2(\zeta^*;0) \notin \Gamma^*$, and $\beta_2(\zeta^*;0)$ is the only point $\beta_2(\zeta^*)$ has in common with the boundary of Δ^*. It follows that $\beta_2(\zeta^*) \cap \Gamma^* = \emptyset$ and consequently $\zeta \in \beta_2(\zeta^*) \subset \beta_2'(\zeta^*)$.

A similar argument shows that ζ is also an element of an arc contained in A. This contradicts the fact that A and B are pointwise disjoint collections of arcs, and the supposition in case 2 has also proved untenable. Hence, our original assumption that Q is of second Baire category is false, and the theorem follows.

If α is an arc at a point $\zeta \in P$, α is said to be *terminally nonrectifiable* if

https://doi.org/10.1017/S1446788700023971 Published online by Cambridge University Press

$\alpha/[t, 1)$ is nonrectifiable for $0 \leqq t < 1$. If S is a planar set and $\zeta \in P$, ζ is termed a *terminally nonrectifiably ambiguous point* relative to S if the arcs of ambiguity may be chosen to be terminally nonrectifiable.

COROLLARY 15. *Let S be a planar set. Then the set of points which are both rectifiably ambiguous points relative to S and terminally nonrectifiably ambiguous points relative to S is a set of first Baire category.*

PROOF. Let A_1 denote the set of rectifiable arcs lying in S and let A_2 denote the set of terminally nonrectifiable arcs in S. Denote by B_1 the set of rectifiable arcs in $P-S$ and by B_2 the set of terminally nonrectifiable arcs in $P-S$. Let $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$. Then A_1, A_2, B_1, B_2, A, and B satisfy the hypothesis of the previous theorem, and the result follows.

As every totally nonrectifiable arc is terminally nonrectifiable, we also obtain the following corollary.

COROLLARY 16. *Let S be a planar set. Then the set of points of P which are both rectifiably ambiguous relative to S and totally nonrectifiably ambiguous relative to S is a set of first Baire category.*

References

F. Bagemihl (1966), 'Ambiguous points of arbitrary planar sets and functions', *Zeitschr. f. math. Logik und Grundlagen d. Math.* **12**, 205–217.

C. Carathéodory (1948), *Vorlesungen über reelle Funktionen.* (New York, 1948).

University of Wisconsin-Milwaukee
Western Illinois University
U.S.A.