Forum of Mathematics, Sigma (2015), Vol. 3, e10, 52 pages
doi:10.1017/fms.2015.7

THE MOONSHINE MODULE FOR CONWAY'S GROUP

JOHN F. R. DUNCAN and SANDER MACK-CRANE
Department of Mathematics, Applied Mathematics and Statistics,
Case Western Reserve University, Cleveland, OH 44106, USA;
email: john.duncan@case.edu, mack-crane@case.edu

Received 26 September 2014; accepted 30 April 2015

Abstract

We exhibit an action of Conway's group – the automorphism group of the Leech lattice – on a distinguished super vertex operator algebra, and we prove that the associated graded trace functions are normalized principal moduli, all having vanishing constant terms in their Fourier expansion. Thus we construct the natural analogue of the Frenkel–Lepowsky–Meurman moonshine module for Conway's group. The super vertex operator algebra we consider admits a natural characterization, in direct analogy with that conjectured to hold for the moonshine module vertex operator algebra. It also admits a unique canonically twisted module, and the action of the Conway group naturally extends. We prove a special case of generalized moonshine for the Conway group, by showing that the graded trace functions arising from its action on the canonically twisted module are constant in the case of Leech lattice automorphisms with fixed points, and are principal moduli for genus-zero groups otherwise.

2010 Mathematics Subject Classification: 11F11, 11F22, 17B69, 20C34

1. Introduction

Taking the upper half-plane $\mathbb{H} := \{\tau \in \mathbb{C} \mid \Im(\tau) > 0\}$, together with the Poincaré metric $ds^2 = y^{-2}(dx^2 + dy^2)$, we obtain the Poincaré half-plane model of the hyperbolic plane. The group of orientation-preserving isometries is the quotient of $\mathrm{SL}_2(\mathbb{R})$ by $\{\pm I\}$, where the action is by Möbius transformations,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \tau := \frac{a\tau + b}{c\tau + d}. \tag{1.1}$$

To any $\tau \in \mathbb{H}$ we may associate a complex elliptic curve $E_\tau := \mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z})$, and from this point of view the modular group $\mathrm{SL}_2(\mathbb{Z})$ is distinguished as the subgroup of $\mathrm{SL}_2(\mathbb{R})$ whose orbits encode the isomorphism types of the curves E_τ. That is, E_τ and $E_{\tau'}$ are isomorphic if and only if $\tau' = \gamma \cdot \tau$ for some $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.

Subgroups of $\mathrm{SL}_2(\mathbb{R})$ that are commensurable with the modular group admit similar interpretations. For example, the orbits of the Hecke congruence group of level N,

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid c = 0 \pmod{N} \right\}, \tag{1.2}$$

correspond to isomorphism types of pairs (E_τ, C), where C is a cyclic subgroup of E_τ of order N (see, for example, [73]).

1.1. Monstrous moonshine. It is a remarkable fact, aspects of which remain mysterious, that certain discrete subgroups of $\mathrm{SL}_2(\mathbb{R})$ in the same commensurability class as the modular group – which is to say, fairly uncomplicated groups – encode detailed knowledge of the representation theory of the largest sporadic simple group, the monster, $\mathbb{M}$. Given that the monster has

$$808017424794512875886459904961710757005754368000000000 \tag{1.3}$$

elements, no nontrivial permutation representations with degree less than

$$97239461142009186000 \tag{1.4}$$

and no nontrivial linear representations with dimension less than 196883 (see [61, 63] or [21]), this is surprising. (Here, representation means ordinary representation, but even over a field of positive characteristic, the minimal dimension of a nontrivial representation is 196882 (see [67, 82]).)

The explanation of this fact relies upon the existence of a graded, infinite-dimensional representation $V^\natural = \bigoplus_{n\geqslant 0} V^\natural_n$ of $\mathbb{M}$, such that, if we define the *McKay–Thompson series*

$$T_m(\tau) := q^{-1} \sum_{n\geqslant 0} \mathrm{tr}_{V^\natural_n} m\, q^n \tag{1.5}$$

for $m \in \mathbb{M}$, where $q := e^{2\pi i\tau}$ for $\tau \in \mathbb{H}$, then the functions T_m are characterized in the following way.

> For each $m \in \mathbb{M}$ there is a discrete group $\Gamma_m < \mathrm{SL}_2(\mathbb{R})$, commensurable with the modular group and having width one at the infinite cusp, such that T_m is the unique Γ_m-invariant holomorphic function on $\mathbb{H}$ satisfying $T_m(\tau) = q^{-1} + O(q)$ as $\Im(\tau) \to \infty$, and remaining bounded as τ approaches any noninfinite cusp of Γ_m.

https://doi.org/10.1017/fms.2015.7 Published online by Cambridge University Press

(We say that a discrete group $\Gamma < \mathrm{SL}_2(\mathbb{R})$ has *width one at the infinite cusp* if the subgroup of upper-triangular matrices in Γ is generated by $\pm\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$. If $\Gamma < \mathrm{SL}_2(\mathbb{R})$ is commensurable with $\mathrm{SL}_2(\mathbb{Z})$ then it acts naturally on $\widehat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$. We define the *cusps* of Γ to be the orbits of Γ on $\widehat{\mathbb{Q}}$. We say that a cusp $\alpha \in \Gamma\backslash\widehat{\mathbb{Q}}$ is *noninfinite* if it does not contain ∞.)

The group Γ_m is $\mathrm{SL}_2(\mathbb{Z})$ in the case that $m = e$ is the identity element. The corresponding McKay–Thompson series T_e, which is the graded dimension of $V^\natural$ by definition, must therefore be the *normalized elliptic modular invariant*,

$$J(\tau) := \frac{\left(1 + 240\sum_{n>0}\sum_{d|n} d^3q^n\right)^3}{q\prod_{n>0}(1-q^n)^{24}} - 744, \tag{1.6}$$

since this is the unique $\mathrm{SL}_2(\mathbb{Z})$-invariant holomorphic function on $\mathbb{H}$ satisfying $J(\tau) = q^{-1} + O(q)$ as $\Im(\tau) \to \infty$.

$$T_e(\tau) = q^{-1}\sum_{n\geqslant 0} \dim V_n^\natural q^n = J(\tau) = q^{-1} + 196884q + 21493760q^2 + \cdots. \tag{1.7}$$

Various groups $\Gamma_0(N)$ occur as Γ_m for elements $m \in \mathbb{M}$. For example, there are two conjugacy classes of involutions in $\mathbb{M}$. If m belongs to the larger of these conjugacy classes, denoted $2B$ in [21], then $\Gamma_{2B} := \Gamma_m = \Gamma_0(2)$, and

$$T_{2B}(\tau) = q^{-1}\prod_{n>0}(1-q^{2n-1})^{24} + 24 = q^{-1} + 276q - 2048q^2 + \cdots. \tag{1.8}$$

The existence of the representation $V^\natural$ was conjectured by Thompson [99] following McKay's observation that $196884 = 1 + 196883$, where the significance of 196884 is clear from (1.7), and the significance of 1 and 196883 is that they are dimensions of irreducible representations of the monster. It is worth noting that, at the time of McKay's observation, the monster group had not yet been proven to exist. So the monster, and therefore also its representation theory, was conjectural. The existence of an irreducible representation with dimension 196883 was conjectured independently by Griess [61] and Conway and Norton [19], and the existence of the monster was ultimately proven by Griess [63], via an explicit tour de force construction of a monster-invariant (commutative but nonassociative) algebra structure on the unique nontrivial 196884-dimensional representation.

Thompson's conjecture was first confirmed, in an indirect fashion, by Atkin, Fong, and Smith [97], but was subsequently established in a strong sense by Frenkel, Lepowsky, and Meurman [49–51], who furnished a concrete construction of the *moonshine module* $V^\natural$ (see (1.5)), together with rich monster-invariant algebraic structure, constituting an infinite-dimensional extension of (a slight modification of) Griess' 196884-dimensional algebra. More particularly,

https://doi.org/10.1017/fms.2015.7 Published online by Cambridge University Press

Frenkel, Lepowsky, and Meurman equipped $V^\natural$ with vertex operators, which had appeared originally in the dual-resonance theory of mathematical physics (see [83, 95] for reviews), and had subsequently found application (see [47, 79]) in the representation theory of affine Lie algebras.

Borcherds generalized the known constructions of vertex operators, and also derived rules for composing them in [2]. Using these rules he was able to demonstrate that $V^\natural$, together with examples arising from certain infinite-dimensional Lie algebras, admits a kind of commutative associative algebra structure – namely, *vertex algebra* structure – which we review in Section 2 (see [52, 71, 78] for more thorough introductions). Frenkel, Lepowsky, and Meurman used the fact that $V^\natural$ supports a representation of the Virasoro algebra to formulate the notion of *vertex operator algebra* in [51], and showed that $\mathbb{M}$ is precisely the group of automorphisms of the vertex algebra structure on $V^\natural$ that commute with the Virasoro action. Vertex operator algebras were subsequently recognized to be 'chiral halves' of two-dimensional conformal field theories (see [53, 54]), and the construction of $V^\natural$ by Frenkel, Lepowsky, and Meurman counts as one of the earliest examples of an orbifold conformal field theory (see [23–25]).

The characterization of the McKay–Thompson series T_m quoted above is the main content of the *monstrous moonshine conjectures*, formulated by Conway and Norton in [19] (see also [98]), and solved by Borcherds in [3]. It is often referred to as the *genus-zero property* of monstrous moonshine, because the existence of a function T_m satisfying the given conditions implies that Γ_m has *genus zero*, in the sense that the orbit space $\Gamma_m\backslash\widehat{\mathbb{H}}$ is isomorphic as a Riemann surface to the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C}\cup\{\infty\}$, where $\widehat{\mathbb{H}} := \mathbb{H}\cup\mathbb{Q}\cup\{\infty\}$ (see, for example, [96] for the Riemann surface structure on $\Gamma\backslash\widehat{\mathbb{H}}$). Indeed, the function T_m witnesses this, as it induces an embedding $\Gamma_m\backslash\mathbb{H} \to \mathbb{C}$ which extends uniquely to an isomorphism $\Gamma_m\backslash\widehat{\mathbb{H}} \to \widehat{\mathbb{C}}$.

Conversely, if $\Gamma < \mathrm{SL}_2(\mathbb{R})$ has genus zero in the above sense, then there is an isomorphism of Riemann surfaces $\Gamma\backslash\widehat{\mathbb{H}} \to \widehat{\mathbb{C}}$, and the composition $\mathbb{H} \to \widehat{\mathbb{H}} \to \Gamma\backslash\widehat{\mathbb{H}} \to \widehat{\mathbb{C}}$ maps $\mathbb{H}$ to $\mathbb{C}$, thereby defining a Γ-invariant holomorphic function T_Γ on $\mathbb{H}$. We call a Γ-invariant holomorphic function $T_\Gamma : \mathbb{H} \to \mathbb{C}$ a *principal modulus* for Γ if it arises in this way from an isomorphism $\Gamma\backslash\widehat{\mathbb{H}} \to \widehat{\mathbb{C}}$. If Γ has width one at the infinite cusp, then, after applying an automorphism of $\widehat{\mathbb{C}}$ if necessary, we have

$$T_\Gamma(\tau) = q^{-1} + O(q) \tag{1.9}$$

as $\Im(\tau) \to \infty$, and no poles at any noninfinite cusps of Γ. Such a function T_Γ – we call it a *normalized principal modulus* for Γ – is unique because the difference between any two defines a holomorphic function on the compact Riemann surface $\Gamma\backslash\widehat{\mathbb{H}}$ which vanishes at the infinite cusp by force of (1.9). The only holomorphic

https://doi.org/10.1017/fms.2015.7 Published online by Cambridge University Press

functions on a compact Riemann surface are constants (see, for example, [64]), and hence this difference vanishes identically.

So knowledge of the McKay–Thompson series T_m is equivalent to knowledge of the discrete groups $\Gamma_m < \mathrm{SL}_2(\mathbb{R})$, according to the characterization furnished by monstrous moonshine. This explains the claim we made above, that subgroups of $\mathrm{SL}_2(\mathbb{R})$ 'know' about the representation theory of the monster. For, according to definition (1.5) of T_m, we can compute the graded trace of a monster element $m \in \mathbb{M}$ on the moonshine module $V^\natural$ as soon as we know the group Γ_m. In particular, we can compute the traces of monster elements on its infinite-dimensional representation $V^\natural$ without doing any computations in the monster itself.

As has been mentioned above, a concrete realization of the McKay–Thompson series T_m is furnished by the Frenkel–Lepowsky–Meurman construction of the moonshine module $V^\natural$. Their method was inspired in part by the original construction of the monster due to Griess [62, 63], and takes Leech's lattice [76, 77] as a starting point. Conway has proven [17] that the Leech lattice Λ is the unique up to isomorphism even positive-definite lattice such that

- the rank of Λ is 24;
- Λ is self-dual; and
- $\langle\lambda, \lambda\rangle \neq 2$ for any $\lambda \in \Lambda$.

Conway also studied the automorphism group of Λ and discovered three new sporadic simple groups in the process [16, 18]. We set $Co_0 := \mathrm{Aut}(\Lambda)$ and call it Conway's group. The largest of Conway's sporadic simple groups is the quotient

$$Co_1 := \mathrm{Aut}(\Lambda)/\{\pm \mathrm{Id}\} \tag{1.10}$$

of Co_0 by its centre.

1.2. Conway moonshine. In their paper [19], Conway and Norton also described an assignment of genus-zero groups $\Gamma_g < \mathrm{SL}_2(\mathbb{R})$ to elements g of the Conway group, Co_0. Their prescription is very concrete and may be described as follows. If $g \in Co_0 = \mathrm{Aut}(\Lambda)$ acts on $\Lambda \otimes_{\mathbb{Z}} \mathbb{C}$ with eigenvalues $\{\varepsilon_i\}_{i=1}^{24}$, then Γ_g is the invariance group of the holomorphic function

$$t_g(\tau) := q^{-1} \prod_{n>0} \prod_{i=1}^{24} (1 - \varepsilon_i q^{2n-1}) \tag{1.11}$$

on $\mathbb{H}$. Observe that for $g = e$ the identity element of Co_0, the function t_e almost coincides with the monstrous McKay–Thompson series T_{2B} of (1.8): the latter has

https://doi.org/10.1017/fms.2015.7 Published online by Cambridge University Press

vanishing constant term, whereas $t_e(\tau) = q^{-1} - 24 + O(q)$. That the invariance groups of the t_g actually are genus-zero subgroups of $SL_2(\mathbb{R})$, and that the t_g are principal moduli, was verified in part in [19], and in full in [91]. (One may see [74] or Table A.1 in Appendix A of this paper for an explicit description of all the groups Γ_g for $g \in Co_0$.)

So discrete subgroups of isometries of the hyperbolic plane also know a lot about the representation theory of Conway's group Co_0, but to fully justify this statement we should construct the appropriate analogue of $V^\natural$, that is, a graded infinite-dimensional Co_0-module whose graded trace functions recover the t_g.

Note at this point that the functions t_g are not distinguished to quite the extent that the monstrous McKay–Thompson series T_m are, for $m \in \mathbb{M}$, because for $g \in Co_0$ we have

$$t_g(\tau) = q^{-1} - \chi_g + O(q), \tag{1.12}$$

where $\chi_g = \sum_{i=1}^{24} \varepsilon_i$ is the trace of g attached to its action on $\Lambda \otimes_{\mathbb{Z}} \mathbb{C}$. In particular, t_g generally has a nonvanishing constant term, and does not satisfy criterion (1.9) defining normalized principal moduli. So for $g \in Co_0$ let us define

$$T_g^s(\tau) := t_g(\tau/2) + \chi_g = q^{-1/2} \prod_{n>0} \prod_{i=1}^{24} (1 - \varepsilon_i q^{n-1/2}) + \chi_g, \tag{1.13}$$

so that $T_g^s(2\tau)$ is the unique normalized principal modulus attached to the genus-zero group Γ_g.

In this article, building upon previous work [4, 36, 50], we construct a $\frac{1}{2}\mathbb{Z}$-graded infinite-dimensional Co_0-module, $V^{s\natural} = \bigoplus_{n \geqslant 0} V^{s\natural}_{n/2}$, such that the normalized principal moduli T_g^s, for $g \in Co_0$, are obtained via an analogue of (1.5). That is, we construct the natural analogue of the moonshine module $V^\natural$ for the Conway group Co_0.

In fact we will do better than this, by establishing a characterization of the algebraic structure underlying $V^{s\natural}$. In [51] it is conjectured that $V^\natural$ is the unique vertex operator algebra such that

- the central charge of $V^\natural$ is 24;
- $V^\natural$ is self-dual; and
- $\deg(v) \neq 1$ for any nonzero $v \in V^\natural$.

Just as vertex operator algebra structure is a crucial feature of the moonshine module $V^\natural$, super vertex operator algebra furnishes the correct framework for understanding $V^{s\natural}$ (and also motivates the rescaling of τ in (1.13)). In Section 4.2 we prove the following result.

https://doi.org/10.1017/fms.2015.7 Published online by Cambridge University Press

THEOREM **4.5**. *There is a unique up to isomorphism C_2-cofinite rational super vertex operator algebra of CFT type $V^{s\natural}$ such that*

- *the central charge of $V^{s\natural}$ is* 12*;*
- *$V^{s\natural}$ is self-dual; and*
- $\deg(v) \neq \frac{1}{2}$ *for any nonzero $v \in V^{s\natural}$.*

(We refer to Section 2.1 for explanations of the technical terms appearing in the statement of Theorem 4.5. We say that a super vertex operator algebra is self-dual if it is rational, irreducible as a module for itself, and if it is its only irreducible module up to isomorphism. We write $\deg(v) = n$ in the case when $L(0)v = nv$.)

As explained in detail in [**51**], the conjectural characterization of $V^\natural$ above puts the monster, and $V^\natural$, at the top tier of a three-tier tower

$$\begin{aligned} \mathbb{M} \curvearrowright V^\natural &\text{ — vertex operator algebras} \\ Co_0 \curvearrowright \Lambda &\text{ — even positive-definite lattices} \\ M_{24} \curvearrowright \mathcal{G} &\text{ — doubly even linear binary codes} \end{aligned} \tag{1.14}$$

involving three sporadic groups and three distinguished structures, arising in vertex algebra, lattice theory, and coding theory, respectively.

An important structural feature of this tower is the evident parallel between the conjectural characterization of $V^\natural$ and Conway's characterization of the Leech lattice quoted earlier. In (1.14) we write $\mathcal{G}$ for the *extended binary Golay code*, introduced (essentially) by Golay in [**60**], which is the unique (see, for example, [**20**, **93**]) doubly even linear binary code such that

- the length of $\mathcal{G}$ is 24;
- $\mathcal{G}$ is self-dual; and
- $wt(C) \neq 4$ for any word $C \in \mathcal{G}$.

The automorphism group of $\mathcal{G}$ is the largest sporadic simple group discovered by Mathieu [**85**, **86**], denoted here by M_{24}.

Theorem 4.5 now suggests that $V^{s\natural}$ may serve as a replacement for Λ in the tower (1.14). With the recent development of Mathieu moonshine (see [**40**] for an account of the original observation, and [**12**] for a review), one may speculate about the existence of a vertex algebraic replacement for $\mathcal{G}$, and an alternative tower to (1.14), with tiers corresponding to moonshine in three forms.

$$\begin{aligned} \mathbb{M} \curvearrowright V^\natural &\text{ — monstrous moonshine} \\ Co_0 \curvearrowright V^{s\natural} &\text{ — Conway moonshine} \\ M_{24} \curvearrowright \text{??} &\text{ — Mathieu moonshine.} \end{aligned} \tag{1.15}$$

https://doi.org/10.1017/fms.2015.7 Published online by Cambridge University Press

Note that a natural analogue of the genus-zero property of monstrous moonshine, and the moonshine for Conway's group considered here, has been obtained for Mathieu moonshine in [13].

We give an explicit construction for $V^{s\natural}$ in Section 4.1. The proof of the characterization result, Theorem 4.5, demonstrates that the even part of $V^{s\natural}$ is isomorphic to the lattice vertex algebra of type D_{12}, which, according to the boson–fermion correspondence (see [32, 45]), is the even part of the Clifford module super vertex operator algebra $A(\mathfrak{a})$ attached to a 24-dimensional orthogonal space $\mathfrak{a}$. From this point of view the connection to the Conway group is reasonably transparent, for we may identify $\mathfrak{a}$ with the space $\Lambda\otimes_{\mathbb{Z}}\mathbb{C}$ enveloping the Leech lattice. We realize $V^{s\natural}$ by performing a $\mathbb{Z}/2$-orbifold of the Clifford module super vertex operator algebra $A(\mathfrak{a})$. Our method also produces an explicit construction of the unique (up to isomorphism) canonically twisted $V^{s\natural}$-module, which we denote $V^{s\natural}_{\rm tw}$, and which is also naturally a Co_0-module. (Canonically twisted modules for super vertex algebras are reviewed in Section 2.1, and the Clifford module super vertex operator algebra construction is reviewed in Section 2.3.)

Our main results appear in Section 4.3, where we consider the graded super trace functions

$$T^s_g(\tau) := q^{-1/2}\sum_{n\geqslant 0} \operatorname{str}_{V^{s\natural}_{n/2}} g\, q^{n/2}, \tag{1.16}$$

$$T^s_{g,\rm tw}(\tau) := q^{-1/2}\sum_{n\geqslant 0} \operatorname{str}_{V^{s\natural}_{{\rm tw},n+1/2}} g\, q^{n+1/2}, \tag{1.17}$$

arising naturally from the actions of Co_0 on $V^{s\natural}$ and $V^{s\natural}_{\rm tw}$. (Necessary facts about the Conway group are reviewed in Section 3.2.)

THEOREM **4.9**. *Let $g\in Co_0$. Then T^s_g is the normalized principal modulus for a genus-zero subgroup of* $\mathrm{SL}_2(\mathbb{R})$.

The statement that the T^s_g are normalized principal moduli is a direct analogue, for the Conway group, of the monstrous moonshine conjectures, formulated by Conway and Norton in [19]. The moonshine conjectures were broadly expanded by Norton in [84, 88], to an association of functions $T_{(m,m')}(\tau)$ to pairs (m,m') of commuting elements in the monster. Norton's *generalized moonshine conjectures* state, among other things (see [89] for a revised formulation), that $T_{(m,m')}$ should be a principal modulus for a genus-zero group $\Gamma_{(m,m')}$, or a constant function, for every commuting pair $m,m'\in\mathbb{M}$. In terms of vertex operator algebra theory, the functions $T_{(m,m')}$ should be defined by traces on twisted modules for $V^\natural$ (see [30]).

https://doi.org/10.1017/fms.2015.7 Published online by Cambridge University Press

The argument used to prove Theorem 4.9 also establishes the following result, which we may regard as confirming a special case of generalized moonshine for the Conway group.

THEOREM **4.10**. *Let $g \in Co_0$. Then $T^s_{g,\mathrm{tw}}$ is constant, with constant value $-\chi_g$, when g has a fixed point in its action on the Leech lattice. If g has no fixed points then $T^s_{g,\mathrm{tw}}$ is a principal modulus for a genus-zero subgroup of* $\mathrm{SL}_2(\mathbb{R})$.

The problem of precisely formulating, and proving, generalized moonshine for the Conway group is an important direction for future work.

Generalized moonshine for the monster remains unproven in general, although a number of special cases have been established, by Dong, Li, and Mason in [**28**, **30**], Ivanov and Tuite in [**69**, **70**], and Höhn in [**65**]. The most general results on generalized moonshine are due to Carnahan [**6**–**9**].

As indicated above, the present paper is closely related to earlier work [**36**], in which a vertex algebraic construction of Conway's sporadic simple group Co_1 (see (1.10)) was obtained. In [**36**], an $N = 1$ super vertex operator algebra $V^{f\natural} = \bigoplus_{n\geqslant 0} V^{f\natural}_{n/2}$ is defined (over the real numbers; we work here over $\mathbb{C}$), and it is proven that the automorphism group of $V^{f\natural}$ – meaning the group of vertex algebra automorphisms which commute with the $N = 1$ structure; see Section 2.1 – is precisely Co_1. In fact, $V^{s\natural}$ is isomorphic to the super vertex operator algebra underlying $V^{f\natural}$ (when defined over $\mathbb{C}$). The essential difference between $V^{s\natural}$ and $V^{f\natural}$ (and between this paper and [**36**]) is that $V^{s\natural}$ admits a faithful action of Co_0, whereby the central element acts as $-\mathrm{Id}$ on any candidate $N = 1$ element.

Both this work and [**36**] rest upon the important antecedents [**4**, **50**]. In Section 15 of [**50**], the construction of the super vertex operator algebra underlying $V^{f\natural}$ is described for the first time, and it is conjectured that the simple Conway group Co_1 should act as automorphisms. Later, in Section 5 of [**4**], a lattice super vertex operator algebra is identified, which turns out to be isomorphic to both $V^{s\natural}$ and the super vertex operator algebra underlying $V^{f\natural}$, and it is explained that both Co_0 and Co_1 can act faithfully on this object. The construction of $V^{s\natural}$ given in Section 4.1 differs significantly from that of $V^{f\natural}$ described in [**50**], but is closely connected, via the boson–fermion correspondence, to the description given in [**4**].

Although the graded trace functions attached to the action of Co_1 on $V^{f\natural}$ are computed explicitly in [**36**], their modular properties are not considered in detail. From the point of view of moonshine, the trace functions arising from $V^{s\natural}$ are better: the functions

$$T^f_g(\tau) := q^{-1/2} \sum_{n\geqslant 0} \mathrm{str}_{V^{f\natural}_{n/2}}\, g\, q^{n/2}, \tag{1.18}$$

https://doi.org/10.1017/fms.2015.7 Published online by Cambridge University Press

defined for $g \in Co_1$, are generally not principal moduli, even though they satisfy the normalization condition (1.9). Nonetheless, the Co_1-module structure on $V^{f\natural}$ plays an important role in the present paper. A crucial step in our proof of Theorems 4.9 and 4.10 is the verification of a Co_0-family of eta-product identities (4.31), which we establish in Lemma 4.8. We are able to prove these in a uniform manner by utilizing the unique (see Proposition 4.4) Co_1-invariant $N = 1$ structure on $V^{f\natural}$.

Another work of particular relevance to the moonshine for Conway's group we consider here is [37], in which the McKay–Thompson series (1.5) of monstrous moonshine are characterized, following earlier work [22], in terms of certain regularized Poincaré series, called Rademacher sums. The theorems in Section 6 of [37] imply that a discrete group $\Gamma < \mathrm{SL}_2(\mathbb{R})$, commensurable with $\mathrm{SL}_2(\mathbb{Z})$ and having width one at the infinite cusp, has genus zero if and only if the associated Rademacher sum

$$R_\Gamma(\tau) := q^{-1} + \lim_{K\to\infty} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma^\times_{<K}} (e^{-2\pi \mathbf{i}\gamma\tau} - e^{-2\pi \mathbf{i}\gamma\infty}) \tag{1.19}$$

is a principal modulus for Γ. (In (1.19) we write Γ_∞ for the subgroup of upper-triangular matrices in Γ, and $\Gamma_\infty\backslash\Gamma^\times_{<K}$ denotes the set of nontrivial cosets for Γ_∞ in Γ such that, if a representative γ is rescaled to a matrix $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$ with integer entries and $c > 0$, then $c < K$ and $-K^2 < d < K^2$. In the case when $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, the right-hand side of (1.19) is exactly the expression given for $J(\tau) + 12$ by Rademacher in [92].)

So, in particular, the results of [37] imply that all the McKay–Thompson series T^s_g of Conway moonshine, attached to the Conway group via its action on $V^{s\natural}$, can be realized as Rademacher sums. (Strictly speaking, the Rademacher sum $R_{\Gamma_g}(\tau)$ generally differs from $T^s_g(2\tau)$ by an additive constant, and a similar statement is true for the functions of monstrous moonshine. The normalized Rademacher sums, defined in Section 4 of [37], have vanishing constant terms, and thus recover the T_m and T^s_g precisely.) Thus we obtain a uniform construction of the T^s_g as the (normalized) Rademacher sums attached to their invariance groups Γ_g.

The formulation of a characterization of the T^s_g, in analogy with that given for the T_m in Theorem 6.5.1 of [37], is another important problem for future work.

1.3. Mathieu moonshine. The significance of the super vertex operator algebra $V^{s\natural}$ is further demonstrated by recent developments in Mathieu moonshine – mentioned above, in connection with the tower (1.15) – which features an assignment of weak Jacobi forms of weight zero and index one, to conjugacy classes in the sporadic simple Mathieu group, M_{24} (the standard reference for the theory of Jacobi forms is [42]).

https://doi.org/10.1017/fms.2015.7 Published online by Cambridge University Press

In forthcoming work [38] we demonstrate how the canonically twisted $V^{s\natural}$-module $V^{s\natural}_{\rm tw}$ may be used to attach weak Jacobi forms of weight zero and index one to conjugacy classes in Co_0 that fix a four-space in $\Lambda \otimes_{\mathbb{Z}} \mathbb{C}$. The group M_{24} is a subgroup of Co_0, and our assignment recovers many (but not all) of the weak Jacobi forms attached to M_{24} by Mathieu moonshine. Mathieu moonshine is identified as a special case of 23 similar moonshine phenomena, collectively known as umbral moonshine, in [14, 15], and our construction generalizes naturally, so as to attach weak Jacobi forms of index greater than one to suitable elements of Co_0. Several of the higher-index Jacobi forms of umbral moonshine arise in this way, from the faithful action of Co_0 on $V^{s\natural}_{\rm tw}$.

The weak Jacobi forms of Mathieu moonshine may be replaced with mock modular forms of weight $1/2$, by utilizing the irreducible unitary are characters of the small $N = 4$ superconformal algebra. (See [12] for a review of this, including an introductory discussion of mock modular forms.) One of the main conjectures of Mathieu moonshine is that these mock modular forms – namely, those prescribed in [10, 39, 55, 56], and defined in a uniform way, via Rademacher sums, in [13] – are the graded traces attached to the action of M_{24} on some graded infinite-dimensional M_{24}-module. Despite the work of Gannon [59], proving the existence of a such an M_{24}-module, an explicit construction of the Mathieu moonshine module is still lacking. The result of [38] stated above, that many of the weak Jacobi forms of Mathieu moonshine may be recovered from an action of M_{24} on $V^{s\natural}_{\rm tw}$, demonstrates that $V^{s\natural}$ may serve as an important tool in the construction of this moonshine module for M_{24}.

Strong evidence in support of the idea that $V^{s\natural}$ can play a role in the construction of modules for Mathieu moonshine, and umbral moonshine more generally, is given in [11], where $V^{s\natural}_{\rm tw}$ is used to attach weak Jacobi forms of weight zero and index two to all of the elements of the sporadic simple Mathieu groups M_{23} and M_{22} (characterized as point stabilizers in M_{24} and M_{23}, respectively). Further, it is shown that the representation theory of the $N = 2$ and small $N = 4$ superconformal algebras (see [41]) naturally leads to assignments of (vector-valued) mock modular forms to the elements of these groups. Thus the first examples of explicitly realized modules underlying moonshine phenomena relating mock modular forms to sporadic simple groups are obtained via $V^{s\natural}$ in [11].

Aside from the interesting connections to umbral moonshine, the main result of the forthcoming work [38] is the assignment of a weak Jacobi form of weight zero and index one to any symplectic derived autoequivalence of a projective complex K3 surface that fixes a stability condition in the distinguished space defined by Bridgeland in [5]. Conjecturally, the data of such a stability condition is equivalent to the physical notion of a supersymmetric nonlinear sigma model on the corresponding K3 surface. (See [1] for a detailed discussion of the moduli space of K3 sigma models, [57] for a concise treatment, and [68] for the relationship with stability conditions.) As demonstrated by Witten in [75], a supersymmetric nonlinear sigma model defines a weak Jacobi form, called the *elliptic genus* of the sigma model in question, and it turns out that the Jacobi form one obtains in the case of a (any) K3 sigma model is precisely that arising from the identity element of Co_0 in the construction of [38].

More generally, one expects, on physical grounds (see [57]), to obtain a weak Jacobi form (with level) from any supersymmetry-preserving automorphism of a nonlinear sigma model – we call it a *twined elliptic genus* – and it is shown in [57] that the automorphism groups of K3 sigma models are the subgroups of Co_0 that fix four-spaces in $\Lambda \otimes_{\mathbb{Z}} \mathbb{C}$.

This is a quantum analogue of the celebrated result of Mukai [87], that the finite groups of symplectic automorphisms of a K3 surface are the subgroups of the sporadic simple Mathieu group M_{23} that have at least five orbits in their action on 24 points. In general it is hard to compute twined elliptic genera, for the Hilbert spaces underlying nonlinear sigma models can, so far, only be constructed for certain special examples. Nonetheless, we find that the construction of [38] agrees precisely with the explicit computations of [57, 58, 100], which account for about half the conjugacy classes of Co_0 that fix a four-space in $\Lambda \otimes_{\mathbb{Z}} \mathbb{C}$. Thus the main result of [38] indicates that $V^{s\natural}$ may serve as a kind of universal object for understanding the twined elliptic genera of K3 sigma models. It may develop that $V^{s\natural}$ can shed light on more subtle structural aspects of K3 sigma models also.

The discussion here indicates that $V^{s\natural}$ plays an important role in Mathieu moonshine, and umbral moonshine more generally. On the other hand, that Conway moonshine and monstrous moonshine are closely related is evident from the discussions in Sections 1.1 and 1.2. Thus the results of this paper furnish further evidence – see also [90], and the introduction to [14] – that monstrous moonshine and umbral moonshine are related in a deep way, possibly having a common origin.

1.4. Organization. The organization of the paper is as follows. We review facts from vertex algebra theory in Section 2. Basic notions are recalled in Section 2.1, invariant bilinear forms on super vertex algebras are discussed in Section 2.2, and the Clifford module super vertex operator algebra construction is reviewed in Section 2.3. Spin groups act naturally on Clifford module super vertex operator algebras, and we review this in detail in Section 3.1. All necessary facts about the Conway group are explained in Section 3.2. The main results of the paper appear in Section 4, which features an explicit construction of $V^{s\natural}$ in Section 4.1, the characterization of $V^{s\natural}$ in Section 4.2, and the analysis of its trace

https://doi.org/10.1017/fms.2015.7 Published online by Cambridge University Press

functions in Section 4.3. The paper concludes with tables in Appendix A, one for the T_g^s, and one for the $T_{g,\mathrm{tw}}^s$, which can be used to facilitate explicit computations.

2. Vertex algebra

In this section we recall some preliminary facts from vertex algebra. In addition to the specific references that follow, we refer to the texts [52, 71, 78] for more background on vertex algebras, vertex operator algebras, and the various kinds of modules over these objects.

2.1. Fundamental notions.

A *super vertex algebra* is a super vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$ equipped with a *vacuum vector* $\mathbf{1} \in V_{\bar{0}}$, a linear operator $T : V \to V$, and a linear map

$$V \to \mathrm{End}(V)[[z^{\pm 1}]]$$
$$a \mapsto Y(a, z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1} \tag{2.1}$$

associating to each $a \in V$ a *vertex operator* $Y(a, z)$, which satisfy the following axioms for any $a, b, c \in V$:

(1) $Y(a, z)b \in V((z))$ and, if $a \in V_{\bar{0}}$ (respectively, $a \in V_{\bar{1}}$), then $a_{(n)}$ is an even (respectively, odd) operator for all n;

(2) $Y(\mathbf{1}, z) = \mathrm{Id}_V$ and $Y(a, z)\mathbf{1} \in a + zV[[z]]$;

(3) $[T, Y(a, z)] = \partial_z Y(a, z)$, $T\mathbf{1} = 0$, and T is an even operator; and,

(4) if $a \in V_{p(a)}$ and $b \in V_{p(b)}$ are $\mathbb{Z}/2$ homogeneous, there exists an element

$$f \in V[[z, w]][z^{-1}, w^{-1}, (z - w)^{-1}]$$

depending on a, b, and c, such that

$$Y(a, z)Y(b, w)c, \quad (-1)^{p(a)p(b)} Y(b, w)Y(a, z)c, \quad \text{and}$$
$$Y(Y(a, z - w)b, w)c$$

are the expansions of f in $V((z))((w))$, $V((w))((z))$, and $V((w))((z-w))$, respectively.

In items (1) and (4) above we write $V((z))$ for the vector space $V((z)) = V[[z]][z^{-1}]$ whose elements are formal Laurent series in z with coefficients in V. Note that $\mathbb{C}((z))$ is naturally a field, and if V is a vector space over $\mathbb{C}$ then $V((z))$ is naturally a vector space over $\mathbb{C}((z))$.

https://doi.org/10.1017/fms.2015.7 Published online by Cambridge University Press

A *module* over a super vertex algebra V is a super vector space $M = M_{\bar{0}} \oplus M_{\bar{1}}$ equipped with a linear map

$$\begin{aligned} V &\to \mathrm{End}(M)[[z^{\pm 1}]] \\ a &\mapsto Y_M(a, z) = \sum_{n\in\mathbb{Z}} a_{(n),M} z^{-n-1} \end{aligned} \tag{2.2}$$

which satisfies the following axioms for any $a, b \in V$, $u \in M$:

(1) $Y_M(a, z)u \in M((z))$ and, if $a \in V_{\bar{0}}$ (respectively, $a \in V_{\bar{1}}$), then $a_{(n),M}$ is an even (respectively, odd) operator for all n;

(2) $Y_M(\mathbf{1}, z) = \mathrm{Id}_M$; and,

(3) if $a \in V_{p(a)}$ and $b \in V_{p(b)}$, there exists an element

$$f \in M[[z, w]][z^{-1}, w^{-1}, (z - w)^{-1}]$$

depending on a, b, and u, such that

$$Y_M(a, z)Y_M(b, w)u, \quad (-1)^{p(a)p(b)} Y_M(b, w)Y_M(a, z)u, \quad \text{and} \quad Y_M(Y(a, z - w)b, w)u$$

are the expansions of f in the corresponding spaces, $M((z))((w))$, $M((w))((z))$, and $M((w))((z - w))$, respectively.

One can also define modules which are twisted by a symmetry of the vertex algebra; we shall use the following special case. Let $\theta := \mathrm{Id}_{V_{\bar{0}}} \oplus (-\mathrm{Id}_{V_{\bar{1}}})$ be the parity involution on a super vertex operator algebra $V = V_{\bar{0}} \oplus V_{\bar{1}}$. A *canonically twisted module* over V is a super vector space $M = M_{\bar{0}} \oplus M_{\bar{1}}$ equipped with a linear map

$$\begin{aligned} V &\to \mathrm{End}(M)[[z^{\pm 1/2}]] \\ a &\mapsto Y_{\mathrm{tw}}(a, z^{1/2}) = \sum_{n\in\frac{1}{2}\mathbb{Z}} a_{(n),\mathrm{tw}} z^{-n-1}, \end{aligned} \tag{2.3}$$

associating to each $a \in V$ a *twisted vertex operator* $Y_{\mathrm{tw}}(a, z^{1/2})$, which satisfies the following axioms for any $a, b \in V$, $u \in M$:

(1) $Y_{\mathrm{tw}}(a, z^{1/2})u \in M((z^{1/2}))$ and, if $a \in V_{\bar{0}}$ (respectively, $a \in V_{\bar{1}}$), then $a_{(n),\mathrm{tw}}$ is an even (respectively, odd) operator for all n;

(2) $Y_{\mathrm{tw}}(\mathbf{1}, z^{1/2}) = \mathrm{Id}_M$;

https://doi.org/10.1017/fms.2015.7 Published online by Cambridge University Press

(3) if $a \in V_{p(a)}$ and $b \in V_{p(b)}$, there exists an element

$$f \in M[[z^{1/2}, w^{1/2}]][z^{-1/2}, w^{-1/2}, (z-w)^{-1}]$$

depending on a, b, and u, such that

$$Y_{\mathrm{tw}}(a, z^{1/2})Y_{\mathrm{tw}}(b, w^{1/2})u, \quad (-1)^{p(a)p(b)}Y_{\mathrm{tw}}(b, w^{1/2})Y_{\mathrm{tw}}(a, z^{1/2})u, \\ \text{and} \quad Y_{\mathrm{tw}}(Y(a, z-w)b, w^{1/2})u$$

are the expansions of f in the three spaces $M((z^{1/2}))((w^{1/2}))$, in $M((w^{1/2}))((z^{1/2}))$, and in $M((w^{1/2}))((z-w))$, respectively; and,

(4) if $\theta(a) = (-1)^m a$, then $a_{(n),\mathrm{tw}} = 0$ for $n \notin \mathbb{Z} + (m/2)$.

More details can be found in [81].

The notion of super vertex algebra may be refined by introducing representations of certain Lie algebras. The *Virasoro* algebra is the Lie algebra spanned by $L(m)$, $m \in \mathbb{Z}$ and a central element $\mathbf{c}$, with Lie bracket

$$[L(m), L(n)] = (m-n)L(m+n) + \frac{m^3 - m}{12}\delta_{m+n,0}\mathbf{c}. \tag{2.4}$$

A super vertex *operator* algebra is a super vertex algebra containing a *Virasoro element* (or *conformal element*) $\omega \in V_{\bar{0}}$ such that, if $L(n) := \omega_{(n+1)}$ for $n \in \mathbb{Z}$, then

(5) $L(-1) = T$;

(6) $[L(m), L(n)] = (m-n)L_{m+n} + ((m^3 - m)/12)\delta_{m+n,0}c\,\mathrm{Id}_V$ for some $c \in \mathbb{C}$, called the *central charge* of V;

(7) $L(0)$ is a diagonalizable operator on V, with eigenvalues contained in $\frac{1}{2}\mathbb{Z}$ and bounded from below, and with finite-dimensional eigenspaces; and

(8) the super space structure on V is recovered from the $L(0)$-eigendata according to the rule that $p(a) = 2n \pmod 2$ when $L(0)v = nv$.

According to item (6), the components of $Y(\omega, z)$ generate a representation of the Virasoro algebra on V with central charge c.

For V a super vertex operator algebra, we write

$$V = \bigoplus_{n \in \frac{1}{2}\mathbb{Z}} V_n, \quad V_n = \{v \in V \mid L(0)v = nv\}, \tag{2.5}$$

for the decomposition of V into eigenspaces for $L(0)$, and we call V_n the homogeneous subspace of *degree* n.

https://doi.org/10.1017/fms.2015.7 Published online by Cambridge University Press

Following [29, 35], a V-module $M = (M, Y_M)$ for a super vertex operator algebra V is called *admissible* if there exists a grading $M = \bigoplus_{n\in\frac{1}{2}\mathbb{Z}} M(n)$, with $M(n) = \{0\}$ for $n < 0$, such that $a_{(n)}M(k) \subset M(k+m-n-1)$ when $a \in V_m$. An admissible V-module is *irreducible* if it has no nontrivial proper graded submodules. A super vertex operator algebra V is called *rational* if any admissible module is a direct sum of irreducible admissible modules, and we say that V is *self-dual* if V is rational, irreducible as a V-module, and if V is the only irreducible admissible V-module, up to isomorphism.

There are two particularly important extensions of the Virasoro algebra to a super Lie algebra. The *Neveu–Schwarz* algebra is a super Lie algebra spanned by $L(m), m \in \mathbb{Z}$, $G(n+1/2), n \in \mathbb{Z}$, and a central element $\mathbf{c}$; the $L(m)$ and $\mathbf{c}$ span the even subalgebra, isomorphic to the Virasoro algebra, and the $G(n+1/2)$ span the odd subspace. The Lie bracket is defined by Equation (2.4) and

$$[L(m), G(n+1/2)] = \frac{m-2(n+1/2)}{2}G(m+n+1/2), \tag{2.6}$$

$$[G(m+1/2), G(n-1/2)] = 2L(m+n) + \frac{4(m+1/2)^2-1}{12}\delta_{m+n,0}\mathbf{c}. \tag{2.7}$$

An $N = 1$ super vertex operator algebra is a super vertex algebra containing an $N = 1$ *element* $\tau \in V_{\bar{1}}$ such that, if $G(n+1/2) := \tau_{(n+1)}$ for $n \in \mathbb{Z}$, then $\omega := \frac{1}{2}G(-1/2)\tau$ is a Virasoro element (with components $L(n) := \omega_{(n+1)}$) as above, and the $L(m)$, $G(n+1/2)$ generate a representation of the Neveu–Schwarz algebra; in particular, the $L(m)$, $G(n+1/2)$ satisfy Equation (2.4), Equation (2.6), and Equation (2.7), where the role of $\mathbf{c}$ is played by $c\,\mathrm{Id}_V$ for some $c \in \mathbb{C}$. For further discussion we refer to [72].

Another extension of the Virasoro algebra to a super Lie algebra is the *Ramond* algebra, spanned by $L(m), m \in \mathbb{Z}$, $G(n), n \in \mathbb{Z}$, and a central element $\mathbf{c}$; as in the case of the Neveu–Schwarz algebra the $L(m)$ and $\mathbf{c}$ span the even subalgebra, isomorphic to the Virasoro algebra, and the $G(n)$ span the odd subspace. The Lie bracket is defined by Equation (2.4) and

$$[L(m), G(n)] = \frac{m-2n}{2}G(m+n), \tag{2.8}$$

$$[G(m), G(n)] = 2L(m+n) + \frac{4m^2-1}{12}\delta_{m+n,0}\mathbf{c}. \tag{2.9}$$

If V is an $N = 1$ super vertex operator algebra (with $N = 1$ element τ and Virasoro element $\omega = \frac{1}{2}G(-1/2)\tau$), and M is a canonically twisted module for V, then the operators $L(m) := \omega_{(n+1),\mathrm{tw}}$ and $G(n) := \tau_{(n+1/2),\mathrm{tw}}$ generate a representation of the Ramond algebra on M.

https://doi.org/10.1017/fms.2015.7 Published online by Cambridge University Press

2.2. Invariant bilinear forms. The notion of an invariant bilinear form on a vertex operator algebra module was introduced in [48]. We say that a bilinear form $\langle\cdot\,,\,\cdot\rangle : M \otimes M \to \mathbb{C}$ on a module (M, Y_M) for a super vertex operator algebra V is *invariant* if

$$\langle Y_M(a,z)b, c\rangle = \langle b, Y_M^{\dagger}(a,z)c\rangle \tag{2.10}$$

for $a \in V$ and $b, c \in M$, where $Y_M^{\dagger}(a,z)$ denotes the *opposite* vertex operator, defined by setting

$$Y_M^{\dagger}(a,z) := (-1)^n Y_M(e^{zL(1)}z^{-2L(0)}a, z^{-1}) \tag{2.11}$$

for a in $V_{n-1/2}$ or V_n. In the right-hand side of (2.11) we have extended the definition of Y_M from V to $V((z))$ by requiring $\mathbb{C}((z))$-linearity. That is, we define $Y_M(f(z)a, z) = f(z)Y_M(a,z)$ for $f(z) \in \mathbb{C}((z))$ and $a \in V$.

Suppose that $M = \bigoplus_{n\in\frac{1}{2}\mathbb{Z}} M(n)$ is an admissible V-module. Then the *restricted dual* of M is the graded vector space $M' = \bigoplus_{n\in\frac{1}{2}\mathbb{Z}} M'(n)$ obtained by setting $M'(n) = M(n)^* := \hom_{\mathbb{C}}(M(n), \mathbb{C})$. According to Proposition 2.5 of [36] (see also Lemma 2.7 of [35], and Theorem 5.2.1 of [48]), M' is naturally an admissible V-module, called the *contragredient* of M. To define the V-module structure on M', write $(\cdot\,,\,\cdot)_M$ for the natural pairing $M' \otimes M \to \mathbb{C}$, and define $Y'_M : V \to \mathrm{End}(M')[[z^{\pm 1}]]$ – the vertex operator correspondence *adjoint* to Y_M – by requiring that

$$(Y'_M(a,z)b', c)_M = (b', Y^{\dagger}(a,z)c)_M \tag{2.12}$$

for $a \in V$, $b' \in M'$ and $c \in M$.

Following the discussion in Section 5.3 of [48] we observe that the datum of a nondegenerate invariant bilinear form on an admissible V-module M is the same as the datum of a V-module isomorphism $M \to M'$. For if $\phi : M \to M'$ is a V-module isomorphism then we obtain a bilinear form $\langle\cdot\,,\,\cdot\rangle$ on M by setting $\langle b, c\rangle = (\phi(b), c)_M$ for $b, c \in M$. It is easily seen to be invariant and nondegenerate. Conversely, if $\langle\cdot\,,\,\cdot\rangle$ is a nondegenerate invariant bilinear form on M then invariance implies that $\langle M(m), M(n)\rangle \subset \{0\}$ unless $m = n$ (see Proposition 2.12 of [94]), and so we obtain a linear grading-preserving isomorphism $\phi : M \to M'$ by requiring $(\phi(b), c)_M = \langle b, c\rangle$ for $b, c \in M(n)$, $n \in \frac{1}{2}\mathbb{Z}$. The invariance of $\langle\cdot\,,\,\cdot\rangle$ then implies that $\phi(Y_M(a,z)b) = Y'_M(a,z)\phi(b)$ for $a \in V$ and $b \in M$, so ϕ is an isomorphism of V-modules.

The following theorem of Scheithauer is the super vertex operator algebra version of a result first proved for vertex operator algebras by Li in [80].

THEOREM 2.1 [94]. *The space of invariant bilinear forms on a super vertex operator algebra V is naturally isomorphic to the dual of $V_0/L(1)V_1$.*

https://doi.org/10.1017/fms.2015.7 Published online by Cambridge University Press

Note that there is some flexibility available in the definitions of invariant bilinear form and opposite vertex operator in the super case. For in [94], a bilinear form $\langle\cdot\,,\,\cdot\rangle^*$ is said to be invariant if

$$\langle Y(a,z)b,c\rangle^* = (-1)^{|a||b|}\langle b, Y^*(a,z)c\rangle^* \tag{2.13}$$

for $Y^*(a,z) := Y(e^{-\lambda^{-2}zL(1)}(-\lambda^{-1}z)^{-2L(0)}a, -\lambda^2 z^{-1})$. Taking $\lambda = \pm\mathbf{i}$, we recover the usual notion of opposite vertex operator for a vertex algebra (see (5.2.4) of [48]), upon restriction to the even sub vertex algebra of V. Observe that this notion of invariant bilinear form is equivalent to (2.10). For if we take $\lambda = -\mathbf{i}$ in the definition of Y^*, for example, then, given a bilinear form $\langle\cdot\,,\,\cdot\rangle^*$ satisfying (2.13), we obtain a bilinear form $\langle\cdot\,,\,\cdot\rangle$ that satisfies (2.10), upon setting $\langle a,b\rangle = \langle a,b\rangle^*$ for $p(a)=0$, and $\langle a,b\rangle = -\mathbf{i}\langle a,b\rangle^*$ for $p(a)=1$. The case that $\lambda = \mathbf{i}$ is directly similar.

2.3. Clifford module construction. We now review the standard construction of vertex operator algebras via Clifford algebra modules.

Let $\mathfrak{a}$ be a finite-dimensional complex vector space equipped with a nondegenerate symmetric bilinear form $\langle\cdot\,,\cdot\rangle$. For each $n\in\mathbb{Z}$ let $\mathfrak{a}(n+1/2)$ be a vector space isomorphic to $\mathfrak{a}$, with a chosen isomorphism $\mathfrak{a}\to\mathfrak{a}(n+1/2)$, denoted $u\mapsto u(n+1/2)$, and define

$$\hat{\mathfrak{a}} = \bigoplus_{n\in\mathbb{Z}} \mathfrak{a}(n+1/2). \tag{2.14}$$

We can extend $\langle\cdot\,,\cdot\rangle$ to a nondegenerate symmetric bilinear form on $\hat{\mathfrak{a}}$ by $\langle u(r), v(s)\rangle = \langle u,v\rangle\delta_{r+s,0}$. We obtain a *polarization* of $\hat{\mathfrak{a}}$ with respect to this bilinear form – that is, a decomposition $\hat{\mathfrak{a}} = \hat{\mathfrak{a}}^-\oplus\hat{\mathfrak{a}}^+$ into a direct sum of maximal isotropic subspaces – by setting

$$\hat{\mathfrak{a}}^- = \bigoplus_{n<0}\mathfrak{a}(n+1/2) \quad\text{and}\quad \hat{\mathfrak{a}}^+ = \bigoplus_{n\geqslant 0}\mathfrak{a}(n+1/2). \tag{2.15}$$

Define the *Clifford algebra* of $\hat{\mathfrak{a}}$ by $\mathrm{Cliff}(\hat{\mathfrak{a}}) = T(\hat{\mathfrak{a}})/I(\hat{\mathfrak{a}})$, where $T(\hat{\mathfrak{a}})$ is the tensor algebra of $\hat{\mathfrak{a}}$, with unity denoted $\mathbf{1}$, and $I(\hat{\mathfrak{a}})$ is the (two-sided) ideal of $T(\hat{\mathfrak{a}})$ generated by elements of the form $u\otimes u + \langle u,u\rangle\mathbf{1}$ for $u\in\hat{\mathfrak{a}}$. Denote by B^- and B^+ the subalgebras of $\mathrm{Cliff}(\hat{\mathfrak{a}})$ generated by $\hat{\mathfrak{a}}^-$ and $\hat{\mathfrak{a}}^+$, respectively. The linear map $-\mathrm{Id}$ on $\hat{\mathfrak{a}}$ induces an involution θ on $\mathrm{Cliff}(\hat{\mathfrak{a}})$ according to the universal property of Clifford algebras. We call θ the *parity* involution and write $\mathrm{Cliff}(\hat{\mathfrak{a}}) = \mathrm{Cliff}(\hat{\mathfrak{a}})^0\oplus\mathrm{Cliff}(\hat{\mathfrak{a}})^1$ for the corresponding decomposition into eigenspaces, where $\mathrm{Cliff}(\hat{\mathfrak{a}})^j$ denotes the θ-eigenspace with eigenvalue $(-1)^j$.

https://doi.org/10.1017/fms.2015.7 Published online by Cambridge University Press

Let $\mathbb{C}\mathbf{v}$ be a one-dimensional vector space equipped with the trivial action from B^+, that is, $\mathbf{1v} = \mathbf{v}$ and $u\mathbf{v} = 0$ for any $u \in \hat{\mathfrak{a}}^+$. Define $A(\mathfrak{a})$ to be the induced $\mathrm{Cliff}(\hat{\mathfrak{a}})$-module, $A(\mathfrak{a}) = \mathrm{Cliff}(\hat{\mathfrak{a}}) \otimes_{B^+} \mathbb{C}\mathbf{v}$. We have a natural isomorphism of B^--modules

$$A(\mathfrak{a}) \simeq \bigwedge(\hat{\mathfrak{a}}^-)\mathbf{v}. \tag{2.16}$$

For $a \in \mathfrak{a}$, define a vertex operator for $a(-1/2)\mathbf{v}$ by

$$Y(a(-1/2)\mathbf{v}, z) = \sum_{n\in\mathbb{Z}} a(n + 1/2)z^{-n-1}. \tag{2.17}$$

There is a *reconstruction theorem* (Theorem 4.4.1 of [52]) which ensures that these vertex operators extend uniquely to a super vertex algebra structure on $A(\mathfrak{a})$. The super space structure $A(\mathfrak{a}) = A(\mathfrak{a})^0 \oplus A(\mathfrak{a})^1$ is given by the parity decomposition on $\bigwedge(\hat{\mathfrak{a}}^-)\mathbf{v}$. That is, restricting the isomorphism of (2.16), we have

$$A(\mathfrak{a})^0 \simeq \bigwedge^{\text{even}}(\hat{\mathfrak{a}}^-)\mathbf{v}, \quad A(\mathfrak{a})^1 \simeq \bigwedge^{\text{odd}}(\hat{\mathfrak{a}}^-)\mathbf{v}. \tag{2.18}$$

Choose an orthonormal basis $\{e_i : 1 \leqslant i \leqslant \dim \mathfrak{a}\}$ for $\mathfrak{a}$. The Virasoro element

$$\omega = -\frac{1}{4}\sum_{i=1}^{\dim \mathfrak{a}} e_i(-3/2)e_i(-1/2)\mathbf{v} \tag{2.19}$$

gives $A(\mathfrak{a})$ the structure of a super vertex operator algebra with central charge $c = \frac{1}{2}\dim \mathfrak{a}$.

Observe that $A(\mathfrak{a})_0$ is spanned by the vacuum $\mathbf{v}$. We compute $L(1)a = 0$ for all $a \in A(\mathfrak{a})_1$, and conclude from Theorem 2.1 that there is a unique nonzero invariant bilinear form on $A(\mathfrak{a})$ up to scale. Scale it so that $\langle \mathbf{v}, \mathbf{v}\rangle = 1$. Then, taking $a = u(-1/2)\mathbf{v}$ for $u \in \mathfrak{a}$, we compute $Y'(a, z) = -Y(a, z^{-1})z^{-1}$ (see (2.11)), and conclude from (2.10) that

$$\langle u(-m - 1/2)a, b\rangle + \langle a, u(m + 1/2)b\rangle = 0 \tag{2.20}$$

for $u \in \mathfrak{a}$, $m \in \mathbb{Z}$, and $a, b \in A(\mathfrak{a})$. This identity is useful for computations. For example, taking $a = \mathbf{v}$ and $b = v(-m - 1/2)\mathbf{v}$ for $v \in \mathfrak{a}$, we see that

$$\langle u(-m - 1/2)\mathbf{v}, v(-m - 1/2)\mathbf{v}\rangle = \langle u, v\rangle \tag{2.21}$$

for $u, v \in \mathfrak{a}$ and $m \geqslant 0$.

A construction similar to $A(\mathfrak{a})$ produces a canonically twisted module for $A(\mathfrak{a})$, which we call $A(\mathfrak{a})_{\mathrm{tw}}$. For the sake of simplicity, let us assume that the dimension of $\mathfrak{a}$ is even.

https://doi.org/10.1017/fms.2015.7 Published online by Cambridge University Press

For each $n \in \mathbb{Z}$ let $\mathfrak{a}(n)$ be a vector space isomorphic to $\mathfrak{a}$, with a chosen isomorphism $\mathfrak{a} \to \mathfrak{a}(n)$ denoted $u \mapsto u(n)$, and define

$$\hat{\mathfrak{a}}_{\mathrm{tw}} = \bigoplus_{n\in\mathbb{Z}} \mathfrak{a}(n). \tag{2.22}$$

The bilinear form on $\mathfrak{a}$ extends to a bilinear form on $\hat{\mathfrak{a}}_{\mathrm{tw}}$ in the same way as $\hat{\mathfrak{a}}$; namely, $\langle u(m), v(n)\rangle = \langle u, v\rangle \delta_{m+n,0}$. We again require a decomposition of $\hat{\mathfrak{a}}_{\mathrm{tw}}$ into maximal isotropic subspaces $\hat{\mathfrak{a}}^+_{\mathrm{tw}} \oplus \hat{\mathfrak{a}}^-_{\mathrm{tw}}$. For this, first choose a polarization $\mathfrak{a} = \mathfrak{a}^- \oplus \mathfrak{a}^+$, and then define $\hat{\mathfrak{a}}^\pm_{\mathrm{tw}}$ by setting

$$\hat{\mathfrak{a}}^-_{\mathrm{tw}} = \mathfrak{a}(0)^- \oplus \left(\bigoplus_{n<0} \mathfrak{a}(n)\right) \quad \text{and} \quad \hat{\mathfrak{a}}^+_{\mathrm{tw}} = \mathfrak{a}(0)^+ \oplus \left(\bigoplus_{n>0} \mathfrak{a}(n)\right), \tag{2.23}$$

where $\mathfrak{a}(0)^\pm$ is the image of $\mathfrak{a}^\pm$ under the isomorphism $u \mapsto u(0)$.

Denote by B^-_{tw} and B^+_{tw} the subalgebras of $\mathrm{Cliff}(\hat{\mathfrak{a}}_{\mathrm{tw}})$ generated by $\hat{\mathfrak{a}}^-_{\mathrm{tw}}$ and $\hat{\mathfrak{a}}^+_{\mathrm{tw}}$, respectively. Define the trivial action of B^+_{tw} on a one-dimensional space $\mathbb{C}\mathbf{v}_{\mathrm{tw}}$, and set $A(\mathfrak{a})_{\mathrm{tw}} = \mathrm{Cliff}(\hat{\mathfrak{a}}_{\mathrm{tw}}) \otimes_{B^+_{\mathrm{tw}}} \mathbb{C}\mathbf{v}_{\mathrm{tw}}$. There is a natural B^-_{tw}-module isomorphism

$$A(\mathfrak{a})_{\mathrm{tw}} \simeq \bigwedge(\hat{\mathfrak{a}}^-_{\mathrm{tw}})\mathbf{v}_{\mathrm{tw}}. \tag{2.24}$$

For $a \in \mathfrak{a}$, define a twisted vertex operator for $a(-1/2)\mathbf{v} \in A(\mathfrak{a})$ on $A(\mathfrak{a})_{\mathrm{tw}}$ by

$$Y_{\mathrm{tw}}(a(-1/2)\mathbf{v}, z^{1/2}) = \sum_{n\in\mathbb{Z}} a(n) z^{-n-1/2}. \tag{2.25}$$

An analogue of the reconstruction theorem for modules (see [46]) ensures that this collection of twisted vertex operators extends uniquely to a canonically twisted $A(\mathfrak{a})$-module structure on $A(\mathfrak{a})_{\mathrm{tw}}$. In particular, the twisted vertex operator

$$Y_{\mathrm{tw}}(\omega, z^{1/2}) = \sum_{n\in\mathbb{Z}} L(n) z^{-n-2} \tag{2.26}$$

equips $A(\mathfrak{a})_{\mathrm{tw}}$ with a representation of the Virasoro algebra, and $L(0) = \omega_{(1),\mathrm{tw}}$ acts diagonalizably. An explicit computation yields that the eigenvalues of $L(0)$ on $A(\mathfrak{a})_{\mathrm{tw}}$ are contained in $\mathbb{Z} + \frac{1}{16}\dim \mathfrak{a}$.

The finite-dimensional Clifford algebra $\mathrm{Cliff}(\mathfrak{a})$ embeds in $\mathrm{Cliff}(\hat{\mathfrak{a}}_{\mathrm{tw}})$ as the subalgebra generated by $\mathfrak{a}(0)$. Through this identification, $\mathrm{Cliff}(\mathfrak{a})$ acts on $A(\mathfrak{a})_{\mathrm{tw}}$, and the $\mathrm{Cliff}(\mathfrak{a})$-submodule of $A(\mathfrak{a})_{\mathrm{tw}}$ generated by $\mathbf{v}_{\mathrm{tw}}$ is the unique (up to isomorphism) nontrivial irreducible representation of $\mathrm{Cliff}(\mathfrak{a})$. We shall denote this subspace of $A(\mathfrak{a})_{\mathrm{tw}}$ by CM. By restricting the isomorphism of (2.24), we obtain

$$A(\mathfrak{a})_{\mathrm{tw}} \simeq \bigwedge\left(\bigoplus_{n<0} \mathfrak{a}(n)\right) \otimes \mathrm{CM}, \quad \mathrm{CM} \simeq \bigwedge(\mathfrak{a}(0)^-)\mathbf{v}_{\mathrm{tw}}. \tag{2.27}$$

https://doi.org/10.1017/fms.2015.7 Published online by Cambridge University Press

There is a unique (up to scale) bilinear form $\langle\cdot\,,\cdot\rangle_{\mathrm{tw}}$ on CM satisfying $\langle ua, b\rangle_{\mathrm{tw}} + \langle a, ub\rangle_{\mathrm{tw}} = 0$ for $u \in \mathfrak{a}$ and $a, b \in \mathrm{CM}$. To choose a scaling, let $\{a_i^-\}_{i=1}^c$ be a basis for $\mathfrak{a}^-$, where $c = \frac{1}{2}\dim\mathfrak{a}$, and set

$$\langle a_1^- \cdots a_c^- \mathbf{v}_{\mathrm{tw}}, \mathbf{v}_{\mathrm{tw}}\rangle_{\mathrm{tw}} = 1. \tag{2.28}$$

We may extend this form uniquely to a bilinear form $\langle\cdot\,,\cdot\rangle_{\mathrm{tw}}$ on $A(\mathfrak{a})_{\mathrm{tw}}$ by requiring that

$$\langle u(-m)a, b\rangle_{\mathrm{tw}} + \langle a, u(m)b\rangle_{\mathrm{tw}} = 0 \tag{2.29}$$

for $u \in \mathfrak{a}$, $m \in \mathbb{Z}$, and $a, b \in A(\mathfrak{a})_{\mathrm{tw}}$ (see (2.20)).

3. Groups

In Section 3.1 we discuss the spin group of a complex vector space of even dimension, and in Section 3.2 we recall the definition and some features of the automorphism group of the Leech lattice, also known as the Conway group.

3.1. The spin groups. Define the *main antiautomorphism* α on $\mathrm{Cliff}(\mathfrak{a})$ by setting $\alpha(u_1\cdots u_k) := u_k\cdots u_1$ for $u_i \in \mathfrak{a}$. Recall that the *spin group* of $\mathfrak{a}$, denoted $\mathrm{Spin}(\mathfrak{a})$, is the set of even invertible elements $x \in \mathrm{Cliff}(\mathfrak{a})$ with $\alpha(x)x = \mathbf{1}$ (that is, the unit element of $\mathrm{Cliff}(\mathfrak{a})$) such that $xux^{-1} \in \mathfrak{a}$ whenever $u \in \mathfrak{a}$.

It is useful to be able to construct some elements of $\mathrm{Spin}(\mathfrak{a})$ explicitly. The expressions $\frac{1}{2}(uv - vu) \in \mathrm{Cliff}(\mathfrak{a})$, for $u, v \in \mathfrak{a}$, span a $\binom{\dim\mathfrak{a}}{2}$-dimensional subspace $\mathfrak{g} < \mathrm{Cliff}(\mathfrak{a})$ which closes under the commutator $[x, y] = xy - yx$ on $\mathrm{Cliff}(\mathfrak{a})$, and forms a simple Lie algebra of type D_c, for $c = \frac{1}{2}\dim\mathfrak{a}$. (Recall our assumption that $\dim\mathfrak{a}$ is even.) The exponentials $\exp(\frac{1}{2}(uv - vu)) \in \mathrm{Cliff}(\mathfrak{a})$ generate $\mathrm{Spin}(\mathfrak{a})$. For example, if $a^+, a^- \in \mathfrak{a}$ are chosen so that

$$\langle a^\pm, a^\pm\rangle = 0, \quad \langle a^-, a^+\rangle = 1, \tag{3.1}$$

then $X = (\mathbf{i}/2)(a^-a^+ - a^+a^-)$ satisfies $X^2 = -\mathbf{1}$, so $e^{\alpha X} = (\cos\alpha)\mathbf{1} + (\sin\alpha)X$.

Set $x(u) = xux^{-1}$ for $x \in \mathrm{Spin}(\mathfrak{a})$ and $u \in \mathfrak{a}$. Then $u \mapsto x(u)$ is a linear transformation on $\mathfrak{a}$ belonging to $\mathrm{SO}(\mathfrak{a})$, and the assignment $x \mapsto x(\cdot)$ defines a map $\mathrm{Spin}(\mathfrak{a}) \to \mathrm{SO}(\mathfrak{a})$ with kernel $\{\pm\mathbf{1}\}$. We say that $\widehat{g} \in \mathrm{Spin}(\mathfrak{a})$ is a *lift* of $g \in \mathrm{SO}(\mathfrak{a})$ if $\widehat{g}(\cdot) = g$. For

$$X = \frac{\mathbf{i}}{2}(a^-a^+ - a^+a^-) \tag{3.2}$$

with $a^\pm$ as in (3.1) we have $Xa^\pm = \pm\mathbf{i}a^\pm = -a^\pm X$ in $\mathrm{Cliff}(\mathfrak{a})$, so

$$e^{\alpha X}(a^\pm) = e^{\alpha X}a^\pm e^{-\alpha X} = e^{\pm 2\alpha\mathbf{i}}a^\pm, \tag{3.3}$$

https://doi.org/10.1017/fms.2015.7 Published online by Cambridge University Press

which is to say, $e^{\alpha X}$ is a lift of the orthogonal transformation on $\mathfrak{a}$ which acts as multiplication by $e^{\pm 2\alpha\mathbf{i}}$ on $a^{\pm}$, and as the identity on vectors orthogonal to a^+ and a^-. For future reference we note here also that $X\mathbf{v}_{\rm tw} = \mathbf{i}\mathbf{v}_{\rm tw}$, so the action of $e^{\alpha X} \in \mathrm{Spin}(\mathfrak{a})$ on $\mathbf{v}_{\rm tw}$ is given by

$$e^{\alpha X}\mathbf{v}_{\rm tw} = e^{\alpha\mathbf{i}}\mathbf{v}_{\rm tw}. \tag{3.4}$$

The group $\mathrm{Spin}(\mathfrak{a})$ acts naturally on $A(\mathfrak{a})$ and $A(\mathfrak{a})_{\rm tw}$. Indeed, writing $A(\mathfrak{a})_1$ for the $L(0)$-eigenspace of $A(\mathfrak{a})$ with eigenvalue equal to 1, the map $u(-\frac{1}{2})v(-\frac{1}{2})\mathbf{v} \mapsto \frac{1}{2}(uv - vu)$ defines an isomorphism of vector spaces $A(\mathfrak{a})_1 \to \mathfrak{g}$, which becomes an isomorphism of Lie algebras once we equip $A(\mathfrak{a})_1$ with the bracket $[X, Y] := X_{(0)}Y$. (It follows from the vertex algebra axioms that $[a_{(0)}, b_{(n)}] = (a_{(0)}b)_{(n)}$ in $\mathrm{End}\, A(\mathfrak{a})$, for any $a, b \in A(\mathfrak{a})$ and $n \in \mathbb{Z}$.) Accordingly, the exponentials $e^{X_{(0)}}$ and $e^{X_{(0),\rm tw}}$ for $X \in A(\mathfrak{a})_1$ generate an action of $\mathrm{Spin}(\mathfrak{a})$ on $A(\mathfrak{a})$ and $A(\mathfrak{a})_{\rm tw}$, respectively. Explicitly, if $a \in A(\mathfrak{a})$ has the form $a = u_1(-n_1 + \frac{1}{2}) \cdots u_k(-n_k + \frac{1}{2})\mathbf{v}$ for some $u_i \in \mathfrak{a}$ and $n_i \in \mathbb{Z}^+$, then

$$xa = u_1'(-n_1 + \tfrac{1}{2}) \cdots u_k'(-n_k + \tfrac{1}{2})\mathbf{v}, \tag{3.5}$$

for $x \in \mathrm{Spin}(\mathfrak{a})$, where $u_i' = x(u_i)$. Evidently $-\mathbf{1}$ is in the kernel of this assignment $\mathrm{Spin}(\mathfrak{a}) \to \mathrm{Aut}(A(\mathfrak{a}))$, so the action factors through $\mathrm{SO}(\mathfrak{a})$.

For $A(\mathfrak{a})_{\rm tw}$ we use (2.27) to identify the elements of the form

$$u_1(-n_1) \cdots u_k(-n_k) \otimes y \tag{3.6}$$

as a spanning set, where $u_i \in \mathfrak{a}$ and $n_i \in \mathbb{Z}^+$ as above, and $y \in \mathrm{CM}$. The image of such an element under $x \in \mathrm{Spin}(\mathfrak{a})$ is given by $u_1'(-n_1) \cdots u_k'(-n_k) \otimes xy$, where $u_i' = x(u_i)$ as before. Since CM is a faithful $\mathrm{Spin}(\mathfrak{a})$-module, so too is $A(\mathfrak{a})_{\rm tw}$.

In terms of the vertex operator correspondences we have

$$\begin{aligned} Y(xa, z)xb &= xY(a, z)b = \sum_{n\in\mathbb{Z}} x(a_{(n)}b)z^{-n-1}, \\ Y_{\rm tw}(xa, z^{1/2})xc &= xY_{\rm tw}(a, z^{1/2})c = \sum_{n\in\frac{1}{2}\mathbb{Z}} x(a_{(n),\rm tw}c)z^{-n-1}, \end{aligned} \tag{3.7}$$

for $x \in \mathrm{Spin}(\mathfrak{a})$, $a, b \in A(\mathfrak{a})$, and $c \in A_{\rm tw}(\mathfrak{a})$.

Recall that our construction of CM depends upon a choice of polarization $\mathfrak{a} = \mathfrak{a}^- \oplus \mathfrak{a}^+$. Observe that if $x \in \mathrm{Spin}(\mathfrak{a})$ is a lift of $-\mathrm{Id}_{\mathfrak{a}} \in \mathrm{SO}(\mathfrak{a})$ then the vector $\mathbf{v}_{\rm tw} \in \mathrm{CM}$, characterized by the condition $u\mathbf{v}_{\rm tw} = 0$ for all $u \in \mathfrak{a}^+$, satisfies $x\mathbf{v}_{\rm tw} = \pm\mathbf{i}^c\mathbf{v}_{\rm tw}$, where $c = \frac{1}{2}\dim\mathfrak{a}$. Indeed, if $\{a_i^{\pm}\}$ is a basis for $\mathfrak{a}^{\pm}$, chosen so that $\langle a_i^-, a_j^+\rangle = \delta_{i,j}$, then

$$\mathfrak{z} := \prod_{i=1}^{c} e^{\frac{\pi}{2}X_i} \tag{3.8}$$

https://doi.org/10.1017/fms.2015.7 Published online by Cambridge University Press

is a lift of $-\mathrm{Id}_{\mathfrak{a}}$, for $X_i = (\mathbf{i}/2)(a_i^- a_i^+ - a_i^+ a_i^-)$, according to (3.3). From (3.4) it follows that $\mathfrak{z}\mathbf{v}_{\mathrm{tw}} = \mathbf{i}^c \mathbf{v}_{\mathrm{tw}}$.

Thus we see that a choice of polarization $\mathfrak{a} = \mathfrak{a}^- \oplus \mathfrak{a}^+$ distinguishes one of the two lifts of $-\mathrm{Id}_{\mathfrak{a}}$ to $\mathrm{Spin}(\mathfrak{a})$, namely, the unique element $\mathfrak{z} \in \mathrm{Spin}(\mathfrak{a})$ such that $\mathfrak{z}(\cdot) = -\mathrm{Id}_{\mathfrak{a}}$ and

$$\mathfrak{z}\mathbf{v}_{\mathrm{tw}} = \mathbf{i}^c \mathbf{v}_{\mathrm{tw}}, \tag{3.9}$$

where $c = \frac{1}{2} \dim \mathfrak{a}$. We call this $\mathfrak{z}$ the lift of $-\mathrm{Id}_{\mathfrak{a}}$ *associated* to the polarization $\mathfrak{a} = \mathfrak{a}^- \oplus \mathfrak{a}^+$. The element $\mathfrak{z}$ acts with order two on $A(\mathfrak{a})_{\mathrm{tw}}$ when $\dim \mathfrak{a}$ is divisible by 4. In this case, we write

$$A(\mathfrak{a})_{\mathrm{tw}} = A(\mathfrak{a})^0_{\mathrm{tw}} \oplus A(\mathfrak{a})^1_{\mathrm{tw}} \tag{3.10}$$

for the decomposition into eigenspaces for $\mathfrak{z}$, where $\mathfrak{z}$ acts as $(-1)^j \,\mathrm{Id}$ on $A(\mathfrak{a})^j_{\mathrm{tw}}$. The element $\mathfrak{z}$ is central, so the action of $\mathrm{Spin}(\mathfrak{a})$ on $A(\mathfrak{a})_{\mathrm{tw}}$ preserves the decomposition (3.10).

Note here the difference between writing $-(xa)$ and $(-x)a$ for $x \in \mathrm{Spin}(\mathfrak{a})$ and $a \in A(\mathfrak{a})$. The former is just the additive inverse of the vector xa in $A(\mathfrak{a})$, whereas the latter is the image of a under the action of $-x = (-\mathbf{1})x$, an element of $\mathrm{Spin}(\mathfrak{a}) < \mathrm{Cliff}(\mathfrak{a})$. So, for example, $\mathfrak{z}a = (-\mathfrak{z})a = a$ for $a \in A(\mathfrak{a})$, and in particular $(-\mathfrak{z})a \neq -a$ unless $a = 0$. On the other hand, $\mathfrak{z}(u) = (-\mathfrak{z})(u) = -u$ for $u \in \mathfrak{a}$. So, from the description (3.5), we see that writing $A(\mathfrak{a})^j$ for the $(-1)^j$ eigenspace of either $\mathfrak{z}$ or $-\mathfrak{z}$ recovers the super space decomposition

$$A(\mathfrak{a}) = A(\mathfrak{a})^0 \oplus A(\mathfrak{a})^1 \tag{3.11}$$

of $A(\mathfrak{a})$. (See (2.18).)

Suppose that $\mathfrak{a} = V \otimes_{\mathbb{R}} \mathbb{C}$ for some real vector space V, and that $\langle \cdot\,, \cdot \rangle$ restricts to an $\mathbb{R}$-valued bilinear form on $V \subset \mathfrak{a}$. Then we obtain another way to determine a lift of $-\mathrm{Id}_{\mathfrak{a}}$ to $\mathrm{Spin}(\mathfrak{a})$ by choosing an orientation $\mathbb{R}^+\omega \subset \bigwedge^{\dim V}(V)$ of V. For if $\{e_i\}_{i=1}^{\dim V}$ is an ordered basis of V satisfying $\langle e_i, e_j \rangle = \pm\delta_{i,j}$ then $\mathfrak{z} = e_1 \cdots e_{\dim V}$ belongs to $\mathrm{Spin}(\mathfrak{a})$ and satisfies $\mathfrak{z}(\cdot) = -\mathrm{Id}_{\mathfrak{a}}$. (Recall that $\dim \mathfrak{a} = \dim_{\mathbb{C}} \mathfrak{a}$ is assumed to be even.) On the other hand, $e_1 \wedge \cdots \wedge e_{\dim V}$ belongs either to $\mathbb{R}^+\omega$ or to $\mathbb{R}^-\omega$, and so we can say that $\mathfrak{z}$ is *consistent* with the chosen orientation of V in the former case, and *inconsistent* in the latter. Since a polarization $\mathfrak{a} = \mathfrak{a}^- \oplus \mathfrak{a}^+$ of $\mathfrak{a} = V \otimes_{\mathbb{R}} \mathbb{C}$ also determines a lift $\mathfrak{z}$ of $-\mathrm{Id}_{\mathfrak{a}}$, characterized by the condition $\mathfrak{z}\mathbf{v}_{\mathrm{tw}} = \mathbf{i}^c \mathbf{v}_{\mathrm{tw}}$, we can say that it too is either consistent or not with a given orientation of V, according as the associated lift $\mathfrak{z}$ is or is not consistent.

3.2. The Conway group. The Leech lattice, denoted Λ, is the unique self-dual positive-definite even lattice of rank 24 with no roots. That is, $\langle \lambda, \lambda \rangle < 4$

https://doi.org/10.1017/fms.2015.7 Published online by Cambridge University Press

for $\lambda \in \Lambda$ implies $\lambda = 0$. It was discovered in 1965 by Leech [76, 77], and the uniqueness statement is a consequence of a (somewhat stronger) theorem due to Conway [17].

Conway also calculated [16, 18] the automorphism group of Λ, which turns out to be a nontrivial 2-fold cover of the sporadic simple group that we denote Co_1. We call $Co_0 := \mathrm{Aut}(\Lambda)$ the *Conway group*. The centre of Co_0 is $Z(Co_0) = \{\pm \mathrm{Id}\}$, and we have $Co_1 = Co_0/Z(Co_0)$.

Set $\Lambda_n := \{\lambda \in \Lambda \mid \langle\lambda, \lambda\rangle = 2n\}$, the set of vectors of *type n* in Λ. Conway's uniqueness proof shows that any type 4 vector is equivalent modulo 2Λ to exactly 47 other vectors of type 4 in Λ, and if $\lambda, \mu \in \Lambda_4$ are equivalent modulo 2Λ then $\lambda = \pm\mu$ or $\langle\lambda, \mu\rangle = 0$. We call a set $\{\lambda_i\}_{i=1}^{24} \subset \Lambda_4$ a *coordinate frame* for Λ when the λ_i are mutually orthogonal, but equivalent modulo 2Λ.

Set $\Omega = \{1, \ldots, 24\}$, and write $\mathcal{P}(\Omega)$ for the power set of Ω. Given a coordinate frame $S = \{\lambda_i\}_{i\in\Omega}$ for Λ, let $E = E_S$ be the subgroup of Co_0 whose elements act as sign changes on the λ_i.

$$E = E_S := \{g \in Co_0 \mid g(\lambda_i) \in \{\pm\lambda_i\}, \forall i \in \Omega\}. \tag{3.12}$$

Then E is an elementary abelian 2-group of order 2^{12}. If we attach a subset $C(g) \subset \Omega$ to each $g \in E$ by setting

$$C(g) := \{i \in \Omega \mid g(\lambda_i) = -\lambda_i\}, \tag{3.13}$$

then the symmetric difference operation equips $\mathcal{G} := \{C(g) \mid g \in E\} \subset \mathcal{P}(\Omega)$ with a group structure naturally isomorphic to that of E, in the sense that $g \mapsto C(g)$ is an isomorphism, $C(gh) = C(g)+C(h)$ for $g, h \in E$. The weight function $C \mapsto \#C$ equips $\mathcal{G}$ with the structure of a binary linear code, and it turns out that $\mathcal{G}$ is a copy of the extended binary *Golay code*, being the unique (see [20, 93]) self-dual doubly even binary linear code of length 24 with no codewords of weight 4.

A choice of identification $\mathfrak{a} = \Lambda \otimes_{\mathbb{Z}} \mathbb{C}$ allows us to embed the Conway group $Co_0 = \mathrm{Aut}(\Lambda)$ in $\mathrm{SO}(\mathfrak{a})$. Given such a choice let us write G for the corresponding subgroup of $\mathrm{SO}(\mathfrak{a})$, isomorphic to Co_0. We write $g \mapsto \chi_g$ for the character of the corresponding representation of G.

$$\chi_g := \mathrm{tr}_{\mathfrak{a}}\, g \tag{3.14}$$

Given a subgroup $H < \mathrm{SO}(\mathfrak{a})$, we say that $\widehat{H} < \mathrm{Spin}(\mathfrak{a})$ is a *lift* of H if the natural map $\mathrm{Spin}(\mathfrak{a}) \to \mathrm{SO}(\mathfrak{a})$ restricts to an isomorphism $\widehat{H} \xrightarrow{\sim} H$.

PROPOSITION 3.1. *Let $G < \mathrm{SO}(\mathfrak{a})$, and suppose that G is isomorphic to Co_0. Then there is a unique lift of G to* $\mathrm{Spin}(\mathfrak{a})$.

https://doi.org/10.1017/fms.2015.7 Published online by Cambridge University Press

Proof. Since the Schur multiplier of Co_0 is trivial (see [21]), the preimage of G under the natural map $\mathrm{Spin}(\mathfrak{a}) \to \mathrm{SO}(\mathfrak{a})$ contains a copy of Co_0. So there is at least one lift. If there are two, $\widehat{G}$ and $\widehat{G}'$ say, then, given $g \in G$, we write $\widehat{g}$ for the corresponding element of $\widehat{G}$, and interpret $\widehat{g}'$ similarly, so that $\widehat{g}(\cdot) = \widehat{g}'(\cdot) = g$. Now $\widehat{g}' = \pm\widehat{g}$ as elements of $\mathrm{Spin}(\mathfrak{a})$, so $\widehat{G} \cap \widehat{G}'$ is a normal subgroup of $\widehat{G}$ (and of $\widehat{G}'$) containing all of its elements of odd order. The only proper nontrivial normal subgroup of Co_0 its centre, which has order two, so $\widehat{G} \cap \widehat{G}' = \widehat{G}$. That is, $\widehat{G} = \widehat{G}'$, as we required. □

Given $G < \mathrm{SO}(\mathfrak{a})$, isomorphic to Co_0, we write $\widehat{G}$ for the unique lift of G to $\mathrm{Spin}(\mathfrak{a})$ whose existence, and uniqueness, is guaranteed by Proposition 3.1. Then $\widehat{G}$ is a copy of the Conway group acting naturally on $A(\mathfrak{a})$ and $A(\mathfrak{a})_{\mathrm{tw}}$. We write

$$\begin{aligned} G &\xrightarrow{\sim} \widehat{G} \\ g &\mapsto \widehat{g} \end{aligned} \tag{3.15}$$

for the inverse of the isomorphism $\widehat{G} \xrightarrow{\sim} G$ obtained by restricting the natural map $\mathrm{Spin}(\mathfrak{a}) \to \mathrm{SO}(\mathfrak{a})$.

Observe that the action of $\widehat{G} \simeq Co_0$ on $A(\mathfrak{a})_{\mathrm{tw}}$ depends upon the choice of polarization $\mathfrak{a} = \mathfrak{a}^- \oplus \mathfrak{a}^+$ used to define $A(\mathfrak{a})_{\mathrm{tw}}$, for the central element of $\widehat{G}$ will be $\mathfrak{z}$ or $-\mathfrak{z}$, depending upon this choice, where $\mathfrak{z}$ denotes the lift of $-\mathrm{Id}_{\mathfrak{a}} \in \mathrm{SO}(\mathfrak{a})$ to $\mathrm{Spin}(\mathfrak{a})$ associated to the chosen polarization (see Section 3.1). We may assume that $\mathfrak{z} \in \widehat{G}$, so long as we allow ourselves the ability to modify the polarization slightly, replacing $a_j^{\pm}$ with $a_j^{\mp}$ for some j, for example, given basis vectors $a_i^{\pm} \in \mathfrak{a}^{\pm}$ satisfying $\langle a_i^-, a_j^+ \rangle = \delta_{i,j}$. (See (3.8).)

In practice we will take $\mathfrak{a}^{\pm}$ to be the span of isotropic eigenvectors $a_i^{\pm}$ for the action of some $g \in G$ on $\mathfrak{a}$, satisfying $\langle a_i^-, a_j^+ \rangle = \delta_{i,j}$. Since these conditions still hold after swapping a_j^- with a_j^+ for some j, we may apply the following convention with no loss of generality: given a choice of identification $\mathfrak{a} = \Lambda \otimes_{\mathbb{Z}} \mathbb{C}$, with G the corresponding copy of Co_0 in $\mathrm{SO}(\mathfrak{a})$, and $\widehat{G}$ the unique lift of G to $\mathrm{Spin}(\mathfrak{a})$, we assume that any polarization $\mathfrak{a} = \mathfrak{a}^- \oplus \mathfrak{a}^+$ is chosen so that the associated lift $\mathfrak{z}$ of $-\mathrm{Id}_{\mathfrak{a}}$ belongs to $\widehat{G}$.

4. Moonshine

This section contains the main results of the paper. In Section 4.1 we describe the construction of a distinguished super vertex operator algebra $V^{s\natural}$, and its unique canonically twisted module $V^{s\natural}_{\mathrm{tw}}$. We equip both $V^{s\natural}$ and $V^{s\natural}_{\mathrm{tw}}$ with actions by the Conway group Co_0. We establish a characterization of the super vertex operator algebra structure on $V^{s\natural}$ in Section 4.2. In Section 4.3 we compute the

https://doi.org/10.1017/fms.2015.7 Published online by Cambridge University Press

graded traces attached to elements of Co_0 via its actions on $V^{s\natural}$ and $V^{s\natural}_{\mathrm{tw}}$. We identify these functions as normalized principal moduli in the case of $V^{s\natural}$, and as constant or principal moduli in the case of $V^{s\natural}_{\mathrm{tw}}$, according as there are fixed points or not in the corresponding action on the Leech lattice.

4.1. Construction. From now on we take $\mathfrak{a}$ to be 24-dimensional. Given a polarization $\mathfrak{a} = \mathfrak{a}^- \oplus \mathfrak{a}^+$, we let $\mathfrak{z}$ be the associated lift of $-\mathrm{Id}_{\mathfrak{a}}$, so that $\mathfrak{z}\mathbf{v}_{\mathrm{tw}} = \mathbf{v}_{\mathrm{tw}}$ (see (3.9)). We write $A(\mathfrak{a})_{\mathrm{tw}} = A(\mathfrak{a})^0_{\mathrm{tw}} \oplus A(\mathfrak{a})^1_{\mathrm{tw}}$ for the decomposition of $A(\mathfrak{a})_{\mathrm{tw}}$ into eigenspaces for $\mathfrak{z}$ (see (3.10)).

Because it is the even part of a super vertex algebra, $A(\mathfrak{a})^0$ is itself a vertex algebra, and $A(\mathfrak{a})_{\mathrm{tw}}$ is an (untwisted) $A(\mathfrak{a})^0$-module. The decomposition $A(\mathfrak{a})_{\mathrm{tw}} = A(\mathfrak{a})^0_{\mathrm{tw}} \oplus A(\mathfrak{a})^1_{\mathrm{tw}}$ is a decomposition into submodules for $A(\mathfrak{a})^0$.

Consider the $A(\mathfrak{a})^0$-modules $V^{s\natural}$ and $V^{s\natural}_{\mathrm{tw}}$ defined by setting

$$V^{s\natural} = A(\mathfrak{a})^0 \oplus A(\mathfrak{a})^1_{\mathrm{tw}}, \quad V^{s\natural}_{\mathrm{tw}} = A(\mathfrak{a})^1 \oplus A(\mathfrak{a})^0_{\mathrm{tw}}. \tag{4.1}$$

PROPOSITION 4.1. *The $A(\mathfrak{a})^0$-module structure on $V^{s\natural}$ extends uniquely to a super vertex operator algebra structure on $V^{s\natural}$, and the $A(\mathfrak{a})^0$-module structure on $V^{s\natural}_{\mathrm{tw}}$ extends uniquely to a canonically twisted $V^{s\natural}$-module structure.*

Proof. We proceed along similar lines to the proof of Proposition 4.1 in [36]. Given a choice of polarization $\mathfrak{a} = \mathfrak{a}^- \oplus \mathfrak{a}^+$, the boson–fermion correspondence [45] defines an isomorphism of vertex operator algebras $A(\mathfrak{a})^0 \xrightarrow{\sim} V_L$, according to [32], where V_L is the lattice vertex operator algebra attached to

$$L = \left\{ (n_i) \in \mathbb{Z}^{12} \,\middle|\, \sum n_i = 0 \ (\mathrm{mod}\ 2) \right\}, \tag{4.2}$$

being a copy of the root lattice of type D_{12}. It also extends to compatible isomorphisms of vertex operator algebra modules

$$A(\mathfrak{a})^1 \xrightarrow{\sim} V_{L+\lambda_v}, \quad A(\mathfrak{a})^0_{\mathrm{tw}} \xrightarrow{\sim} V_{L+\lambda_s}, \quad A(\mathfrak{a})^1_{\mathrm{tw}} \xrightarrow{\sim} V_{L+\lambda_c}, \tag{4.3}$$

where the λ_x are representatives for the nontrivial cosets of L in its dual, $L^* = \frac{1}{2}\mathbb{Z}^{12}$,

$$\lambda_v = (1, 0, \ldots, 0), \quad \lambda_s = \tfrac{1}{2}(1, 1, \ldots, 1), \quad \lambda_c = \tfrac{1}{2}(-1, 1, \ldots, 1). \tag{4.4}$$

The irreducible modules for a lattice vertex operator algebra are known [26] to be in correspondence with the cosets of the lattice in its dual – that is, the discriminant group of the lattice, L^*/L – and the associated fusion algebra is

https://doi.org/10.1017/fms.2015.7 Published online by Cambridge University Press

naturally isomorphic to the group algebra $\mathbb{C}[L^*/L]$. (These facts are explained in detail in [27].)

From this we deduce that $V^{s\natural}$ is isomorphic to V_{L^+} as a V_L-module, where $L^+ = L \cup (L + \lambda_c)$. Note that L^+ is a self-dual integral lattice in the particular case at hand. (In fact, L^+ is the unique self-dual positive-definite integral lattice of rank 12 with no vectors of length 1, sometimes denoted D_{12}^+.) So the super vertex operator algebra structure on V_{L^+}, which uniquely extends the V_L-module structure according to the structure of the fusion algebra of V_L, furnishes the claimed super vertex operator algebra structure on $V^{s\natural}$, and $V^{s\natural}$ is self-dual.

By inspection we see that the canonical automorphism of V_{L^+} – arising from the super structure – coincides with that attached to the vector $\lambda_v \in \frac{1}{2}V_{L^+}$, since $e^{2\pi i\langle\lambda_v,\lambda\rangle}$ is 1 or -1 according as $\lambda \in L^+$ belongs to L or $L+\lambda_c$. This shows that the coset module $V_{L^++\lambda_v} = V_{L+\lambda_v} \oplus V_{L+\lambda_s}$ is the unique irreducible canonically twisted module for V_{L^+}, so $V^{s\natural}_{\mathrm{tw}}$ is the unique irreducible canonically twisted module for $V^{s\natural}$, according to (4.3). This completes the proof. □

We now equip $V^{s\natural}$ and $V^{s\natural}_{\mathrm{tw}}$ with module structures for the Conway group, Co_0.

As detailed in Section 3.1, the spin group $\mathrm{Spin}(\mathfrak{a})$ acts naturally on $A(\mathfrak{a})^j$ and $A(\mathfrak{a})^j_{\mathrm{tw}}$, so it acts naturally on $V^{s\natural}$ and $V^{s\natural}_{\mathrm{tw}}$. This action respects the super vertex algebra and canonically twisted module structures defined in Proposition 4.1, in the sense that (3.7) holds for $x \in \mathrm{Spin}(\mathfrak{a})$, $a, b \in V^{s\natural}$, and $c \in V^{s\natural}_{\mathrm{tw}}$.

Recalling the setup of Section 3.2, we now assume to be chosen an identification $\mathfrak{a} = \Lambda \otimes_{\mathbb{Z}} \mathbb{C}$, and write G for the corresponding copy of $Co_0 = \mathrm{Aut}(\Lambda)$ in $\mathrm{SO}(\mathfrak{a})$, isomorphic to Co_0. We take $\widehat{G}$ to be the lift of G to $\mathrm{Spin}(\mathfrak{a})$ (see Proposition 3.1), so that the restriction of the natural map $\mathrm{Spin}(\mathfrak{a}) \to \mathrm{SO}(\mathfrak{a})$ defines an isomorphism $\widehat{G} \xrightarrow{\sim} G$. We write $g \mapsto \widehat{g}$ for the inverse isomorphism, and in this way we obtain actions of the Conway group $\widehat{G} \simeq Co_0$ on $V^{s\natural}$ and $V^{s\natural}_{\mathrm{tw}}$.

Note that the actions on $V^{s\natural}$ and $V^{s\natural}_{\mathrm{tw}}$ depend upon the choice of polarization $\mathfrak{a} = \mathfrak{a}^- \oplus \mathfrak{a}^+$. We assume, according to the convention established in Section 3.2, that the polarization is chosen so that the associated lift $\mathfrak{z} \in \mathrm{Spin}(\mathfrak{a})$ of $-\mathrm{Id}_{\mathfrak{a}} \in \mathrm{SO}(\mathfrak{a})$ is the nontrivial central element of $\widehat{G}$.

$$\mathfrak{z} \in Z(\widehat{G}). \tag{4.5}$$

With this convention both $V^{s\natural}$ and $V^{s\natural}_{\mathrm{tw}}$ are faithful modules for $\widehat{G}$.

If $\mathfrak{z} \notin Z(\widehat{G})$ then we would have $-\mathfrak{z} \in Z(\widehat{G})$, and $-\mathfrak{z}$ acts trivially on both $A(\mathfrak{a})^0$ and $A(\mathfrak{a})^1_{\mathrm{tw}}$ (see Section 3.1). Thus for $-\mathfrak{z} \in Z(\widehat{G})$ the $\widehat{G}$-module structure on $V^{s\natural}$ would factor through to $\widehat{G}/Z(\widehat{G})$, being a copy of the simple group Co_1 (see Section 3.2).

Actually, such an action, not faithful for $\widehat{G}$, is useful for us, and it arises naturally when we consider the spaces $V^{f\natural}$ and $V^{f\natural}_{\mathrm{tw}}$, closely related to $V^{s\natural}$

and $V_{\rm tw}^{s\natural}$, defined by setting

$$V^{f\natural} = A(\mathfrak{a})^0 \oplus A(\mathfrak{a})^0_{\rm tw}, \quad V^{f\natural}_{\rm tw} = A(\mathfrak{a})^1 \oplus A(\mathfrak{a})^1_{\rm tw}. \tag{4.6}$$

Making obvious changes to the proof of Proposition 4.1 (that is, swapping $A(\mathfrak{a})^1_{\rm tw}$ with $A(\mathfrak{a})^0_{\rm tw}$, and λ_c with λ_s, etc.,), we obtain the following direct analogue of that result.

PROPOSITION 4.2. *The $A(\mathfrak{a})^0$-module structure on $V^{f\natural}$ extends uniquely to a super vertex operator algebra structure on $V^{f\natural}$, and the $A(\mathfrak{a})^0$-module structure on $V^{f\natural}_{\rm tw}$ extends uniquely to a canonically twisted $V^{f\natural}$-module structure.*

Actually, $V^{f\natural}$ is isomorphic to $V^{s\natural}$ as a super vertex operator algebra, since the proof of Proposition 4.1 shows that both are isomorphic to the lattice super vertex operator algebra attached to D_{12}^+, being the unique self-dual positive-definite integral lattice of rank 12 with no vectors of length 1.

The group $\widehat{G}$ acts naturally on $V^{f\natural}$ and $V^{f\natural}_{\rm tw}$ via the natural actions of Spin($\mathfrak{a}$), and it is as $\widehat{G}$-modules that the difference between $V^{s\natural}$ and $V^{f\natural}$ manifests: the centre of $\widehat{G}$ acts trivially on the latter, according to our convention (4.5).

To aid the reader in comparing the results here with those of [36], we mention that the $N = 1$ super vertex operator algebra ${}_{\mathbb{C}}A^{f\natural}$ studied there is isomorphic to $V^{s\natural} \simeq V^{f\natural}$ as a super vertex operator algebra. The main results of [36] include the statement that the full automorphism group (fixing the $N = 1$ element) of ${}_{\mathbb{C}}A^{f\natural}$ is a copy of the sporadic simple Conway group Co_1. So as Co_0-modules we have ${}_{\mathbb{C}}A^{f\natural} \simeq V^{f\natural}$, but ${}_{\mathbb{C}}A^{f\natural} \not\simeq V^{s\natural}$. A faithful Co_0-module structure on the super vertex operator algebra underlying ${}_{\mathbb{C}}A^{f\natural}$ is mentioned in Remark 4.12 of [36].

Following the method of [48] we may describe the super vertex operator algebra structure on $V^{s\natural} \simeq V^{f\natural}$ quite explicitly. For the sake of later applications we now present details for the realization $V^{f\natural}$.

So, we seek to describe the vertex operator correspondence on $V^{f\natural}$ explicitly (see (2.1)). According to the vertex algebra axioms we may regard this correspondence as a linear map $V^{f\natural} \otimes V^{f\natural} \to V^{f\natural}((z))$, denoted $a \otimes b \mapsto Y(a, z)b$, whose restriction to $A(\mathfrak{a})^0 \otimes V^{f\natural}$ is already defined by the $A(\mathfrak{a})^0$-module structure on $V^{f\natural} = A(\mathfrak{a})^0 \oplus A(\mathfrak{a})^0_{\rm tw}$. According to the fusion rules described in the proof of Proposition 4.1, we require to specify linear maps

$$A(\mathfrak{a})^0_{\rm tw} \otimes A(\mathfrak{a})^0 \to A(\mathfrak{a})^0_{\rm tw}((z)), \tag{4.7}$$

$$A(\mathfrak{a})^0_{\rm tw} \otimes A(\mathfrak{a})^0_{\rm tw} \to A(\mathfrak{a})^0((z)). \tag{4.8}$$

For (4.7) we apply the fact that a vertex operator correspondence should satisfy skew symmetry (see [48]) to conclude that

$$Y(a, z)b = e^{zL(-1)} Y(b, -z)a \tag{4.9}$$

for $a \in A(\mathfrak{a})^0_{\rm tw}$ and $b \in A(\mathfrak{a})^0$. The right-hand side of (4.9) is already defined, by the $A(\mathfrak{a})^0$-module structure on $V^{f\natural}$, so we may regard it as defining the left-hand side.

For (4.8) we use the nondegenerate bilinear forms on $A(\mathfrak{a})$ and $A(\mathfrak{a})_{\rm tw}$ (see (2.20) and (2.29)) to define $Y(a,z)b$ for $a, b \in A(\mathfrak{a})^0_{\rm tw}$ by requiring that

$$\langle Y(a,z)b, c\rangle = (-1)^n \langle e^{z^{-1}L(1)}b, Y(c,-z^{-1})e^{zL(1)}a\rangle_{\rm tw} z^{1-2n} \tag{4.10}$$

for all $c \in A(\mathfrak{a})^0$ when $a \in (A(\mathfrak{a})^0_{\rm tw})_{n-1/2}$ (see (2.11)).

The identity (4.10) ensures that the bilinear form on $V^{f\natural}$, obtained by restricting those on $A(\mathfrak{a})$ and $A(\mathfrak{a})_{\rm tw}$, is invariant (see (2.10)) for the given super vertex operator algebra structure. According to Theorem 2.1, an invariant bilinear form on $V^{f\natural}$ is uniquely determined, up to scale. Thus we arrive at the following result.

PROPOSITION 4.3. *The super vertex operator algebra $V^{f\natural}$ admits a unique invariant bilinear form such that $\langle \mathbf{v}, \mathbf{v}\rangle = 1$. It coincides with the bilinear form obtained by restriction from those defined above for $A(\mathfrak{a})$ and $A(\mathfrak{a})_{\rm tw}$.*

In [36] it is shown that the super vertex operator algebra $V^{s\natural} \simeq V^{f\natural}$ admits an $N = 1$ structure, with automorphism group isomorphic to Co_1. Our next result is a kind of converse to that, showing how to recover a Co_1-invariant $N = 1$ structure from an action by automorphisms of Co_1 on $V^{f\natural}$.

PROPOSITION 4.4. *Suppose to be given a nontrivial map $Co_1 \to \mathrm{Aut}(V^{f\natural})$. Then the resulting action of the simple Conway group Co_1 on $V^{f\natural}$ fixes a unique one-dimensional subspace of $(V^{f\natural})_{3/2}$. A suitably scaled vector in this subspace defines an $N = 1$ structure on $V^{f\natural}$.*

Proof. The content of Proposition 4.6 in [36] is that the full group of super vertex operator algebra automorphisms of $V^{f\natural}$ is $\mathrm{Spin}(\mathfrak{a})/\langle\mathfrak{z}\rangle$. So a nontrivial action of Co_1 on $V^{f\natural}$ by automorphisms realizes Co_1 as a subgroup of $\mathrm{Spin}(\mathfrak{a})/\langle\mathfrak{z}\rangle$. We write $\widehat{G}$ for the preimage of this copy of Co_1 in $\mathrm{Spin}(\mathfrak{a})$. Then either $\widehat{G} \simeq 2 \times Co_1$, or $\widehat{G}$ is isomorphic to the Conway group Co_0, since these are the only 2-fold covers of Co_1. In either case $\langle\mathfrak{z}\rangle$ is the only normal subgroup of order 2 in $\widehat{G}$, so $\widehat{G}$ has trivial intersection with the kernel of the natural map $\mathrm{Spin}(\mathfrak{a}) \to \mathrm{SO}(\mathfrak{a})$. We conclude that $\widehat{G} \simeq 2 \times Co_1$ is impossible, for otherwise the map $\mathrm{Spin}(\mathfrak{a}) \to \mathrm{SO}(\mathfrak{a})$ would furnish a nontrivial representation of Co_1 on $\mathfrak{a}$, and the minimal dimension of a nontrivial representation for Co_1 is 276 according to [21]. So $\widehat{G} \simeq Co_0$.

Write G for the image of $\widehat{G}$ in $\mathrm{SO}(\mathfrak{a})$. Then G is a copy of Co_0 in $\mathrm{SO}(\mathfrak{a})$ and $\widehat{G}$ is a lift of G to $\mathrm{Spin}(\mathfrak{a})$ such that $\mathfrak{z} \in Z(\widehat{G})$. The group G must preserve a copy Λ of the Leech lattice in $\mathfrak{a}$, so we are in the setup of Section 3.2, and our notation $\widehat{G}$, G, etc., is consistent with the conventions established there.

https://doi.org/10.1017/fms.2015.7 Published online by Cambridge University Press

Now let $S = \{\lambda_i\}_{i\in\Omega}$ be a coordinate frame for $\Lambda \subset \mathfrak{a}$ (see Section 3.2), and let $E = E_S$ be the subgroup of G consisting of elements which act by sign changes on the λ_i. Let $\mathcal{G}$ be the corresponding copy of the Golay code in $\mathcal{P}(\Omega)$. We will use E to construct the desired $N = 1$ element in $(A(\mathfrak{a})_{\text{tw}})^0_{3/2}$.

As explained in Section 2.3 we may use the isomorphism $\mathfrak{a} \xrightarrow{\sim} \mathfrak{a}(0)$ to identify $\text{Cliff}(\mathfrak{a})$ as a subalgebra of $\text{Cliff}(\hat{\mathfrak{a}}_{\text{tw}})$. In this way we may regard $A(\mathfrak{a})_{\text{tw}}$ as a $\text{Cliff}(\mathfrak{a})$-module, and we may identify $(A(\mathfrak{a})_{\text{tw}})_{3/2} = \text{CM}$ as the $\text{Cliff}(\mathfrak{a})$-submodule of $A(\mathfrak{a})_{\text{tw}}$ generated by $\mathbf{v}_{\text{tw}}$.

Define an idempotent element $t \in \text{Cliff}(\mathfrak{a})$ by setting

$$t = \frac{1}{4096} \sum_{g\in E} \widehat{g}, \tag{4.11}$$

where $g \mapsto \widehat{g}$ denotes the inverse of the natural isomorphism $\widehat{G} \to G$. Then t is not in the subalgebra of $\text{Cliff}(\mathfrak{a}) < \text{Cliff}(\hat{\mathfrak{a}}_{\text{tw}})$ generated by $\mathfrak{a}^+$, since the only idempotent in B^+ (see Section 2.3) is $\mathbf{1}$. So $t\mathbf{v}_{\text{tw}} \neq 0$, and

$$(A(\mathfrak{a})_{\text{tw}})_{3/2} = \text{CM} = \text{Cliff}(\mathfrak{a})t\mathbf{v}_{\text{tw}} = \{xt\mathbf{v}_{\text{tw}} \mid x \in \text{Cliff}(\mathfrak{a})\}, \tag{4.12}$$

since CM is an irreducible $\text{Cliff}(\mathfrak{a})$-module.

Now choose an ordering on the index set Ω, let $e_i = \frac{1}{\sqrt{8}}\lambda_i$ for $i \in \Omega$, and, given a subset $C = \{i_1, \ldots, i_k\} \subset \Omega$, with $i_1 < \cdots < i_k$, define an element $e_C \in \text{Cliff}(\mathfrak{a})$ by setting $e_C = e_{i_1}e_{i_2}\cdots e_{i_k}$. Then, taking $e_\emptyset = \mathbf{1}$, the set $\{e_C \mid C \subset \Omega\}$ furnishes a vector space basis for $\text{Cliff}(\hat{\mathfrak{a}})$, so CM is spanned by the vectors $e_C t\mathbf{v}_{\text{tw}}$ for $C \subset \Omega$. Also, $e_C e_D = \pm e_{C+D}$ (where the $+$ in the subscript on the right-hand side denotes the symmetric difference operation on $\mathcal{P}(\Omega)$).

Observe that $e_C t = \pm e_D t$ whenever C and D are equivalent modulo $\mathcal{G}$, since in that case one of e_{C+D} or $-e_{C+D}$ belongs to $\widehat{E} = \{\widehat{g} \mid g \in E\}$. So if $\mathcal{T}$ is a set of representatives for the cosets of $\mathcal{G}$ in $\mathcal{P}(\Omega)$ then CM is spanned by the $e_C t\mathbf{v}_{\text{tw}}$ for $C \in \mathcal{T}$. Since the Golay code is self-dual, $\mathcal{T}$ has cardinality 2^{12}, which is also the dimension of CM, so the $e_C t\mathbf{v}_{\text{tw}}$ for $C \in \mathcal{T}$ must in fact furnish a basis for CM.

We claim that $t\mathbf{v}_{\text{tw}}$ is $\widehat{G}$-invariant. Certainly it is $\widehat{E}$-invariant. Using the fact that the $e_C t\mathbf{v}_{\text{tw}}$ for $C \in \mathcal{T}$ form a basis for CM, we see that $t\mathbf{v}_{\text{tw}}$ is actually the only $\widehat{E}$-invariant vector in CM, because the space spanned by $e_C t\mathbf{v}_{\text{tw}}$ is a one-dimensional representation of $\widehat{E}$ with character given by $\chi(\widehat{g}) = (-1)^{\#(C\cap D)}$ in the case when $g = g(D)$. Since the Golay code is self-dual, we only have $\#(C\cap D) = 0 \pmod 2$ for all $D \in \mathcal{G}$ when $C + \mathcal{G} = \mathcal{G}$.

Consider the action of $\widehat{G}$ on CM^0. The central element of $\widehat{G}$ is $\mathfrak{z}$, which acts trivially on CM^0, so CM^0 is a direct sum of irreducible modules for $\widehat{G}/\langle\mathfrak{z}\rangle \simeq Co_1$. We have $\dim \text{CM}^0 = 2048$, so only irreducible representations of Co_1 with dimension not exceeding 2048 can arise. According to [21] there are exactly four

https://doi.org/10.1017/fms.2015.7 Published online by Cambridge University Press

irreducible representations, up to equivalence, that can appear, and they are each determined by their dimension: 1, 276, 299, or 1771. The equation $2048 = 276a + 299b + 1771c$ has no nonnegative integer solutions, so there must be at least one nonzero $\widehat{G}$-fixed vector in CM^0. Such a vector must also be fixed by $\widehat{E}$, and we have seen that $t\mathbf{v}_{\mathrm{tw}}$ is the only possibility, so $t\mathbf{v}_{\mathrm{tw}}$ is fixed by $\widehat{G}$, as was claimed.

It follows that the decomposition of CM^0 into irreducible representations for Co_1 is given by $2048 = 1 + 276 + 1771$, with the one-dimensional representation spanned by $t\mathbf{v}_{\mathrm{tw}}$. From this we may conclude that $t\mathbf{v}_{\mathrm{tw}}$ is not isotropic with respect to the invariant bilinear form $\langle\cdot\,,\,\cdot\rangle$ on $V^{f\natural}$. (See Proposition 4.3.) For the restriction of $\langle\cdot\,,\,\cdot\rangle$ to $\mathrm{CM}^0 = (A(\mathfrak{a})_{\mathrm{tw}})_{3/2}$ is $\mathrm{Spin}(\mathfrak{a})$-invariant, and hence also $\widehat{G}$-invariant, and so the above decomposition implies that a $\widehat{G}$-invariant map $f : \mathrm{CM}^0 \to \mathbb{C}$ that vanishes on $t\mathbf{v}_{\mathrm{tw}}$ vanishes everywhere. Take $f(v) = \langle v, t\mathbf{v}_{\mathrm{tw}}\rangle$ for $v \in \mathrm{CM}^0$ to conclude that, if $t\mathbf{v}_{\mathrm{tw}}$ is isotropic, then $\langle v, t\mathbf{v}_{\mathrm{tw}}\rangle = 0$ for all $v \in \mathrm{CM}^0$, but this contradicts the nondegeneracy of $\langle\cdot\,,\,\cdot\rangle_{\mathrm{tw}}$ on $A(\mathfrak{a})_{\mathrm{tw}}$, which can be easily checked from the defining identities, (2.28) and (2.29).

Now choose $\alpha \in \mathbb{C}$ such that $\tau = \alpha t\mathbf{v}_{\mathrm{tw}}$ satisfies $\langle\tau, \tau\rangle = 8$. Observe that $\langle e_C\tau, \tau\rangle = 0$ whenever $C \subset \{1, \ldots, 24\}$ has cardinality two or four. We conclude from Proposition 4.3 of [**36**] that τ is an $N = 1$ element for $V^{f\natural}$. This completes the proof. □

4.2. Characterization. In this short section we establish a characterization of the super vertex operator algebra structure on $V^{s\natural}$. This is a strengthening of the main theorem of Section 5.1 in [**36**], for we arrive at the same conclusion without the hypothesis of an $N = 1$ structure.

Recall that a super vertex operator algebra V is said to be *C_2-cofinite* if the subspace $\{a_{(-2)}b \mid a, b \in V\} < V$ has finite codimension in V. Following [**31**], we say that a super vertex operator algebra $V = (V, Y, \mathbf{v}, \omega)$ is of *CFT type* if the $L(0)$-grading $V = \bigoplus_{n\in\frac{1}{2}\mathbb{Z}} V_n$ is bounded from below by 0, and if V_0 is spanned by the vacuum vector $\mathbf{v}$. Note that a super vertex operator algebra (in the sense of Section 2.1) that is C_2-cofinite and of CFT type is *nice* in the sense of [**66**].

THEOREM 4.5. *There is a unique up to isomorphism C_2-cofinite rational super vertex operator algebra of CFT type $V^{s\natural}$ such that*

- *the central charge of $V^{s\natural}$ is* 12*;*
- *$V^{s\natural}$ is self-dual; and*
- $\deg(v) \neq \frac{1}{2}$ *for any nonzero $v \in V^{s\natural}$.*

Proof. We first show that V admits a unique invariant bilinear form. Since V is self-dual, the contragredient module V' is isomorphic to V, so there exists

https://doi.org/10.1017/fms.2015.7 Published online by Cambridge University Press

a V-module isomorphism $\phi : V \to V'$. As explained in Section 2.2, this determines a nondegenerate invariant bilinear form $\langle\cdot\,,\cdot\rangle$ on V. Now let $a \in V_1$. We claim that $L(1)a = 0$. In any case, $L(1)a \in V_0$, so $L(1)a = C\mathbf{v}$ for some $C \in \mathbb{C}$, since V is of CFT type. Thus we have $Y^\dagger(a, z) = (-1)Y(a, z^{-1})z^{-2} + (-C)\,\mathrm{Id}_V\, z^{-1}$. (See (2.11).) Applying axiom (2) from the super vertex algebra definition in Section 2.1, we see that $\langle Y(a, z)\mathbf{v}, \mathbf{v}\rangle = 0$, since $a_{(-n-1)}\mathbf{v} \in V_{n+1}$, and $\langle V_m, V_n\rangle = 0$ unless $m = n$ (see Section 2.2). On the other hand, $\langle Y(a, z)\mathbf{v}, \mathbf{v}\rangle = \langle \mathbf{v}, Y^\dagger(a, z)\mathbf{v}\rangle$ by invariance, so

$$0 = \langle \mathbf{v}, Y^\dagger(a, z)\mathbf{v}\rangle = (-1)\langle \mathbf{v}, Y(a, z^{-1})\mathbf{v}\rangle z^{-2} + (-C)\langle \mathbf{v}, \mathbf{v}\rangle z^{-1}. \tag{4.13}$$

Now the coefficient of z^{-1} in $Y(a, z^{-1})\mathbf{v}z^{-2}$ is $a_{(0)}\mathbf{v}$, which vanishes by another application of axiom (2). Since $\langle\cdot\,,\cdot\rangle$ is nondegenerate, and V_0 is spanned by $\mathbf{v}$, we must have $\langle \mathbf{v}, \mathbf{v}\rangle \neq 0$. So we must have $C = 0$. This verifies our claim that $L(1)V_1 \subset \{0\}$. Now we apply Theorem 2.1 to conclude that the invariant bilinear form $\langle\cdot\,,\cdot\rangle$ just constructed is in fact the unique invariant bilinear form on V.

Since V_1 is in the kernel of $L(1)$, we may conclude that the even sub vertex operator algebra $V_{\bar{0}}$ is *strongly rational* in the sense of Section 3 of [**33**]. This allows us to apply Theorem 1 of [**33**], which tells us that the Lie algebra structure on V_1, obtained by setting $[a, b] := a_{(0)}b$ for $a, b \in V_1$, is reductive. The argument used to prove Theorem 2 of [**33**] – see the proof of Theorem 5.12 in [**36**] for details – then shows that the Lie rank of V_1 is bounded above by the central charge of V. Applying Proposition 5.14 in [**36**], we conclude that V_1 is a semisimple Lie algebra of dimension 276, with Lie rank bounded above by 12.

At this point our argument has converged with that used to establish Theorem 5.15 in [**36**]. Picking up at the second paragraph of the proof of Theorem 5.15 in [**36**], we see that an application of the main result of [**34**] shows that V_1 is the simple complex Lie algebra of type D_{12}, and that the vertex operators on V equip V with a module structure of level 1 for the affine Lie algebra of type $D_{12}^{(1)}$. So $V_{\bar{0}}$ is isomorphic to the lattice vertex operator algebra attached to the D_{12} lattice. Proceeding as in the proof of Proposition 4.1, we see that V itself is isomorphic to a lattice super vertex operator algebra, and the lattice must be that obtained by adjoining some λ_x say, of (4.4), to V. Since $V_{1/2} = \{0\}$, either $\lambda_x = \lambda_s$ or $\lambda_x = \lambda_c$, but both choices define isomorphic lattices, and hence isomorphic super vertex operator algebras. This completes the proof. □

4.3. Principal moduli. The spin group $\mathrm{Spin}(\mathfrak{a})$ acts naturally on $V^{s\natural}$ and $V^{s\natural}_{\mathrm{tw}}$, respecting the super vertex algebra and canonically twisted module structures, in the sense that (3.7) holds for $x \in \mathrm{Spin}(\mathfrak{a})$, $a, b \in V^{s\natural}$, and $c \in V^{s\natural}_{\mathrm{tw}}$. In particular, the $\mathrm{Spin}(\mathfrak{a})$-actions preserve the gradings defined by $L(0)$. We may compute the associated graded traces explicitly, and will do so shortly (see Lemma 4.6).

https://doi.org/10.1017/fms.2015.7 Published online by Cambridge University Press

Since $V^{s\natural}$ and $V^{s\natural}_{\mathrm{tw}}$ are super spaces, it is natural to consider graded super traces. Recall that the *super trace* of a parity-preserving operator X on a super vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$ is defined by setting

$$\mathrm{str}_V\, X := \mathrm{tr}_{V_{\bar{0}}}\, X - \mathrm{tr}_{V_{\bar{1}}}\, X. \tag{4.14}$$

(By parity preserving we just mean $X(V_{\bar{j}}) \subset V_{\bar{j}}$.)

Observe that the super space gradings on $A(\mathfrak{a})$ and $A(\mathfrak{a})_{\mathrm{tw}}$ are given by the eigenspace decompositions for $\mathfrak{z}$, so we have

$$\mathrm{str}_{A(\mathfrak{a})}\, Xq^{L(0)-c/24} = \mathrm{tr}_{A(\mathfrak{a})}\, \mathfrak{z}Xq^{L(0)-c/24}, \tag{4.15}$$

$$\mathrm{str}_{A(\mathfrak{a})_{\mathrm{tw}}}\, Xq^{L(0)-c/24} = \mathrm{tr}_{A(\mathfrak{a})_{\mathrm{tw}}}\, \mathfrak{z}Xq^{L(0)-c/24}, \tag{4.16}$$

for X an operator on $A(\mathfrak{a})$, $A(\mathfrak{a})_{\mathrm{tw}}$, that commutes with $L(0)$ and $\mathfrak{z}$.

Given $g \in \mathrm{SO}(\mathfrak{a})$, define $\eta_g(\tau)$ by setting

$$\eta_g(\tau) := q \prod_{i=1}^{24} \prod_{n>0} (1 - \varepsilon_i q^n), \tag{4.17}$$

where $q = e^{2\pi \mathbf{i}\tau}$ and the ε_i are the eigenvalues for the action of g on $\mathfrak{a}$. Given $x \in \mathrm{Spin}(\mathfrak{a})$, write C_x for the super trace of x (that is, the ordinary trace of $\mathfrak{z}x$) as an operator on CM.

$$C_x := \mathrm{str}_{\mathrm{CM}}\, x = \mathrm{tr}_{\mathrm{CM}}\, \mathfrak{z}x. \tag{4.18}$$

A simple calculation now reveals that the graded super traces of $x \in \mathrm{Spin}(\mathfrak{a})$ on $A(\mathfrak{a})$ and $A(\mathfrak{a})_{\mathrm{tw}}$ are given by

$$\mathrm{str}_{A(\mathfrak{a})}\, xq^{L(0)-c/24} = \frac{\eta_{\overline{x}}(\tau/2)}{\eta_{\overline{x}}(\tau)}, \tag{4.19}$$

$$\mathrm{str}_{A(\mathfrak{a})_{\mathrm{tw}}}\, xq^{L(0)-c/24} = C_x \eta_{\overline{x}}(\tau), \tag{4.20}$$

where $\overline{x}$ is a shorthand for $x(\cdot)$, being the image of x in $\mathrm{SO}(\mathfrak{a})$. Note that $c = 12$ since $\dim \mathfrak{a} = 24$. (See (2.19).)

This leads us quickly to expressions for the graded super traces of an arbitrary $x \in \mathrm{Spin}(\mathfrak{a})$ on $V^{s\natural}$ and $V^{s\natural}_{\mathrm{tw}}$, which we record in the following lemma.

LEMMA 4.6. *For $x \in \mathrm{Spin}(\mathfrak{a})$, the graded super traces for the actions of x on $V^{s\natural}$ and $V^{s\natural}_{\mathrm{tw}}$ are given by*

$$\mathrm{str}_{V^{s\natural}}\, xq^{L(0)-c/24} = \frac{1}{2}\left(\frac{\eta_{\overline{x}}(\tau/2)}{\eta_{\overline{x}}(\tau)} + \frac{\eta_{-\overline{x}}(\tau/2)}{\eta_{-\overline{x}}(\tau)} + C_x\eta_{\overline{x}}(\tau) - C_{\mathfrak{z}x}\eta_{-\overline{x}}(\tau)\right), \tag{4.21}$$

$$\mathrm{str}_{V^{s\natural}_{\mathrm{tw}}}\, xq^{L(0)-c/24} = \frac{1}{2}\left(\frac{\eta_{\overline{x}}(\tau/2)}{\eta_{\overline{x}}(\tau)} - \frac{\eta_{-\overline{x}}(\tau/2)}{\eta_{-\overline{x}}(\tau)} + C_x\eta_{\overline{x}}(\tau) + C_{\mathfrak{z}x}\eta_{-\overline{x}}(\tau)\right). \tag{4.22}$$

https://doi.org/10.1017/fms.2015.7 Published online by Cambridge University Press

The construction of Section 4.1 equips $V^{s\natural}$ and $V^{s\natural}_{\mathrm{tw}}$ with actions by a group $\widehat{G} < \mathrm{Spin}(\mathfrak{a})$ isomorphic to the Conway group, Co_0. We recall our convention (see Section 3.2) that any polarization $\mathfrak{a} = \mathfrak{a}^- \oplus \mathfrak{a}^+$ used to realize $V^{s\natural}$ and $V^{s\natural}_{\mathrm{tw}}$ is chosen so that the associated lift $\mathfrak{z}$ of -1 (see Section 3.1) belongs to $\widehat{G}$.

Lemma 4.6 now attaches two holomorphic functions on the upper half-plane to each conjugacy class $[g] \subset Co_0$, namely, the super traces defined by (4.21) and (4.22).

$$T^s_g(\tau) := \mathrm{str}_{V^{s\natural}}\, \widehat{g} q^{L(0)-c/24} \tag{4.23}$$

$$T^s_{g,\mathrm{tw}}(\tau) := \mathrm{str}_{V^{s\natural}_{\mathrm{tw}}}\, \widehat{g} q^{L(0)-c/24}. \tag{4.24}$$

These functions T^s_g and $T^s_{g,\mathrm{tw}}$ are special. In order to demonstrate this we first recall the following result due to Conway and Norton, and to Queen.

Theorem 4.7 [19, 91]. *For any $g \in Co_0$, regarded as a subgroup of* $\mathrm{SO}(\mathfrak{a})$, *the function*

$$t_g(\tau) := \frac{\eta_g(\tau)}{\eta_g(2\tau)} \tag{4.25}$$

is a principal modulus for a genus-zero group $\Gamma_g < \mathrm{SL}_2(\mathbb{R})$ containing some $\Gamma_0(N)$.

We recall that a subgroup $\Gamma < \mathrm{SL}_2(\mathbb{R})$ commensurable with $\mathrm{SL}_2(\mathbb{Z})$ is said to have *genus zero* if the natural Riemann surface structure on the quotient $\Gamma\backslash\mathbb{H}\cup\hat{\mathbb{Q}}$ has genus zero, where $\hat{\mathbb{Q}} = \mathbb{Q}\cup\{\infty\}$. If Γ has genus zero then the field of Γ-invariant meromorphic functions on $\mathbb{H}$, with possible poles at the cusps $\Gamma\backslash\hat{\mathbb{Q}}$, is a simple transcendental extension of $\mathbb{C}$. A generator is called a *principal modulus* for Γ. (The term Hauptmodul is also commonly used for this.)

Note that a complete description of the invariance groups of the t_g appears in [74]. The reader may see also the tables in Appendix A.

Comparing with (4.19), we see that the principal moduli t_g of Conway and Norton and of Queen are recovered in a simple way from the action of $\widehat{G}$ on $A(\mathfrak{a})$. Namely,

$$t_g(\tau) = \mathrm{str}_{A(\mathfrak{a})}\, \widehat{g} q^{2L(0)-c/12} = \mathrm{tr}_{A(\mathfrak{a})}\, \mathfrak{z}\widehat{g} q^{2L(0)-c/12} \tag{4.26}$$

for $g \in \widehat{G} \simeq Co_0$. So the trace functions obtained from the action of Co_0 on $A(\mathfrak{a})$ are principal moduli, according to Theorem 4.7.

We will show that the functions T^s_g, defined in (4.21) by the action of Co_0 on the super vertex operator algebra $V^{s\natural}$, are also principal moduli for all $g \in \widehat{G} \simeq Co_0$, but are distinguished in that they also satisfy the normalization condition

$$T^s_g(2\tau) = q^{-1} + O(q). \tag{4.27}$$

Note that this condition does not hold in general for t_g, for we have $t_g(\tau) = q^{-1} - \chi_g + O(q)$, where χ_g denotes the character of $G \simeq Co_0$ defined by its action on $\mathfrak{a} = \Lambda \otimes_{\mathbb{Z}} \mathbb{C}$ (see (3.14)). The discrete subgroups of $\mathrm{SL}_2(\mathbb{R})$ attached to Co_0 via the T_g^s are essentially the same as those arising from the t_g of Conway and Norton, and of Queen: it will develop that $T_g^s(2\tau) = t_g(\tau) + \chi_g$ for all $g \in Co_0$. (See Theorem 4.9.)

For explicit computations with T_g^s and $T_{g,\mathrm{tw}}^s$, the notion of Frame shape is useful. Since $G \simeq Co_0$ is the automorphism group of an integral lattice in $\mathfrak{a}$ (see Section 3.2), the traces χ_g for $g \in G$ are all integers. So the characteristic polynomial for the action of $g \in G$ on $\mathfrak{a}$ can be written in the form $\prod_{m>0}(1-x^m)^{k_m}$ for some nonnegative integers k_m (all but finitely many being zero). In this situation then we find $\eta_g(\tau) = \prod_{m>0} \eta(m\tau)^{k_m}$ upon comparing with (4.17).

The formal product

$$\pi_g := \prod_{m>0} m^{k_m} \tag{4.28}$$

is called the *Frame shape* of g. We may define $\eta_\pi(\tau)$ for an arbitrary formal product $\pi = \prod_{m>0} m^{k_m}$ (with all but finitely many k_m equal to zero) by setting

$$\eta_\pi(\tau) := \prod_{m>0} \eta(m\tau)^{k_m}. \tag{4.29}$$

Of course then $\eta_g = \eta_{\pi_g}$ for $g \in G$.

Thus T_g^s and $T_{g,\mathrm{tw}}^s$ can be expressed explicitly in terms of the data $\pi_{\pm g}$ and $C_{\widehat{\pm g}}$. (Note that $\widehat{-g} = \mathfrak{z}\hat{g}$.) This data is collected in the tables of Appendix A for all $g \in Co_0$. Note that the trace of g as an operator on $\mathfrak{a}$ can also be read off from the Frame shape of g, for if $\pi_g = \prod_{m>0} m^{k_m}$ then $\chi_g = k_1$.

The values $C_{\hat{g}}$ are determined up to a sign by the eigenvalues of g. Indeed, suppose that $g \in G \simeq Co_0$ and assume, as in the discussion immediately following Lemma 4.6, that $\mathfrak{a} = \mathfrak{a}^- \oplus \mathfrak{a}^+$ is a polarization such that $\mathfrak{a}^\pm$ is spanned by (isotropic) eigenvectors $a_i^\pm$ for g, constituting a pair of dual bases in the sense that $\langle a_i^-, a_j^+ \rangle = \delta_{i,j}$. Assume also, as usual, that the associated lift $\mathfrak{z}$ of $-\mathrm{Id}_{\mathfrak{a}}$ belongs to our chosen copy $\widehat{G}$ of Co_0 in $\mathrm{Spin}(\mathfrak{a})$. Write $\lambda_i^{\pm 1}$ for the eigenvalue of g attached to $a_i^\pm$. Then, after choosing $\alpha_i \in 2\pi\mathbb{Q}$ such that $\lambda_i^{\pm 1} = e^{\pm 2\alpha_i \mathbf{i}}$, we see that the product $x = \prod_{i=1}^{12} e^{\alpha_i X_i}$ is a lift of g to $\mathrm{Spin}(\mathfrak{a})$, where the $X_i \in \mathfrak{g} < \mathrm{Cliff}(\mathfrak{a})$ are defined as in (3.8). Setting $\nu_i = e^{\alpha_i \mathbf{i}}$, we obtain that the trace of x on CM is given by $\prod_{i=1}^{12}(\nu_i + \nu_i^{-1})$, or, equivalently, by $\nu \prod_{i=1}^{12}(1 + \lambda_i^{-1})$,where $\nu = \prod_{i=1}^{12} \nu_i$ is one of the two square roots of $\prod_{i=1}^{12} \lambda_i$.

We conclude from this that $C_{\hat{g}}$ is given by

$$C_{\hat{g}} = \nu \prod_{i=1}^{12} (1 - \lambda_i^{-1}) \tag{4.30}$$

https://doi.org/10.1017/fms.2015.7 Published online by Cambridge University Press

(see (4.18)), where ν is one of the two square roots of $\prod_{i=1}^{12}\lambda_i$. In particular, $C_{\hat{g}} = 0$ if and only if g has a fixed point in $\mathfrak{a}$.

Our proof that the T_g^s are normalized principal moduli depends upon the following lemma.

LEMMA 4.8. *For $g \in G \simeq Co_0$ we have*

$$2\chi_g - \frac{\eta_{-g}(\tau/2)}{\eta_{-g}(\tau)} + \frac{\eta_g(\tau/2)}{\eta_g(\tau)} + C_{\widehat{-g}}\eta_{-g}(\tau) - C_{\hat{g}}\eta_g(\tau) = 0. \tag{4.31}$$

Proof. Rewrite the required identity (4.31) in the form

$$\frac{1}{2}\left(\frac{\eta_{-g}(\tau/2)}{\eta_{-g}(\tau)} - \frac{\eta_g(\tau/2)}{\eta_g(\tau)}\right) = \chi_g + \frac{1}{2}(C_{\widehat{-g}}\eta_{-g}(\tau) - C_{\hat{g}}\eta_g(\tau)). \tag{4.32}$$

Then, noting the identities (4.19) and (4.20), we recognize the left-hand side of (4.32) as $\operatorname{tr}_{A(\mathfrak{a})^1} \hat{g}q^{L(0)-c/24}$, and the right-hand side as $\chi_g + \operatorname{tr}_{A(\mathfrak{a})^1_{\mathrm{tw}}} \hat{g}q^{L(0)-c/24}$. We now modify slightly the notational convention (2.5) to write

$$A(\mathfrak{a})^1 = \bigoplus_{n\in\mathbb{Z}}(A(\mathfrak{a})^1)_n \tag{4.33}$$

for the grading of $A(\mathfrak{a})^1$ arising from the action of $L(0) - \frac{1}{2}\mathrm{Id}$, and similarly for $A(\mathfrak{a})^1_{\mathrm{tw}}$. Observe that both gradings are concentrated in nonnegative degrees, positive in the case of $A(\mathfrak{a})^1_{\mathrm{tw}}$. Also, $(A(\mathfrak{a})^1)_0$ is isomorphic to $\mathfrak{a}$ as a $\widehat{G}$-module, by construction. So we require to show that $(A(\mathfrak{a})^1)_n \simeq (A(\mathfrak{a})^1_{\mathrm{tw}})_n$ as $\widehat{G}$-modules, for each positive integer n.

Recall (see Proposition 4.2) that $V^{f\natural}_{\mathrm{tw}} = A(\mathfrak{a})^1 \oplus A(\mathfrak{a})^1_{\mathrm{tw}}$ may be regarded as the unique canonically twisted module for $V^{f\natural} = A(\mathfrak{a})^0 \oplus A(\mathfrak{a})^0_{\mathrm{tw}}$. Recall also (see Proposition 4.4) that there is a uniquely determined $\widehat{G}$-invariant $N = 1$ element $\tau \in V^{f\natural}$. Then the Fourier components of the twisted vertex operator $Y_{\mathrm{tw}}(\tau, z) : V^{f\natural}_{\mathrm{tw}} \to V^{f\natural}_{\mathrm{tw}}((z))$ define an action of the Ramond algebra (see Section 2.1) on $V^{f\natural}_{\mathrm{tw}}$. The defining relations show that $G(0) := \tau_{(1/2)}$ commutes with $L(0)$, and therefore preserves the subspaces $(A(\mathfrak{a})^1)_n \oplus (A(\mathfrak{a})^1_{\mathrm{tw}})_n < V^{f\natural}_{\mathrm{tw}}$. According to the fusion rules described in the proof of Proposition 4.1, the restriction of $G(0)$ to $A(\mathfrak{a})^1$ must map to $A(\mathfrak{a})^1_{\mathrm{tw}}$ (and *vice versa*). Now $G(0)^2 = L(0) - \frac{1}{2}\mathrm{Id}$, so $G(0)$ defines an injective map $(A(\mathfrak{a})^1)_n \to (A(\mathfrak{a})^1_{\mathrm{tw}})_n$ for all positive integers n. Since τ is $\widehat{G}$-invariant, these maps $(A(\mathfrak{a})^1)_n \to (A(\mathfrak{a})^1_{\mathrm{tw}})_n$ are embeddings of $\widehat{G}$-modules.

The lemma follows then if we can verify that $(A(\mathfrak{a})^1)_n$ and $(A(\mathfrak{a})^1_{\mathrm{tw}})_n$ have the same dimension, for all $n > 0$. That is, we should verify the $g = e$ case of (4.32), which is the identity

$$\frac{1}{2}\left(\frac{\Delta(\tau)^2}{\Delta(2\tau)\Delta(\tau/2)} - \frac{\Delta(\tau/2)}{\Delta(\tau)}\right) = 24 + 2^{11}\frac{\Delta(2\tau)}{\Delta(\tau)}, \tag{4.34}$$

where $\Delta(\tau) = \eta(\tau)^{24}$ is the Ramanujan Delta function. This can be checked in a number of ways. For example, $f(\tau) = \Delta(2\tau)/\Delta(\tau)$ is a $\Gamma_0(2)$-invariant function, so the same is true of

$$(T_2 f)(\tau) := \frac{1}{2}\left(f\left(\frac{\tau}{2}\right) + f\left(\frac{\tau+1}{2}\right)\right). \tag{4.35}$$

(See [73, Section IX.6].) Now f is actually a principal modulus for $\Gamma_0(2)$, with a simple pole at the unique noninfinite cusp, so $T_2 f$ has a pole of order at most 2 at the noninfinite cusp of $\Gamma_0(2)$, and no other poles. So $T_2 f$ is a polynomial in f, of degree at most 2; that is,

$$T_2 f = af^2 + bf + c \tag{4.36}$$

for some a, b, and c. Inspecting the first four coefficients of f, we see that $a = 2048 = 2^{11}$, $b = 24$, and $c = 0$. Observing that $f((\tau+1)/2) = -f(\tau)/f(\tau/2)$, we now obtain

$$\frac{1}{2}\left(f\left(\frac{\tau}{2}\right) - \frac{f(\tau)}{f(\frac{\tau}{2})}\right) = 24 f(\tau) + 2^{11} f(\tau)^2 \tag{4.37}$$

from (4.36), and (4.34) follows upon division of (4.37) by $f(\tau)$. The proof of the lemma is complete. □

We now come to the main results of this paper.

THEOREM 4.9. *Let $g \in Co_0$. Then T_g^s is the normalized principal modulus for a genus-zero subgroup of* $\mathrm{SL}_2(\mathbb{R})$.

THEOREM 4.10. *Let $g \in Co_0$. Then $T_{g,\mathrm{tw}}^s$ is constant, with constant value $-\chi_g$, when g has a fixed point in its action on the Leech lattice. If g has no fixed points then $T_{g,\mathrm{tw}}^s$ is a principal modulus for a genus-zero subgroup of* $\mathrm{SL}_2(\mathbb{R})$.

It is convenient to prove Theorems 4.9 and 4.10 together.

Proof of Theorems 4.9 and 4.10. With $\widehat{G}$ a lift of $G \simeq Co_0$ to $\mathrm{Spin}(\mathfrak{a})$ as before, define

$$\tilde{t}_g(\tau) := t_g(\tau/2) = \mathrm{str}_{A(\mathfrak{a})}\, \hat{g} q^{L(0)-c/24} \tag{4.38}$$

for $g \in G$ (see (4.25) and (4.26)), and define also the twisted analogues,

$$\tilde{t}_{g,\mathrm{tw}}(\tau) := \mathrm{str}_{A(\mathfrak{a})_{\mathrm{tw}}}\, \hat{g} q^{L(0)-c/24}. \tag{4.39}$$

Then $\tilde{t}_{g,\mathrm{tw}}(\tau) = C_{\hat{g}}\eta_g(\tau)$ according to (4.20), and so $\tilde{t}_{g,\mathrm{tw}}$ vanishes identically if and only if g has a fixed point for its action on $\mathfrak{a} = \Lambda \otimes_{\mathbb{Z}} \mathbb{C}$ according to (4.30).

https://doi.org/10.1017/fms.2015.7 Published online by Cambridge University Press

Using (4.21) and (4.22), we may now write

$$T^s_g = \tfrac{1}{2}(\tilde{t}_g + \tilde{t}_{-g} + \tilde{t}_{g,\mathrm{tw}} - \tilde{t}_{-g,\mathrm{tw}}), \tag{4.40}$$

$$T^s_{g,\mathrm{tw}} = \tfrac{1}{2}(\tilde{t}_g - \tilde{t}_{-g} + \tilde{t}_{g,\mathrm{tw}} + \tilde{t}_{-g,\mathrm{tw}}), \tag{4.41}$$

and the identity (4.31) may be rewritten as $\chi_g + \frac{1}{2}(\tilde{t}_g - \tilde{t}_{-g} - \tilde{t}_{g,\mathrm{tw}} + \tilde{t}_{-g,\mathrm{tw}}) = 0$. So, applying (4.31) to (4.40) and (4.41), we obtain

$$T^s_g = \tilde{t}_g + \chi_g, \tag{4.42}$$

$$T^s_{g,\mathrm{tw}} = \tilde{t}_{g,\mathrm{tw}} - \chi_g. \tag{4.43}$$

Since $t_g = q^{-1} - \chi_g + O(q)$ by inspection, (4.42) verifies that $T^s_g(2\tau) = q^{-1} + O(q)$ for $g \in G$, and so $T^s_g(2\tau)$ is a normalized principal modulus according to Theorem 4.7. This proves Theorem 4.9.

Equation (4.43) verifies that $T^s_{g,\mathrm{tw}}$ is constant, with constant value $-\chi_g$, when g has a fixed point for its action on the Leech lattice according to the first paragraph of this proof, so it remains to understand $T^s_{g,\mathrm{tw}}$ in the case that g has no fixed points.

Observe that $C_{\hat{g}}/\tilde{t}_{g,\mathrm{tw}} = 1/\eta_g$. If g has no fixed points then the Frame shape $\pi_g = \prod_{m>0} m^{k_m}$ satisfies $\sum_{m>0} k_m = 0$, so $1/\eta_g$ is a modular function for some congruence subgroup of $SL_2(\mathbb{Z})$. In fact, it has been verified in [19] that $1/\eta_g$ is, up to an additive constant, the McKay–Thompson series of an element of the monster group, for every such g in the Conway group. So $T^s_{g,\mathrm{tw}} = -\chi_g + C_{\hat{g}}\eta_g$ is indeed a principal modulus for a genus-zero subgroup of $SL_2(\mathbb{R})$, whenever g has no fixed points in Λ. The monster elements corresponding to elements $g \in Co_0$ without fixed points may be read off from Table A.2 in Appendix A. This completes the proof of Theorem 4.10. □

Acknowledgements

We thank Miranda Cheng, Xi Dong, Igor Frenkel, Matthias Gaberdiel, Terry Gannon, Sarah Harrison, Jeff Harvey, Shamit Kachru, and Timm Wrase for discussions on related topics. We also thank Jeff Harvey, Zhong Zheng, and the anonymous referee, for helpful comments and corrections to earlier drafts. The first author gratefully acknowledges support from the Simons Foundation (#316779) and the US National Science Foundation (DMS 1203162).

Appendix A. Data

In Tables A.1 and A.2, we give all the data necessary for explicit computation of the McKay–Thompson series T^s_g (see (4.23)) and $T^s_{g,\mathrm{tw}}$ (see (4.24)), attached to the Conway group Co_0 in this paper.

Table A.1. Data for the T_g^s.

Co_0	Co_1	π_g	$C_{\widehat{-g}}$	Γ_g
1A	1A	1^{24}	4096	$2-$
2A	1A	$2^{24}/1^{24}$	0	$4+$
2B	2A	$1^8 2^8$	0	$4-$
2C	2A	$2^{16}/1^8$	0	$4-$
4A	2B	$4^{12}/2^{12}$	64	$8\|2+$
2D	2C	2^{12}	0	$4\|2-$
3A	3A	$3^{12}/1^{12}$	1	$6+6$
6A	3A	$1^{12}6^{12}/2^{12}3^{12}$	729	$(6+6)^{\triangle\frac{1}{2}}$
3B	3B	$1^6 3^6$	64	$6+3$
6B	3B	$2^6 6^6/1^6 3^6$	0	$12+$
3C	3C	$3^9/1^3$	-8	$6-$
6C	3C	$1^3 6^9/2^3 3^9$	0	$12+4$
3D	3D	3^8	16	$6\|3$
6D	3D	$6^8/3^8$	0	$12\|3+$
4B	4A	$1^8 4^8/2^8$	256	$(8+)^{\triangle\frac{1}{2}}$
4C	4A	$4^8/1^8$	0	$8+$
4D	4B	$4^8/2^4$	0	$8-$
4E	4C	$1^4 2^2 4^4$	0	$8-$
4F	4C	$2^6 4^4/1^4$	0	$8-$
4G	4D	$2^4 4^4$	0	$8\|2-$
8A	4E	$8^6/4^6$	8	$16\|4+$
4H	4F	4^6	0	$8\|4-$
5A	5A	$5^6/1^6$	1	$10+10$
10A	5A	$1^6 10^6/2^6 5^6$	125	$(10+10)^{\triangle\frac{1}{2}}$
5B	5B	$1^4 5^4$	16	$10+5$
10B	5B	$2^4 10^4/1^4 5^4$	0	$20+$
5C	5C	$5^5/1^1$	-4	$10-$
10C	5C	$1^1 10^5/2^1 5^5$	0	$20+4$
6E	6A	$3^4 6^4/1^4 2^4$	9	$12+12$
6F	6A	$1^4 6^8/2^8 3^4$	81	$(12+12)^{\triangle\frac{1}{2}}$
12A	6B	$2^6 12^6/4^6 6^6$	1	$(12\|2+6)^{\triangle\frac{1}{4}}$
6G	6C	$2^5 3^4 6^1/1^4$	0	$12+3\triangle\frac{1}{2}$
6H	6C	$1^4 2^1 6^5/3^4$	0	$12+3\triangle\frac{1}{2}$
6I	6D	$1^5 3^1 6^4/2^4$	72	$(12+12)^{\triangle\frac{1}{2}}$
6J	6D	$2^1 6^5/1^5 3^1$	0	$12+12$

https://doi.org/10.1017/fms.2015.7 Published online by Cambridge University Press

Table A.1. *Continued*.

Co_0	Co_1	π_g	$C_{-\widehat{g}}$	Γ_g
6K	6E	$1^2 2^2 3^2 6^2$	0	$12+3$
6L	6E	$2^4 6^4/1^2 3^2$	0	$12+3$
6M	6F	$3^3 6^3/1^1 2^1$	0	$12-$
6N	6F	$1^1 6^6/2^2 3^3$	0	$12-$
6O	6G	$2^3 6^3$	0	$12\|2+3 \triangle \frac{1}{2}$
12B	6H	$12^4/6^4$	4	$24\|6+$
6P	6I	6^4	0	$12\|6-$
7A	7A	$7^4/1^4$	1	$14+14$
14A	7A	$1^4 14^4/2^4 7^4$	49	$(14+14)^{\triangle\frac{1}{2}}$
7B	7B	$1^3 7^3$	8	$14+7$
14B	7B	$2^3 14^3/1^3 7^3$	0	$28+$
8B	8A	$8^4/2^4$	16	$16\|2+$
8C	8B	$2^4 8^4/4^4$	0	$(16\|2+)^{\triangle\frac{1}{4}}$
8D	8C	$1^4 8^4/2^2 4^2$	32	$(16+)^{\triangle\frac{1}{2}}$
8E	8C	$2^2 8^4/1^4 4^2$	0	$16+$
8F	8D	$8^4/4^2$	0	$16-$
8G	8E	$1^2 2^1 4^1 8^2$	0	$16-$
8H	8E	$2^3 4^1 8^2/1^2$	0	$16-$
8I	8F	$4^2 8^2$	0	$16\|4-$
9A	9A	$9^3/1^3$	1	$18+18$
18A	9A	$1^3 18^3/2^3 9^3$	27	$(18+18)^{\triangle\frac{1}{2}}$
9B	9B	$9^3/3^1$	-2	$18-$
18B	9B	$3^1 18^3/6^1 9^3$	0	$36+4$
9C	9C	$1^3 9^3/3^2$	4	$18+9$
18C	9C	$2^3 3^2 18^3/1^3 6^2 9^3$	0	$36+$
10D	10A	$5^2 10^2/1^2 2^2$	5	$20+20$
10E	10A	$1^2 10^4/2^4 5^2$	25	$(20+20)^{\triangle\frac{1}{2}}$
20A	10B	$2^3 20^3/4^3 10^3$	-1	$(20\|2+10)^{\triangle\frac{1}{4}}$
20B	10C	$4^2 20^2/2^2 10^2$	4	$40\|2+$
10F	10D	$2^3 5^2 10^1/1^2$	0	$20+5 \triangle \frac{1}{2}$
10G	10D	$1^2 2^1 10^3/5^2$	0	$20+5 \triangle \frac{1}{2}$
10H	10E	$1^3 5^1 10^2/2^2$	20	$(20+20)^{\triangle\frac{1}{2}}$
10I	10E	$2^1 10^3/1^3 5^1$	0	$20+20$

https://doi.org/10.1017/fms.2015.7 Published online by Cambridge University Press

Table A.1. *Continued.*

Co_0	Co_1	π_g	$C_{\widehat{-g}}$	Γ_g
10J	10F	$2^2 10^2$	0	$20\|2+5$
11A	11A	$1^2 11^2$	4	$22+11$
22A	11A	$2^2 22^2/1^2 11^2$	0	$44+$
12C	12A	$2^4 3^4 12^4/1^4 4^4 6^4$	1	$24+24, 3 \vartriangle \frac{1}{2} 12 \triangledown$
12D	12A	$1^4 12^4/3^4 4^4$	9	$24+8, 3 \vartriangle \frac{1}{2} 12 \triangledown$
12E	12B	$2^2 12^4/4^4 6^2$	-3	$12-$
12F	12C	$6^2 12^2/2^2 4^2$	9	$24\|2+12$
12G	12D	$2^1 3^3 12^3/1^1 4^1 6^3$	4	$(24+8)^{\vartriangle \frac{1}{2}}$
12H	12D	$1^1 12^3/3^3 4^1$	0	$24+8$
12I	12E	$1^2 3^2 4^2 12^2/2^2 6^2$	16	$(24+)^{\vartriangle \frac{1}{2}}$
12J	12E	$4^2 12^2/1^2 3^2$	0	$24+$
24A	12F	$4^3 24^3/8^3 12^3$	-1	$(24\|4+6)^{\vartriangle \frac{1}{8}}$
12K	12G	$4^2 12^2/2^1 6^1$	0	$24+3 \vartriangle \frac{1}{2}$
12L	12H	$1^1 2^2 3^1 12^2/4^2$	0	$(24\|2+12)^{\vartriangle \frac{1}{4}}$
12M	12H	$2^3 6^1 12^2/1^1 3^1 4^2$	0	$(24\|2+12)^{\vartriangle \frac{1}{4}}$
12N	12I	$2^2 3^2 4^1 12^1/1^2$	0	$24+3 \vartriangle \frac{1}{2}$
12O	12I	$1^2 4^1 6^2 12^1/3^2$	0	$24+3 \vartriangle \frac{1}{2}$
12P	12J	$2^1 4^1 6^1 12^1$	0	$24\|2+3$
12Q	12K	$1^3 12^3/2^1 3^1 4^1 6^1$	12	$(24+24)^{\vartriangle \frac{1}{2}}$
12R	12K	$2^2 3^1 12^3/1^3 4^1 6^2$	0	$24+24$
24B	12L	$24^2/12^2$	2	$48\|12+$
12S	12M	12^2	0	$24\|12-$
13A	13A	$13^2/1^2$	1	$26+26$
26A	13A	$1^2 26^2/2^2 13^2$	13	$(26+26)^{\vartriangle \frac{1}{2}}$
28A	14A	$2^2 28^2/4^2 14^2$	1	$(28\|2+14)^{\vartriangle \frac{1}{4}}$
14C	14B	$1^1 2^1 7^1 14^1$	0	$28+7$
14D	14B	$2^2 14^2/1^1 7^1$	0	$28+7$
15A	15A	$1^3 15^3/3^3 5^3$	1	$30+6, 10, 15$
30A	15A	$2^3 3^3 5^3 30^3/1^3 6^3 10^3 15^3$	-1	$(30+6, 10, 15)^{\vartriangle \frac{1}{2}}$
15B	15B	$3^2 15^2/1^2 5^2$	1	$30+5, 6, 30$
30B	15B	$1^2 5^2 6^2 30^2/2^2 3^2 10^2 15^2$	9	$(30+5, 6, 30)^{\vartriangle \frac{1}{2}}$
15C	15C	$15^2/3^2$	1	$30\|3+10$

https://doi.org/10.1017/fms.2015.7 Published online by Cambridge University Press

Table A.1. *Continued*.

Co_0	Co_1	π_g	$C_{\widehat{-g}}$	Γ_g
30C	15C	$3^2 30^2/6^2 15^2$	5	$(30\|3+10)^{\triangle\frac{1}{2}}$
15D	15D	$1^1 3^1 5^1 15^1$	4	$30+3, 5, 15$
30D	15D	$2^1 6^1 10^1 30^1/1^1 3^1 5^1 15^1$	0	$60+$
15E	15E	$1^2 15^2/3^1 5^1$	2	$30+15$
30E	15E	$2^2 3^1 5^1 30^2/1^2 6^1 10^1 15^2$	0	$(30+15)^{\triangle\frac{1}{2}}$
16A	16A	$2^2 16^2/4^1 8^1$	0	$(32\|2+)^{\triangle\frac{1}{4}}$
16B	16B	$1^2 16^2/2^1 8^1$	8	$(32+)^{\triangle\frac{1}{2}}$
16C	16B	$2^1 16^2/1^2 8^1$	0	$32+$
18D	18A	$9^1 18^1/1^1 2^1$	3	$36+36$
18E	18A	$1^1 18^2/2^2 9^1$	9	$(36+36)^{\triangle\frac{1}{2}}$
18F	18B	$1^2 9^1 18^1/2^1 3^1$	6	$(36+36)^{\triangle\frac{1}{2}}$
18G	18B	$2^1 3^1 18^2/1^2 6^1 9^1$	0	$36+36$
18H	18C	$2^2 9^1 18^1/1^1 6^1$	0	$36+9\triangle\frac{1}{2}$
18I	18C	$1^1 2^1 18^2/6^1 9^1$	0	$36+9\triangle\frac{1}{2}$
20C	20A	$2^2 5^2 20^2/1^2 4^2 10^2$	1	$40+8, 5\triangle\frac{1}{2}20\triangledown$
20D	20A	$1^2 20^2/4^2 5^2$	5	$40+40, 5\triangle\frac{1}{2}20\triangledown$
20E	20B	$4^1 20^1$	0	$40\|4+5\triangle\frac{1}{2}$
20F	20C	$2^2 5^1 20^1/1^1 4^1$	0	$(40\|2+20)^{\triangle\frac{1}{4}}$
20G	20C	$1^1 2^1 10^1 20^1/4^1 5^1$	0	$(40\|2+20)^{\triangle\frac{1}{4}}$
21A	21A	$1^2 21^2/3^2 7^2$	1	$42+6, 14, 21$
42A	21A	$2^2 3^2 7^2 42^2/1^2 6^2 14^2 21^2$	1	$(42+6, 14, 21)^{\triangle\frac{1}{2}}$
21B	21B	$7^1 21^1/1^1 3^1$	1	$42+3, 14, 42$
42B	21B	$1^1 3^1 14^1 42^1/2^1 6^1 7^1 21^1$	7	$(42+3, 14, 42)^{\triangle\frac{1}{2}}$
21C	21C	$3^1 21^1$	2	$42\|3+7$
42C	21C	$6^1 42^1/3^1 21^1$	0	$(42\|3+7)^{\triangle\frac{1}{2}}$
22BC	22A	$2^1 22^1$	0	$44\|2+11\triangle\frac{1}{2}$
23AB	23AB	$1^1 23^1$	2	$46+23$
46AB	23AB	$2^1 46^1/1^1 23^1$	0	$92+$
24C	24A	$2^2 24^2/6^2 8^2$	1	$96+32, 96^{24\triangledown}, \triangle\frac{1}{2}48\triangledown$
24D	24B	$2^1 3^2 4^1 24^2/1^2 6^1 8^2 12^1$	-1	$48+48, 16^{\triangle\frac{1}{2}}$
24E	24B	$1^2 4^1 6^1 24^2/2^1 3^2 8^2 12^1$	3	$48+16, 48^{\triangle\frac{1}{2}}$

https://doi.org/10.1017/fms.2015.7 Published online by Cambridge University Press

Table A.1. *Continued*.

Co_0	Co_1	π_g	$C_{\widehat{-g}}$	Γ_g
24F	24C	$8^1 24^1/2^1 6^1$	4	$48\|2+$
24G	24D	$12^1 24^1/4^1 8^1$	3	$48\|4+12$
24H	24E	$2^1 6^1 8^1 24^1/4^1 12^1$	0	$(48\|2+)^{\triangle\frac{1}{4}}$
24I	24F	$2^1 3^1 4^1 24^1/1^1 8^1$	0	$(48\|4+12)^{\triangle\frac{1}{8}}$
24J	24F	$1^1 4^1 6^1 24^1/3^1 8^1$	0	$(48\|4+12)^{\triangle\frac{1}{8}}$
52A	26A	$2^1 52^1/4^1 26^1$	-1	$(52\|2+26)^{\triangle\frac{1}{4}}$
28B	28A	$1^1 4^1 7^1 28^1/2^1 14^1$	4	$(56+)^{\triangle\frac{1}{2}}$
28C	28A	$4^1 28^1/1^1 7^1$	0	$56+$
56AB	28B	$4^1 56^1/8^1 28^1$	1	$(56\|4+14)^{\triangle\frac{1}{8}}$
30F	30A	$1^1 2^1 15^1 30^1/3^1 5^1 6^1 10^1$	-1	$60+12, 15, 20$
30G	30A	$2^2 3^1 5^1 30^2/1^1 6^2 10^2 15^1$	1	$(60+12, 15, 20)^{\triangle\frac{1}{2}}$
60A	30B	$2^1 10^1 12^1 60^1/4^1 6^1 20^1 30^1$	1	$(60\|2+5, 6, 30)^{\triangle\frac{1}{4}}$
60B	30C	$6^1 60^1/12^1 30^1$	-1	$(60\|6+10)^{\triangle\frac{1}{12}}$
30H	30D	$1^1 6^1 10^1 15^1/3^1 5^1$	0	$60+3\triangle\frac{1}{2}, 5\triangle\frac{1}{2}, 15$
30I	30D	$2^1 3^1 5^1 30^1/1^1 15^1$	0	$60+3\triangle\frac{1}{2}, 5\triangle\frac{1}{2}, 15$
30J	30E	$2^1 3^1 5^1 30^1/6^1 10^1$	2	$60+12, 15, 20$
30K	30E	$2^1 30^1/3^1 5^1$	0	$60+12, 15, 20$
33A	33A	$3^1 33^1/1^1 11^1$	1	$66+6, 11, 66$
66A	33A	$1^1 6^1 11^1 66^1/2^1 3^1 22^1 33^1$	3	$(66+6, 11, 66)^{\triangle\frac{1}{2}}$
35A	35A	$1^1 35^1/5^1 7^1$	1	$70+10, 14, 35$
70A	35A	$2^1 5^1 7^1 70^1/1^1 10^1 14^1 35^1$	-1	$(70+10, 14, 35)^{\triangle\frac{1}{2}}$
36A	36A	$2^1 9^1 36^1/1^1 4^1 18^1$	1	$72+8, 9\triangle\frac{1}{2}36\triangledown$
36B	36A	$1^1 36^1/4^1 9^1$	3	$72+72, 9\triangle\frac{1}{2}36\triangledown$
39AB	39AB	$1^1 39^1/3^1 13^1$	1	$78+6, 26, 39$
78AB	39AB	$2^1 3^1 13^1 78^1/1^1 6^1 26^1 39^1$	1	$(78+6, 26, 39)^{\triangle\frac{1}{2}}$
40AB	40A	$2^1 40^1/8^1 10^1$	1	$160+32, 160^{40\triangledown}, \triangle\frac{1}{2}40\triangledown$
84A	42A	$4^1 6^1 14^1 84^1/2^1 12^1 28^1 42^1$	1	$(84\|2+6, 14, 21)^{\triangle\frac{1}{4}}$
60C	60A	$1^1 4^1 6^1 10^1 15^1 60^1/2^1 3^1 5^1 12^1 20^1 30^1$	1	$120+15, 24, 3\triangle\frac{1}{2}60\triangledown$
60D	60A	$3^1 4^1 5^1 60^1/1^1 12^1 15^1 20^1$	-1	$120+15, 120, 3\triangle\frac{1}{2}60\triangledown$

https://doi.org/10.1017/fms.2015.7 Published online by Cambridge University Press

Table A.2. Data for the $T^{s}_{g,\mathrm{tw}}$.

Co_0	Co_1	π_g	$C_{\hat{g}}$	$\Gamma_{g,\mathrm{tw}}$	$\mathbb{M}$
2A	1A	$2^{24}/1^{24}$	4096	$2-$	2B
4A	2B	$4^{12}/2^{12}$	64	$4\|2-$	4D
3A	3A	$3^{12}/1^{12}$	729	$3-$	3B
6A	3A	$1^{12}6^{12}/2^{12}3^{12}$	1	$6+6$	6B
6B	3B	$2^{6}6^{6}/1^{6}3^{6}$	64	$6+3$	6C
6C	3C	$1^{3}6^{9}/2^{3}3^{9}$	-8	$6-$	6E
6D	3D	$6^{8}/3^{8}$	16	$6\|3-$	6F
4C	4A	$4^{8}/1^{8}$	256	$4-$	4C
8A	4E	$8^{6}/4^{6}$	8	$8\|4-$	8F
5A	5A	$5^{6}/1^{6}$	125	$5-$	5B
10A	5A	$1^{6}10^{6}/2^{6}5^{6}$	1	$10+10$	10D
10B	5B	$2^{4}10^{4}/1^{4}5^{4}$	16	$10+5$	10B
10C	5C	$1^{1}10^{5}/2^{1}5^{5}$	-4	$10-$	10E
6E	6A	$3^{4}6^{4}/1^{4}2^{4}$	81	$6+2$	6D
6F	6A	$1^{4}6^{8}/2^{8}3^{4}$	9	$6-$	6E
12A	6B	$2^{6}12^{6}/4^{6}6^{6}$	1	$12\|2+6$	12F
6J	6D	$2^{1}6^{5}/1^{5}3^{1}$	72	$6-$	6E
12B	6H	$12^{4}/6^{4}$	4	$12\|6-$	12J
7A	7A	$7^{4}/1^{4}$	49	$7-$	7B
14A	7A	$1^{4}14^{4}/2^{4}7^{4}$	1	$14+14$	14C
14B	7B	$2^{3}14^{3}/1^{3}7^{3}$	8	$14+7$	14B
8B	8A	$8^{4}/2^{4}$	16	$8\|2-$	8D
8E	8C	$2^{2}8^{4}/1^{4}4^{2}$	32	$8-$	8E
9A	9A	$9^{3}/1^{3}$	27	$9-$	9B
18A	9A	$1^{3}18^{3}/2^{3}9^{3}$	1	$18+18$	18E
18B	9B	$3^{1}18^{3}/6^{1}9^{3}$	-2	$18-$	18D
18C	9C	$2^{3}3^{2}18^{3}/1^{3}6^{2}9^{3}$	4	$18+9$	18C
10D	10A	$5^{2}10^{2}/1^{2}2^{2}$	25	$10+2$	10C
10E	10A	$1^{2}10^{4}/2^{4}5^{2}$	5	$10-$	10E
20A	10B	$2^{3}20^{3}/4^{3}10^{3}$	-1	$20\|2+10$	20E
20B	10C	$4^{2}20^{2}/2^{2}10^{2}$	4	$20\|2+5$	20D
10I	10E	$2^{1}10^{3}/1^{3}5^{1}$	20	$10-$	10E
22A	11A	$2^{2}22^{2}/1^{2}11^{2}$	4	$22+11$	22B
12C	12A	$2^{4}3^{4}12^{4}/1^{4}4^{4}6^{4}$	9	$12+4$	12B
12D	12A	$1^{4}12^{4}/3^{4}4^{4}$	1	$12+12$	12H
12E	12B	$2^{2}12^{4}/4^{4}6^{2}$	-3	$12-$	12I

https://doi.org/10.1017/fms.2015.7 Published online by Cambridge University Press

Table A.2. *Continued*.

Co_0	Co_1	π_g	$C_{\hat{g}}$	$\Gamma_{g,\mathrm{tw}}$	$\mathbb{M}$
12F	12C	$6^2 12^2/2^2 4^2$	9	$12\|2+2$	12G
12H	12D	$1^1 12^3/3^3 4^1$	4	$12-$	12I
12J	12E	$4^2 12^2/1^2 3^2$	16	$12+3$	12E
24A	12F	$4^3 24^3/8^3 12^3$	-1	$24\|4+6$	24F
12R	12K	$2^2 3^1 12^3/1^3 4^1 6^2$	12	$12-$	12I
24B	12L	$24^2/12^2$	2	$24\|12-$	24J
13A	13A	$13^2/1^2$	13	$13-$	13B
26A	13A	$1^2 26^2/2^2 13^2$	1	$26+26$	26B
28A	14A	$2^2 28^2/4^2 14^2$	1	$28\|2+14$	28D
15A	15A	$1^3 15^3/3^3 5^3$	-1	$15+15$	15C
30A	15A	$2^3 3^3 5^3 30^3/1^3 6^3 10^3 15^3$	1	$30+6, 10, 15$	30A
15B	15B	$3^2 15^2/1^2 5^2$	9	$15+5$	15B
30B	15B	$1^2 5^2 6^2 30^2/2^2 3^2 10^2 15^2$	1	$30+5, 6, 30$	30D
15C	15C	$15^2/3^2$	5	$15\|3-$	15D
30C	15C	$3^2 30^2/6^2 15^2$	1	$30\|3+10$	30E
30D	15D	$2^1 6^1 10^1 30^1/1^1 3^1 5^1 15^1$	4	$30+3, 5, 15$	30C
30E	15E	$2^2 3^1 5^1 30^2/1^2 6^1 10^1 15^2$	2	$30+15$	30G
16C	16B	$2^1 16^2/1^2 8^1$	8	$16-$	16B
18D	18A	$9^1 18^1/1^1 2^1$	9	$18+2$	18A
18E	18A	$1^1 18^2/2^2 9^1$	3	$18-$	18D
18G	18B	$2^1 3^1 18^2/1^2 6^1 9^1$	6	$18-$	18D
20C	20A	$2^2 5^2 20^2/1^2 4^2 10^2$	5	$20+4$	20C
20D	20A	$1^2 20^2/4^2 5^2$	1	$20+20$	20F
21A	21A	$1^2 21^2/3^2 7^2$	1	$21+21$	21D
42A	21A	$2^2 3^2 7^2 42^2/1^2 6^2 14^2 21^2$	1	$42+6, 14, 21$	42B
21B	21B	$7^1 21^1/1^1 3^1$	7	$21+3$	21B
42B	21B	$1^1 3^1 14^1 42^1/2^1 6^1 7^1 21^1$	1	$42+3, 14, 42$	42D
42C	21C	$6^1 42^1/3^1 21^1$	2	$42\|3+7$	42C
46AB	23AB	$2^1 46^1/1^1 23^1$	2	$46+23$	46AB
24C	24A	$2^2 24^2/6^2 8^2$	1	$24\|2+12$	24H
24D	24B	$2^1 3^2 4^1 24^2/1^2 6^1 8^2 12^1$	3	$24+8$	24C
24E	24B	$1^2 4^1 6^1 24^2/2^1 3^2 8^2 12^1$	-1	$24+24$	24I
24F	24C	$8^1 24^1/2^1 6^1$	4	$24\|2+3$	24D
24G	24D	$12^1 24^1/4^1 8^1$	3	$24\|4+2$	24G
52A	26A	$2^1 52^1/4^1 26^1$	-1	$52\|2+26$	52B
28C	28A	$4^1 28^1/1^1 7^1$	4	$28+7$	28C

https://doi.org/10.1017/fms.2015.7 Published online by Cambridge University Press

Table A.2. *Continued.*

Co_0	Co_1	π_g	$C_{\widehat{g}}$	$\Gamma_{g,\mathrm{tw}}$	$\mathbb{M}$
56AB	28B	$4^1 56^1/8^1 28^1$	1	$56\|4+14$	56BC
30F	30A	$1^1 2^1 15^1 30^1/3^1 5^1 6^1 10^1$	1	$30+2,15,30$	30F
30G	30A	$2^2 3^1 5^1 30^2/1^1 6^2 10^2 15^1$	-1	$30+15$	30G
60A	30B	$2^1 10^1 12^1 60^1/4^1 6^1 20^1 30^1$	1	$60\|2+5,6,30$	60E
60B	30C	$6^1 60^1/12^1 30^1$	-1	$60\|6+10$	60F
30K	30E	$2^1 30^1/3^1 5^1$	2	$30+15$	30G
33A	33A	$3^1 33^1/1^1 11^1$	3	$33+11$	33A
66A	33A	$1^1 6^1 11^1 66^1/2^1 3^1 22^1 33^1$	1	$66+6,11,66$	66B
35A	35A	$1^1 35^1/5^1 7^1$	-1	$35+35$	35B
70A	35A	$2^1 5^1 7^1 70^1/1^1 10^1 14^1 35^1$	1	$70+10,14,35$	70B
36A	36A	$2^1 9^1 36^1/1^1 4^1 18^1$	3	$36+4$	36B
36B	36A	$1^1 36^1/4^1 9^1$	1	$36+36$	36D
39AB	39AB	$1^1 39^1/3^1 13^1$	1	$39+39$	39CD
78AB	39AB	$2^1 3^1 13^1 78^1/1^1 6^1 26^1 39^1$	1	$78+6,26,39$	78BC
40AB	40A	$2^1 40^1/8^1 10^1$	1	$40\|2+20$	40CD
84A	42A	$4^1 6^1 14^1 84^1/2^1 12^1 28^1 42^1$	1	$84\|2+6,14,21$	84B
60C	60A	$1^1 4^1 6^1 10^1 15^1 60^1/2^1 3^1 5^1 12^1 20^1 30^1$	-1	$60+4,15,60$	60C
60D	60A	$3^1 4^1 5^1 60^1/1^1 12^1 15^1 20^1$	1	$60+12,15,20$	60D

In both tables, the first column lists the conjugacy class of the element $g \in Co_0$ under consideration, and the next two columns list this element's associated Co_1 class (see Section 3.2) and Frame shape π_g (see Equation (4.28)). These are followed by the super trace $C_{\widehat{-g}}$ of $\widehat{-g}$ on CM (in Table A.1) or the super trace $C_{\widehat{g}}$ of $\widehat{g}$ on CM (in Table A.2; see Equation (4.18)). In the fifth column we describe explicitly the invariance groups of the McKay–Thompson series, writing Γ_g for the invariance group of T^s_g (Table A.1), and $\Gamma_{g,\mathrm{tw}}$ for the invariance group of $T^s_{g,\mathrm{tw}}$ (Table A.2). The invariance groups in Table A.2 are each associated to an element of the monster group by monstrous moonshine, and this is listed in the last column labelled $\mathbb{M}$.

Note that a complete description of the groups Γ_g first appeared in [74], but our notation, in Table A.1, is different in certain cases, adhering more closely to the traditions initiated in [19]. More specifically, we follow the conventions of [44], so that $n|h-$, for example, when h is the largest divisor of 24 such that h^2 divides nh, denotes the subgroup of index h in $\Gamma_0(n/h)$ defined in [19]. (See also [43] for a detailed analysis of the groups $n|h-$, and their extensions by Atkin–Lehner involutions.) So, for example, $12+3$ denotes the group obtained by adjoining an Atkin–Lehner involution $W_3 = (1/\sqrt{3})\begin{pmatrix} 3a & b \\ 12c & 3d \end{pmatrix}$ to $\Gamma_0(12)$, where $9ad - 12bc = 3$.

https://doi.org/10.1017/fms.2015.7 Published online by Cambridge University Press

Not all the groups Γ_g appear in [44], so we need some additional notation. We use $\triangle(1/h)$ and $n\triangledown$ to denote upper-triangular and lower-triangular matrices, respectively:

$$\triangle\frac{1}{h} := \begin{pmatrix} 1 & 1/h \\ 0 & 1 \end{pmatrix}, \quad n\triangledown := \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}. \tag{A.1}$$

We then write $12 + 3 \triangle \frac{1}{2}$, for example, for the group generated by $\Gamma_0(12)$ and the product of W_3 with $\triangle\frac{1}{2}$, where W_3 is an Atkin–Lehner involution for $\Gamma_0(12)$, as in the previous paragraph. Note that this group is also denoted $12 + 3'$ in [44]. Now the group denoted $4|2-$ in [19, 44] can be described as $8 + \triangle\frac{1}{2}4\triangledown$, for it is generated by $\Gamma_0(8)$ together with the product of $\triangle\frac{1}{2}$ and $4\triangledown$. For $8|2+$, we may write $16 + 16, \triangle\frac{1}{2}8\triangledown$, meaning the group generated by $\Gamma_0(16)$, the Fricke involution $\frac{1}{4}\left(\begin{smallmatrix} 0 & -1 \\ 16 & 0 \end{smallmatrix}\right)$, and the product of $\triangle\frac{1}{2}$ with $8\triangledown$.

Note that Γ_g and Γ_{-g} are related by conjugation by $\triangle\frac{1}{2}$: for every $g \in Co_0$,

$$\Gamma_{-g} = \Gamma_g^{\triangle\frac{1}{2}} = \begin{pmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \end{pmatrix} \Gamma_g \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{pmatrix}, \tag{A.2}$$

since the Fourier expansions of the functions $T_g^s(2\tau)$ and $T_{-g}^s(2\tau)$ differ exactly by signs on even powers of q.

As mentioned in Section 4.3, the invariance groups $\Gamma_{g,\mathrm{tw}}$, of the canonically twisted McKay–Thompson series $T^s_{g,\mathrm{tw}}$, are all genus-zero groups that arise in monstrous moonshine. We include the corresponding monstrous class names in Table A.2, where the $\Gamma_{g,\mathrm{tw}}$ are described explicitly.

References

[1] P. S. Aspinwall, '$K3$ surfaces and string duality', in *Fields, Strings and Duality (Boulder, CO, 1996)* (World Scientific, River Edge, NJ, 1997), 421–540.

[2] R. Borcherds, 'Vertex algebras, Kac–Moody algebras, and the monster', *Proc. Natl Acad. Sci. USA* **83**(10) (1986), 3068–3071.

[3] R. E. Borcherds, 'Monstrous moonshine and monstrous Lie superalgebras', *Invent. Math.* **109**(2) (1992), 405–444.

[4] R. E. Borcherds and A. J. E. Ryba, 'Modular moonshine II', *Duke Math. J.* **83**(2) (1996), 435–459.

[5] T. Bridgeland, 'Stability conditions on $K3$ surfaces', *Duke Math. J.* **141**(2) (2008), 241–291.

[6] S. Carnahan, 'Generalized moonshine I: genus-zero functions', *Algebra Number Theory* **4**(6) (2010), 649–679.

[7] S. Carnahan, 'Generalized moonshine, II: Borcherds products', *Duke Math. J.* **161**(5) (2012), 893–950.

[8] S. Carnahan, 'Generalized moonshine IV: monstrous lie algebras', Preprint, 2012, arXiv:1208.6254.

[9] S. Carnahan, 'Monstrous lie algebras', *RIMS Kôkyûroku* **1872** (2014).

https://doi.org/10.1017/fms.2015.7 Published online by Cambridge University Press

[10] M. C. N. Cheng, '$K3$ surfaces, $N = 4$ dyons and the Mathieu group M_{24}', *Commun. Number Theory Phys.* **4**(4) (2010), 623–657.
[11] M. C. N. Cheng, X. Dong, J. F. R. Duncan, S. Harrison, S. Kachru and T. Wrase, 'Mock modular Mathieu moonshine modules', Preprint, 2014, arXiv:1406.5502 [hep-th].
[12] M. C. N. Cheng and J. F. R. Duncan, 'The largest Mathieu group and (mock) automorphic forms', in *String-Math 2011*, Proceedings of Symposia in Pure Mathematics, 85 (American Mathematical Society, Providence, RI, 2012), 53–82.
[13] M. C. N. Cheng and J. F. R. Duncan, 'On Rademacher sums, the largest Mathieu group, and the holographic modularity of moonshine', *Commun. Number Theory Phys.* **6**(3) (2012), 697–758.
[14] M. C. N. Cheng, J. F. R. Duncan and J. A. Harvey, 'Umbral moonshine', *Commun. Number Theory Phys.* **8**(2) (2014), 101–242.
[15] M. C. N. Cheng, J. F. R. Duncan and J. A. Harvey, 'Umbral moonshine and the Niemeier lattices', *Res. Math. Sci.* **1**(3) (2014), 1–81.
[16] J. H. Conway, 'A perfect group of order 8, 315, 553, 613, 086, 720, 000 and the sporadic simple groups', *Proc. Natl Acad. Sci. USA* **61** (1968), 398–400.
[17] J. H. Conway, 'A characterisation of Leech's lattice', *Invent. Math.* **7** (1969), 137–142.
[18] J. H. Conway, 'A group of order 8, 315, 553, 613, 086, 720, 000', *Bull. Lond. Math. Soc.* **1** (1969), 79–88.
[19] J. H. Conway and S. P. Norton, 'Monstrous moonshine', *Bull. Lond. Math. Soc.* **11**(3) (1979), 308–339.
[20] J. H. Conway and N. J. A. Sloane, *Sphere Packings, Lattices and Groups*, 3rd edn, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 290 (Springer, New York, 1999). With additional contributions by E. Bannai, R. E. Borcherds, J. Leech, S. P. Norton, A. M. Odlyzko, R. A. Parker, L. Queen and B. B. Venkov.
[21] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson, *Atlas of Finite Groups* (Clarendon, Oxford, 1985). (Maximal Subgroups and Ordinary Characters for Simple Groups. With comput. assist. from J. G. Thackray).
[22] J. Conway, J. McKay and A. Sebbar, 'On the discrete groups of moonshine', *Proc. Amer. Math. Soc.* **132** (2004), 2233–2240.
[23] L. Dixon, P. Ginsparg and J. Harvey, 'Beauty and the beast: superconformal symmetry in a monster module', *Comm. Math. Phys.* **119**(2) (1988), 221–241.
[24] L. Dixon, J. A. Harvey, C. Vafa and E. Witten, 'Strings on orbifolds', *Nuclear Phys.* B **261**(4) (1985), 678–686.
[25] L. Dixon, J. A. Harvey, C. Vafa and E. Witten, 'Strings on orbifolds II', *Nuclear Phys.* B **274**(2) (1986), 285–314.
[26] C. Dong, 'Vertex algebras associated with even lattices', *J. Algebra* **161**(1) (1993), 245–265.
[27] C. Dong and J. Lepowsky, *Generalized Vertex Algebras and Relative Vertex Operators*, Progress in Mathematics, 112 (Birkhäuser Boston, Inc., Boston, MA, 1993).
[28] C. Dong, H. Li and G. Mason, 'Some twisted sectors for the moonshine module', in *Moonshine, The Monster, and Related Topics (South Hadley, MA, 1994)*, Contemporary Mathematics, 193 (American Mathematical Society, Providence, RI, 1996), 25–43.
[29] C. Dong, H. Li and G. Mason, 'Twisted representations of vertex operator algebras', *Math. Ann.* **310**(3) (1998), 571–600.
[30] C. Dong, H. Li and G. Mason, 'Modular invariance of trace functions in orbifold theory and generalized moonshine', *Comm. Math. Phys.* **214** (2000), 1–56.

[31] C. Dong, H. Li, G. Mason and P. S. Montague, 'The radical of a vertex operator algebra', in *The Monster and Lie Algebras (Columbus, OH, 1996)*, Ohio State Univ. Math. Res. Inst. Publ., 7 (de Gruyter, Berlin, 1998), 17–25.

[32] C. Dong and G. Mason, 'Nonabelian orbifolds and the boson-fermion correspondence', *Comm. Math. Phys.* **163**(3) (1994), 523–559.

[33] C. Dong and G. Mason, 'Rational vertex operator algebras and the effective central charge', *Int. Math. Res. Not.* **56** (2004), 2989–3008.

[34] C. Dong and G. Mason, 'Integrability of C_2-cofinite vertex operator algebras', *Int. Math. Res. Not.* (2006), 1–15. Art. ID 80468.

[35] C. Dong and Z. Zhao, 'Twisted representations of vertex operator superalgebras', *Commun. Contemp. Math.* **8**(1) (2006), 101–121.

[36] J. F. Duncan, 'Super-moonshine for Conway's largest sporadic group', *Duke Math. J.* **139**(2) (2007), 255–315.

[37] J. F. R. Duncan and I. B. Frenkel, 'Rademacher sums, moonshine and gravity', *Commun. Number Theory Phys.* **5**(4) (2011), 1–128.

[38] J. F. R. Duncan and S. Mack-Crane, *Derived Equivalences of K3 Surfaces and Twined Elliptic Genera*, to appear (2015).

[39] T. Eguchi and K. Hikami, 'Note on twisted elliptic genus of K3 surface', *Phys. Lett.* B **694** (2011), 446–455.

[40] T. Eguchi, H. Ooguri and Y. Tachikawa, 'Notes on the K3 surface and the Mathieu group M_{24}', *Exp. Math.* **20** (2011), 91–96.

[41] T. Eguchi and A. Taormina, 'On the unitary representations of $N = 2$ and $N = 4$ superconformal algebras', *Phys. Lett.* B **210**(1–2) (1988), 125–132.

[42] M. Eichler and D. Zagier, *The Theory of Jacobi Forms* (Birkhäuser, Boston, MA, 1985).

[43] C. R. Ferenbaugh, 'The genus-zero problem for $n|h$-type groups', *Duke Math. J.* **72**(1) (1993), 31–63.

[44] D. Ford, J. McKay and S. Norton, 'More on replicable functions', *Comm. Algebra* **22**(13) (1994), 5175–5193.

[45] I. B. Frenkel, 'Two constructions of affine Lie algebra representations and boson-fermion correspondence in quantum field theory', *J. Funct. Anal.* **44**(3) (1981), 259–327.

[46] E. Frenkel and M. Szczesny, 'Twisted modules over vertex algebras on algebraic curves', *Adv. Math.* **187**(1) (2004), 195–227.

[47] I. B. Frenkel and V. G. Kac, 'Basic representations of affine Lie algebras and dual resonance models', *Invent. Math.* **62**(1) (1980/81), 23–66.

[48] I. B. Frenkel, Y.-Z. Huang and J. Lepowsky, 'On axiomatic approaches to vertex operator algebras and modules', *Mem. Amer. Math. Soc.* **104**(494) (1993).

[49] I. B. Frenkel, J. Lepowsky and A. Meurman, 'A natural representation of the Fischer–Griess monster with the modular function J as character', *Proc. Natl Acad. Sci. USA* **81**(10) (1984), 3256–3260.

[50] I. B. Frenkel, J. Lepowsky and A. Meurman, 'A moonshine module for the monster', in *Vertex Operators in Mathematics and Physics (Berkeley, CA, 1983)*, Mathematical Sciences Research Institute Publications, 3 (Springer, New York, 1985), 231–273.

[51] I. B. Frenkel, J. Lepowsky and A. Meurman, *Vertex Operator Algebras and the Monster*, Pure and Applied Mathematics, 134 (Academic, Boston, MA, 1988).

[52] E. Frenkel and D. B. Zvi, *Vertex Algebras and Algebraic Curves*, 2nd edn, Mathematical Surveys and Monographs, 88 (American Mathematical Society, Providence, RI, 2004).

[53] M. R. Gaberdiel, 'An introduction to conformal field theory', *Rep. Progr. Phys.* **63** (2000), 607–667.
[54] M. R. Gaberdiel, '2D conformal field theory and vertex operator algebras', Preprint, 2005, arXiv:hep-th/0509027.
[55] M. R. Gaberdiel, S. Hohenegger and R. Volpato, 'Mathieu moonshine in the elliptic genus of K3', *JHEP* **1010** (2010) 062.
[56] M. R. Gaberdiel, S. Hohenegger and R. Volpato, 'Mathieu twining characters for K3', *JHEP* **1009** (2010) 058 19 pages.
[57] M. R. Gaberdiel, S. Hohenegger and R. Volpato, 'Symmetries of K3 sigma models', *Commun. Number Theory Phys.* **6**(1) (2012), 1–50.
[58] M. R. Gaberdiel and R. Volpato, 'Mathieu moonshine and orbifold K3s', *Contrib. Math. Comput. Sci.* **8** (2014), 109–141.
[59] T. Gannon, 'Much ado about Mathieu', Preprint, 2012, arXiv:1211.5531 [math.RT].
[60] M. J. E. Golay, 'Notes on digital coding', *Proc. IRE* **37** (1949), 657.
[61] R. L. Griess Jr., 'The structure of the 'monster' simple group', in *Proceedings of the Conference on Finite Groups (University of Utah, Park City, UT, 1975)* (Academic, New York, 1976), 113–118.
[62] R. L. Griess Jr., 'A construction of F_1 as automorphisms of a 196, 883-dimensional algebra', *Proc. Natl Acad. Sci. USA* **78**(2, Part 1) (1981), 686–691.
[63] R. L. Griess Jr., 'The friendly giant', *Invent. Math.* **69**(1) (1982), 1–102.
[64] P. A. Griffiths, *Introduction to Algebraic Curves*, Translations of Mathematical Monographs, 76 (American Mathematical Society, Providence, RI, 1989). Translated from the Chinese by Kuniko Weltin.
[65] G. Hoehn, 'Generalized moonshine for the baby monster', 2003, available at http://www.math.ksu.edu/ gerald/papers/.
[66] G. Hoehn, 'Selbstduale vertexoperatorsuperalgebren und das babymonster (self-dual vertex operator super algebras and the baby monster)', in *Bonner Mathematische Schriften (Bonn, 1996)* Vol. 286 (Library of Mathematics, Bonn, 2007), 1–85.
[67] P. E. Holmes and R. A. Wilson, 'A new computer construction of the Monster using 2-local subgroups', *J. Lond. Math. Soc.* (2) **67**(2) (2003), 349–364.
[68] D. Huybrechts, 'On derived categories of K3 surfaces, symplectic automorphisms and the Conway group', Preprint, 2013, arXiv:1309.6528 [math.AG].
[69] R. Ivanov and M. Tuite, 'Rational generalised moonshine from abelian orbifoldings of the moonshine module', *Nuclear Phys.* B **635**(3) (2002), 435–472.
[70] R. Ivanov and M. Tuite, 'Some irrational generalised moonshine from orbifolds', *Nuclear Phys.* B **635**(3) (2002), 473–491.
[71] V. Kac, *Vertex Algebras for Beginners*, 2nd edn, University Lecture Series, 10 (American Mathematical Society, Providence, RI, 1998).
[72] V. Kac and W. Wang, 'Vertex operator superalgebras and their representations', in *Mathematical Aspects of Conformal and Topological Field Theories and Quantum Groups (South Hadley, MA, 1992)*, Contemporary Mathematics, 175 (American Mathematical Society, Providence, RI, 1994), 161–191.
[73] A. W. Knapp, *Elliptic Curves*, Mathematical Notes, 40 (Princeton University Press, Princeton, NJ, 1992).
[74] T. Kondo, 'The automorphism group of Leech lattice and elliptic modular functions', *J. Math. Soc. Japan* **37**(2) (1985), 337–362.

https://doi.org/10.1017/fms.2015.7 Published online by Cambridge University Press

[75] P. S. Landweber (Ed.), *Elliptic Curves and Modular Forms in Algebraic Topology*, Lecture Notes in Mathematics, 1326 (Springer, Berlin 1988).
[76] J. Leech, 'Some sphere packings in higher space', *Canad. J. Math.* **16** (1964), 657–682.
[77] J. Leech, 'Notes on sphere packings', *Canad. J. Math.* **19** (1967), 251–267.
[78] J. Lepowsky and H. Li, *Introduction to Vertex Operator Algebras and Their Representations*, Progress in Mathematics, 227 (Birkhäuser Boston Inc., Boston, MA, 2004).
[79] J. Lepowsky and R. L. Wilson, 'Construction of the affine Lie algebra $A_1^{(1)}$', *Comm. Math. Phys.* **62**(1) (1978), 43–53.
[80] H. S. Li, 'Symmetric invariant bilinear forms on vertex operator algebras', *J. Pure Appl. Algebra* **96**(3) (1994), 279–297.
[81] H.-S. Li, 'Local systems of twisted vertex operators, vertex operator superalgebras and twisted modules', in *Moonshine, the Monster, and Related Topics (South Hadley, MA, 1994)*, Contemporary Mathematics, 193 (American Mathematical Society, Providence, RI, 1996), 203–236.
[82] S. Linton, R. Parker, P. Walsh and R. Wilson, 'Computer construction of the monster', *J. Group Theory* **1**(4) (1998), 307–337.
[83] S. Mandelstam, 'Dual-resonance models', *Phys. Rep.* **13** (1974), 259–353.
[84] G. Mason, 'Finite groups and modular functions', in *The Arcata Conference on Representations of Finite Groups (Arcata, CA, 1986)*, Proceedings of Symposia in Pure Mathematics, 47 (American Mathematical Society, Providence, RI, 1987), 181–210. With an appendix by S. P. Norton.
[85] É. Mathieu, 'Mémoire sur l'étude des fonctions de plusiers quantités, sur la manière de les former et sur les substitutions qui les laissent invariables', *J. Math. Pures Appl.* (9) **6** (1861), 241–323.
[86] É. Mathieu, 'Sur la fonction cinq fois transitive de 24 quantités', *J. Math. Pures Appl.* (9) **18** (1873), 25–46.
[87] S. Mukai, 'Finite groups of automorphisms of K3 surfaces and the Mathieu group', *Invent. Math.* **94**(1) (1988), 183–221.
[88] S. Norton, 'Generalized moonshine', *Proc. Sympos. Pure Math.* **47** (1987), 208–209.
[89] S. Norton, 'From moonshine to the monster', in *Proceedings on Moonshine and Related Topics (Montréal, QC, 1999)*, CRM Proceedings & Lecture Notes, 30 (American Mathematical Society, Providence, RI, 2001), 163–171.
[90] K. Ono, L. Rolen and S. Trebat-Leder, 'Classical and umbral moonshine: connections and p-adic properties', Preprint, 2014, arXiv:1403.3712 [math.NT].
[91] L. Queen, 'Modular functions arising from some finite groups', *Math. Comp.* **37**(156) (1981), 547–580.
[92] H. Rademacher, 'The Fourier series and the functional equation of the absolute modular invariant $J(\tau)$', *Amer. J. Math.* **61**(1) (1939), 237–248.
[93] E. M. Rains and N. J. A. Sloane, 'Self-dual codes', in *Handbook of Coding Theory*, Vol. I, II (North-Holland, Amsterdam, 1998), 177–294.
[94] N. R. Scheithauer, 'Vertex algebras, Lie algebras, and superstrings', *J. Algebra* **200**(2) (1998), 363–403.
[95] J. H. Schwarz, 'Dual-resonance theory', *Phys. Rep.* **8** (1973), 269–335.
[96] G. Shimura, *Introduction to The Arithmetic Theory of Automorphic Functions*, Publications of the Mathematical Society of Japan, 11. Kanô Memorial Lectures, 1 (Iwanami Shoten, Tokyo, 1971).

https://doi.org/10.1017/fms.2015.7 Published online by Cambridge University Press

[97] S. D. Smith, 'On the head characters of the monster simple group', in *Finite Groups – Coming of Age (Montreal, QC, 1982)*, Contemporary Mathematics, 45 (American Mathematical Society, Providence, RI, 1985), 303–313.

[98] J. G. Thompson, 'Finite groups and modular functions', *Bull. Lond. Math. Soc.* **11**(3) (1979), 347–351.

[99] J. G. Thompson, 'Some numerology between the Fischer–Griess monster and the elliptic modular function', *Bull. Lond. Math. Soc.* **11**(3) (1979), 352–353.

[100] R. Volpato, 'On symmetries of $N = (4, 4)$ sigma models on T4', *JHEP* **1408** (2014) 094.