Canad. Math. Bull. Vol. **26** (4), 1983

THE SHRINKING PROPERTY

BY
MARY ELLEN RUDIN

ABSTRACT. A space has the shrinking property if, for every open cover $\{V_a \mid a \in A\}$, there is an open cover $\{W_a \mid a \in A\}$ with $\overline{W_a} \subset V_a$ for each $a \in A$. It is strangely difficult to find an example of a normal space without the shrinking property. It is proved here that any Σ-product of metric spaces has the shrinking property.

By a space we mean a T_3 topological space.

We say that a space has the *shrinking* property if, for every set Δ and open cover $\{V_\delta \mid \delta \in \Delta\}$ of the space, there is an open cover $\{W_\delta \mid \delta \in \Delta\}$ with $\bar{W}_\delta \subset V_\delta$ for each δ, the W_δ's being the "shrinking" of V_δ's. A space is normal precisely if every open cover of cardinality *two* has a shrinking; every paracompact space has the shrinking property. The usual order topology on ω_1 yields a normal space with the shrinking property which is not paracompact. But it is strangely difficult to find an example of a normal space *without* the shrinking property.

Suppose X is a normal space. It is well known [1] that X has a *countable* open cover which cannot be shrunk if and only if X is Dowker, i.e. $(X \times I)$ is not normal or equivalently X is not countably paracompact. We define X to be κ-Dowker for a cardinal κ provided there is a *nested* open cover of X of cardinality κ which cannot be shrunk: then $X \times Y$ is not normal for every normal space Y for which κ is the minimal cardinality of a subset of Y with a limit point. For each κ we know of (essentially one real) κ-Dowker space [2, 3], the examples involve box products of increasing sequences of cardinals and the various cardinal functions are large. We also have a number of consistency examples if $\kappa = \omega$. I know of no other examples of normal spaces without the shrinking property; I would hope for more useful examples.

The rest of this paper shows that any Σ-product of metric spaces (known to be normal and not paracompact [4]) has the shrinking property. A parallel paper [5] shows that for Σ-products of compact spaces, normality, the shrinking property, and countable tightness in the factors, are all equivalent. Both of these theorems indicate the degree to which normality carries the shrinking property with it.

Received by the editors July 31, 1982.
A.M.S. Subg. Class.: Primary 54D20, 54D18
Keywords: shrinking, normal, paracompact, Dowker

THEOREM. *A Σ-product of metric spaces has the shrinking property.*

Proof. Let Γ be an uncountable set and, for each $\gamma \in \Gamma$, let X_γ be a metric space having at least two points "0" and "1" at a distance 1 apart. Let $\Sigma = \{p \in \prod_{\gamma \in \Gamma} X_\gamma \mid p(\gamma) = 0$ for all but countably many $\gamma \in \Gamma\}$. Let $\{V_\delta \mid \delta \in \Delta\}$ be an open cover of Σ. We want to define an open cover $\{W_\delta \mid \delta \in \Delta\}$ of Σ with $\bar{W}_\delta \subset V_\delta$ for each δ.

For each $\gamma \in \Gamma$ and $n \in \omega$, let $\mathscr{B}_n(\gamma)$ be a locally finite open cover of X_γ by sets of diameter less than $1/2^n$.

For each $n \in \omega$, let F_n be the set of all functions f whose domain $D(f)$ is a finite subset of Γ such that $f(\gamma) \in \mathscr{B}_n(\gamma)$ for every $\gamma \in D(f)$. Let $F = \bigcup_{n \in \omega} F_n$. If $f \in F$, let $U_f = \{p \in \Sigma \mid p(\gamma) \in f(\gamma)$ for all $\gamma \in D(f)\}$. Thus $\{U_f \mid f \in F\}$ is an open basis for Σ.

If $f \in F$, define

$$G_f = \left\{ g \in F \;\middle|\; \begin{array}{l} \forall \gamma \in (D(g) \cap D(f)), f(\gamma) \subset g(x) \\ \forall \gamma \in (D(g) - D(f)), 0 \in g(\gamma); \text{ and} \\ \exists \delta \in \Delta, \bar{U}_g \subset V_\delta. \end{array} \right\}.$$

If possible, choose $q_f \in (\bar{U}_f - \bigcup \{U_g \mid g \in G_f\})$.

Choose $\gamma_0 \in \Gamma$; and, for $q \in \Sigma$, let $\{\gamma_j(q) \mid j \in \omega\}$ be an indexing of the support $\{\gamma \in \Gamma \mid q(\gamma) \neq 0\}$ of q. Define $\mathscr{S} = \{f_0, f_1, \ldots, f_n \mid \forall i \leq n,\ f_i \in F_i$ and

$$D(f_i) = \{\gamma_0\} \cup \{\gamma_j(q_{f_k}) \mid j < i, k < i\}, \quad \text{and} \quad \forall i < n,\ q_{f_i} \text{ is defined}\}.$$

We say that $(f_0, \ldots, f_n) \in \mathscr{S}_n$ if $\bar{U}_{f_n} \subset \bigcup \{U_g \mid g \in G_{f_n}\}$.

LEMMA. *If $p \in \Sigma$, there is $n \in \omega$ and $(f_0, \ldots, f_n) \in \mathscr{S}_n$ such that $p \in \bigcap_{i \leq n} U_{f_i}$.*

Proof of lemma. Otherwise, for each $n \in \omega$, we can choose $f_n \in F_n$ with $p \in U_{f_n}$ and $D(f_n) = \{\gamma_0\} \cup \{\gamma_j(q_{f_k}) \mid j < n, k < n\}$ and q_{f_n} defined.

Let $E = \{\gamma_j(q_{f_k}) \mid j \in \omega, k \in \omega\}$. Define $q \in \Sigma$ by $q(\gamma) = p(\gamma)$ for $\gamma \in E$, and $q(\gamma) = 0$ otherwise. Since $q \in \Sigma$, there is $g \in F$ with $q \in U_g \subset \overline{U_g} \subset V_\delta$ for some $\delta \in \Delta$. Since $D(g)$ is finite, there is $n \in \omega$ such that, if $\gamma \in (E \cap D(g))$, then $\gamma = \gamma_j(q_k)$ for some $j < n$ and $k < n$ and $\{x \in X_\gamma \mid \text{distance } (x, p(\gamma)) \leq 1/2^n\} \subset g(\gamma)$. Thus $g \in G_{f_n}$ and $q_{f_n} \in U_g$ contrary to our choice of q_{f_n}.

By our lemma then, if $p \in \Sigma$ there is a minimal $n \in \omega$, called n_p, for which there is an $(f_0, \ldots, f_n) \in \mathscr{S}_n$ with $p \in \bigcap_{i \leq n} U_{f_i}$. We let S_p denote one such $(f_0, \ldots, f_{n_p})$. Finally choose $m_p \in \omega$ sufficiently large that if $\gamma \in D(f_{n_p})$, then $\{x \in X_\gamma \mid \text{distance } (x, p(\gamma)) \leq 1/2^m\} \subset f_{n_p}(\gamma)$. Observe that $n_p < m_p$.

Suppose that $S = (f_0, \ldots, f_n) \in \mathscr{S}_n$. By definition $\bar{U}_{f_n} \subset \bigcup \{U_g \mid g \in G_{f_n}\}$. Either: Case (1) There is $g \in G_{f_n}$ with $D(g) \subset D(f_n)$.
Or: Case (2) For all $g \in G_{f_n}$, $E_g = (D(g) - D(f_n)) \neq \emptyset$.

https://doi.org/10.4153/CMB-1983-064-x Published online by Cambridge University Press

Suppose case (2). For each $\alpha\in a_1$, we choose $g_\alpha\in G_{f_n}$ by induction such that $E_{g_\alpha}\cap E_{g_\beta}=\emptyset$ for all $\beta<\alpha$. To see that this is possible let $E=\bigcup\{E_{g_\beta}\mid\beta<\alpha\}$ and $G=\{g\in G_{f_n}\mid E\cap E_g\neq\emptyset\}$. There is $q\in\Sigma$ with $q(\gamma)\in f_n(\gamma)$ for all $\gamma\in D(f_n)$, $q(\gamma)=1$ for all $\gamma\in E$, and $q(\gamma)=0$ otherwise. Since $q\in(U_{f_n}-\{U_g\mid g\in G\})$, $(G_{f_n}-G)\neq\emptyset$. For all $\alpha\in\omega_1$ there is $\delta_\alpha\in\Delta$ with $\bar{U}_{g_\alpha}\subset V_{\delta_\alpha}$. Without loss of generality we have either:

Case (2a) All δ_α are the same. Or

Case (2b) All δ_α are different.

In cases (1) and (2a) there is $\delta\in\Delta$, called δ_S, with $\bar{U}_{f_n}\subset V_\delta$. In case (2b), for each $\alpha\in\omega_1$, we let $g_{\alpha S}$ and $\delta_{\alpha S}$ denote the chosen g_α and δ_α, respectively.

Now suppose that $p\in\Sigma$. Let $\mathcal{S}_p=\{(f_0,\ldots,f_j)\in\mathcal{S}\mid j\leq m_p$ and $p\in\bigcap_{i\leq j}\bar{U}_{f_i}\}$. If $f_0,\ldots,f_j\in\mathcal{S}_p$, the domain of f_i is completely determined by $f_0,\ldots,f_{i-1}$, and, for $\gamma\in D(f_i)$, since $p(\gamma)\in\overline{f_i(\gamma)}$ and $\mathcal{B}_i(\gamma)$ is locally finite, the number of choices for $f_i(\gamma)$ is finite. So $\mathcal{S}_p$ is finite. Let $\Gamma_p=\bigcup\{D(f)\mid f\in S\in\mathcal{S}_p\}$. For $\gamma\in\Gamma_p$, let $A_p(\gamma)=\bigcap\{A\subset X_\gamma\mid p(\gamma)\in A$ and, for some $B\in\bigcup_{i\leq m_p}\beta_i(\gamma)$, either $A=B$ or $A=(X_\gamma-B)\}$. If S_p has case (1) or (2a), choose $f_p\in F$ with $D(f_p)=\Gamma_p$ and $p(\gamma)\in f_p(\gamma)\subset A_p(\gamma)$ for all $\gamma\in\Gamma_p$. If S_p has case (2b), choose and $\alpha\in\omega_1$ with $p\in U_{g_{\alpha,S_p}}$; then choose $f_p\in F$ with $D(f_p)=\Gamma_p\cup D(g_{\alpha,S_p})$ and $p(\gamma)\in f_p(\gamma)\subset A_p(\gamma)$ for all $\gamma\in\Gamma_p$, and $p(\gamma)\in f_p(\gamma)\subset g_{\alpha,S_p}(\gamma)$ for all $\gamma\in D(g_{\alpha,S_p})$; let α_p denote this α.

For $\delta\in\Delta$, let $P_\delta=\{p\in\Sigma\mid\delta=\delta_{S_p}$ if S_p has case (1) or (2a), and $\delta=\delta_{\alpha_p,S_p}$ if S_p has case (2b)$\}$. Let $W_\delta=\bigcup\{U_{f_p}\mid p\in P_\delta\}$. Clearly $\{W_\delta\mid\delta\in\Delta\}$ is an open cover of Σ; we must show that $\bar{W}_\delta\subset V_\delta$.

So fix δ and $q\in(\Sigma-V_\delta)$; we must show that $q\notin\overline{\bigcup\{U_{f_p}\mid p\in P_\delta\}}$.

We first prove that $q\notin\overline{\bigcup\{U_{f_p}\mid p\in P_\delta}$ and $n_p\leq m_q\}$. Suppose $p\in P_\delta$, $S_p=(f_0,\ldots,f_{n_p})$ and $n_p\leq m_q$. If there is a smallest $i\leq n_p$ with $q\notin\bar{U}_{f_i}$, there is $\gamma\in D(f_i)$ with $q\notin\overline{f_i(\gamma)}$. Since $D(f_i)$ is determined by $(f_0,\ldots,f_{i-1})$ and $i\leq m_q$, $\gamma\in\Gamma_q$ and $f_i(\gamma)\cap A_q(\gamma)=\emptyset$. But $\gamma\in\Gamma_p$ and $i\leq n_p\leq m_p$, so $A_p(\gamma)\subset f_i(\gamma)$. Thus $U_{f_p}\cap U_{f_q}=\emptyset$. So we can assume that $q\in\bigcap_{i\leq n_p}\bar{U}_{f_i}$. By definition, since $n_p\leq m_q$, $S_p\in\mathcal{S}_q$.

Since $\mathcal{S}_q$ is finite, if $q\in\overline{\bigcup\{U_{f_p}\mid p\in P_\delta}$ and $n_p\leq m_q\}$ there is an $S\in\mathcal{S}_q$ such that $q\in\overline{\bigcup\{U_{f_p}\mid p\in P_\delta}$ and $S_p=S\}$. Let $S=(f_0,\ldots,f_n)$. If S has cases (1) or (2a), $\bar{U}_{f_n}\subset V_\delta$. But $q\in\bar{U}_{f_n}$ and this contradicts $q\notin V_\delta$. If case (2b), there is a unique $\alpha\in\omega_1$ with $\delta_{\alpha S}=\delta$; and if $p\in P_\delta$ has $S_p=S$, then $\alpha_p=\alpha$ and $U_{f_p}\subset U_{g_{\alpha S}}$ and $\bar{U}_{g_{\alpha S}}\subset V_\delta$. Since $q\notin V_\delta$, $q\notin\overline{\bigcup\{U_{f_p}\mid p\in P_\delta}$ and $S_p=S\}$.

It remains to prove that $q\notin\overline{\bigcup\{U_{f_p}\mid p\in P_\delta}$ and $m_\delta<n_p\}$. We assume that $p\in P_\delta$ and $m_q<n_p$ and prove that $U_{f_p}\cap U_{f_q}=\emptyset$. Let $S_q=f_0,\ldots,f_{n_q}$. Since $n_q<m_q<n_p$, by the argument given before, interchanging p and q, $S_q\in\mathcal{S}_p$. However, by the minimality of n_p, there must be an $i\leq n_q$ and $\gamma\in D(f_i)$ such that $p(\gamma)\notin f_i(\gamma)$. By our choice of m_q, the distance $(p(\gamma),q(\gamma))>1/2^{m_q}$. Choose $B\in\mathcal{B}_{m_q}(\gamma)$ with $q(\gamma)\in B$. Then $p(\gamma)\in(X_\gamma-\bar{B})$. Since $\gamma\in\Gamma_q$, $f_q(\gamma)\subset B$; and since $\gamma\in\Gamma_p$ and $m_q<n_p<m_p$, $f_p(\gamma)\subset(X_\gamma-B)$. Thus $U_{f_p}\cap U_{f_q}=\emptyset$.

https://doi.org/10.4153/CMB-1983-064-x Published online by Cambridge University Press

Bibliography

1. C. H. Dowker, *On countably paracompact spaces*, Can. J. Math. **3** (1971) 214–224.

2. M. E. Rudin, *A normal space X for which $X \times I$ is not normal*, Fund. Math. LXXIII (1971) 179–186.

3. M. E. Rudin, *κ-Dowker spaces*, Czech. Math. J. **28** (1978) Praha, 324–326.

4. S. P. Gul'ko, *On properties of subsets of Σ-products*, Soviet Math. Dokl. **18** (1977), 1438–1442.

5. A. LeDonne, Normality and shrinking property in Σ-product of spaces, this Journal, to appear.

University of Wisconsin
Madison, Wisconsin
53706

https://doi.org/10.4153/CMB-1983-064-x Published online by Cambridge University Press