DARBOUX PROPERTIES AND APPLICATIONS TO NON-ABSOLUTELY CONVERGENT INTEGRALS

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1. Introduction and summary of results. This paper consists of two parts. The first contains an outline of the theorems and principal results and the second (§§2-6) gives proofs of the theorems and additional details. The theorems concern properties of Darboux continuous functions and functions having generalized Darboux properties. The corresponding results are shown to have interesting applications to the theory of non-absolutely convergent integrals.

1.1. Definitions and notation. We consider only finite single valued functions of a single real variable. We denote the set of points on the y-axis constituting the image of the set E or closed interval $[a, b]$ defined by the function $F(x)$ by $F[E]$ or $F[a, b]$ respectively. The inverse function $F^{-1}(y)$ denotes the set of points on the x axis for which $F(x) = y$. We denote the closed interval on the y-axis with end points $F(a)$ and $F(b)$ by $[a, b]Fy$, omitting the F when no confusion results.

DEFINITION 1. *A finite function $F(x)$ fulfils Lusin's condition* (N) *on the set E if $|F[S]| = 0$ for every subset S of E for which $|S| = 0$* [**13**, p. 224].

DEFINITION 2. *The function $F(x)$ has the Darboux properties* D, D_k $(0 < k \leqslant 1)$, D_*, D_d *on $[a, b]$ if the following conditions are satisfied for every interval $[l, m]$, $a \leqslant l < m \leqslant b$, where μ^* denotes outer Lebesgue measure*:

(D) $\quad F[l, m] \supset [l, m]y \quad$ (*Darboux continuity*),

(D_k) $\quad \mu^*\{F[l, m] \,.\, [l, m]y\} \geqslant k|[l, m]y|, \quad 0 < k \leqslant 1,$

(D_*) $\quad \mu^*\{F[l,m] \,.\, [l, m]y\} > 0, \quad$ *when* $F(l) \neq F(m)$,

(D_d) $\quad F[l, m]$ *is everywhere dense on* $[l, m]y$.

THEOREM A. *For measurable functions the condition* (N) *is necessary and sufficient in order that the function should transform every measurable set into a measurable set.*

If, therefore, in definition 2 $F(x)$ is measurable and fulfils the condition (N) on $[l, m]$ then $F[l, m]$ is a measurable set.

A set E is dense in the sense of order if there is a point of E between every two points of E.

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DEFINITION 3. *The function $F(x)$ has the property* D' *on* $[a, b]$ *if* $F(x)$ *is Darboux continuous on* $[a, b]$ *and if the values* $y \in F[a, b]$ *for which* $F^{-1}(y)$ *is countably infinite and dense (in the ordered sense) form a null set. The function* $F(x)$ *has the property* D'' *on* $[a, b]$ *if it has the property* D' *on every interval* $[l, m]$, $a \leqslant l < m \leqslant b$.

DEFINITION 4. *Where* D_x *denotes an arbitrary Darboux property we say that* x_0 *is a point of* D_x*-discontinuity of* $F(x)$ *on the right (left) if there exists no interval* $[x_0, x_0 + h]$ ($[x_0 - h, x_0]$) *over which* $F(x)$ *has the property* D_x.

DEFINITION 5. *The function* $F(x)$ *is* CG *on a set* E *if* E *can be expressed as the sum of a denumerable sequence of sets* E_n *over each of which* $F(x)$ *is continuous. If each set* E_n *is closed we say that* $F(x)$ *is* [CG] *on* E. *With continuity replaced by absolute continuity we define* ACG *and* [ACG] *in the same way.*

Properties ACG and [ACG] imply the condition (N) [**13**, p. 225], CG and [CG] do not.

1.2. Examples. The following examples, given in the appendix, are listed here for purposes of reference:

Example 1. A function $F(x)$ that has property D_d and is ACG on [0, 1] with $DF = 0$ almost everywhere on [0, 1] where $F(x)$ is not constant and has none of the properties D_*, D_k ($k \leqslant 1$), D, D' or D'' on [0, 1].

Example 2. A function that is Darboux continuous and ACG on [0, 1] with $DF > 0$ almost everywhere on [0, 1] where $F(x)$ does not have properties D' or D'' and is not non-decreasing on [0, 1].

Example 3. A function $F(x)$ that has property D_1 and is ACG on [0, 1] with $DF > 0$ almost everywhere on [0, 1] where $F(x)$ is neither Darboux continuous nor non-decreasing on [0, 1].

Example 4. A function $G(x)$ that is Darboux continuous and ACG on [0, 1] and a function $H(x)$ that is continuous and [ACG] on [0, 1] where $G(x) + H(x) = F(x)$ is the function defined in Example 1.

1.3. Theorems and results concerning Darboux properties. From Definitions 2 and 3 it follows that, in the sequence of Darboux properties D'', D', D, D_{k_2}, D_{k_1} ($1 \geqslant k_2 > k_1$), each implies all those following it. Furthermore, property D_k ($k > 0$) implies both D_* and D_d. That D_* does not imply any of the others even for functions that are [ACG] is shown by the example: $F(x) = x$, $x \leqslant a$; $F(x) = x + 1$, $x > a$. Example 1 shows that D_d does not imply any of the others when $F(x)$ is ACG, Example 2 that D does not imply D' or D'' and Example 3 that D_1 does not imply D for functions that are ACG.

THEOREM I. *Property* D_k, $k < 1$, *implies property* D_1.

The next theorem is of more general interest as it gives conditions that are sufficient to ensure that a function is Darboux continuous.

THEOREM II. *For $F(x)$ to be Darboux continuous on $[a, b]$ it is sufficient that $F(x)$ be* [CG] *on* $[a, b]$ *and fulfil the condition*

$$(I) \qquad \begin{aligned} &\liminf_{h\to 0+} F(x+h) \leqslant F(x) \leqslant \limsup_{h\to 0+} F(x+h), \qquad && a \leqslant x < b, \\ &\liminf_{h\to 0+} F(x-h) \leqslant F(x) \leqslant \limsup_{h\to 0+} F(x-h), \qquad && a < x \leqslant b. \end{aligned}$$

It is shown by an example that the condition [CG] is not necessary. The characteristic function of the set of rational points on $[a, b]$ gives an example showing that, with [CG] replaced by ACG, the function $F(x)$ need have none of the Darboux properties defined above.

THEOREM II′. *If $F(x)$ fulfils the condition* (I) *of the preceding theorem on $[a, b]$ and is such that, where E denotes the set of points of* D-*discontinuity* (*on either side*) *of $F(x)$ on $[a, b]$, every closed subset of E contains a portion on which $F(x)$ is continuous then E is empty and $F(x)$ is Darboux continuous on $[a, b]$.*

THEOREM III. *If $F(x)$ is Darboux continuous and* [CG] *on* $[a, b]$, *then* $F(x)$ *has property* D′ *on* $[a, b]$, i.e. *the points $y \in F[a, b]$ for which $F^{-1}(y)$ is countably infinite and dense* (*in the ordered sense*) *form a null set.*

COROLLARY OF THEOREMS II AND III. *For functions that are* [CG], *properties* D″, D′,D, D_k ($k \leqslant 1$) *and* D_d *are equivalent.*

Banach has defined $F(x)$ to have the property (T_2) on $[a, b]$ if almost every value $y \in F[a, b]$ is taken not more than a countable number of times on $[a, b]$ [**13**, p. 277].

THEOREM IV. *If $F(x)$ is measurable and fulfils the condition* (N) *on the interval* $[a, b]$ *then $F(x)$ necessarily fulfils the condition* (T_2) *on* $[a, b]$.

Theorem IV is well known for continuous functions [**13**, p. 284]. From Theorems III and IV it follows that if $F(x)$ is a function that is Darboux continuous, [CG] on $[a, b]$ and fulfils the condition (N) on $[a, b]$, then the set E_y of points $y \in F[a, b]$ for which F^{-1} (y) is dense is a null set.

The sum of two functions that are [ACG] and Darboux continuous need retain none of the Darboux properties defined above as is shown by the example: $F(x) = - G(x) = \sin 1/x, x \neq 0$; $F(0) = 1, G(0) = 0$. Lebesgue [**10**, p. 98] has given an example of a function $F(x)$ that is measurable and Darboux continuous without $F(x) + x$ being Darboux continuous. The next theorem shows that when one of the functions is continuous the sum must retain some traces of Darboux continuity.

THEOREM V. *If $F(x)$ is Darboux continuous* (*not necessarily measurable*) *and $G(x)$ is continuous on $[a, b]$ then $H(x) = F(x) + G(x)$ has the property* D_d *on* $[a, b]$.

Example 4 shows that this is the only Darboux property that $H(x)$ need have even when both $F(x)$ and $G(x)$ are ACG. From Theorem V and the corollary after Theorem III we obtain

THEOREM VI. *If $F(x)$ is Darboux continuous and* [CG] *on* $[a, b]$ *and $G(x)$ is continuous on* $[a, b]$ *then $H(x) = F(x) + G(x)$ is Darboux continuous on* $[a, b]$.

The next theorems establish some differential properties of functions that are Darboux continuous or have some other Darboux property.

THEOREM VII. If *$F(x)$ fulfils Lusin's condition* (N) *and has property* D_* *on* $[a, b]$ *and if $ADF = 0$ almost everywhere on* $[a, b]$ *then $F(x)$ is constant on* $[a, b]$.

Example 1 shows that Theorem VII is not true with D_* replaced by D_d. It might be presumed that the corresponding theorem that $F(x)$ is non-decreasing when $ADF \geqslant 0$ almost everywhere on $[a, b]$ would be true. Example 2 shows that this is not the case even when $F(x)$ is Darboux continuous and ACG and $DF > 0$ almost everywhere on $[a, b]$.

THEOREM VIII. *If $F(x)$ fulfils the condition* (N) *on* $[a, b]$ *and if $ADF > 0$ almost everywhere on* $[a, b]$ *then a necessary and sufficient condition for $F(x)$ to be* AC *and increasing is that $F(x)$ have the property* D' *on* $[a, b]$.

COROLLARY. *If $F(x)$ is* ACG *on* $[a, b]$ *and has property* D' *on* $[a, b]$ *and if $ADF \geqslant 0$ almost everywhere on* $[a, b]$ *then $F(x)$ is* AC *and non-decreasing on* $[a, b]$.

THEOREM IX. *If $F(x)$ is* [ACG] *and has property* D_* *on* $[a,b]$ *and if $ADF \geqslant 0$ almost everywhere on* $[a, b]$ *then $F(x)$ is non-decreasing on* $[a, b]$. *With* D_* *replaced by* D, *$F(x)$ is also* AC.

THEOREM X. *If $F(x)$ is* [ACG] *and Darboux continuous on* $[a, b]$ *then a sufficient condition for $F(x)$ to be* (i) AC, (ii) *continuous and* [ACG] *on* $[a, b]$ *is the existence of a function $G(x)$ that is* (i) AC, (ii) *continuous and* [ACG] *respectively on* $[a, b]$ *and such that $ADG(x) \geqslant ADF(x)$ almost everywhere on* $[a, b]$.

Functions that are C_r- continuous [**3**] and M_r- continuous [**5**] and also [ACG] are Darboux continuous. Denjoy [**4**, p. 179] has proved the following theorem for which we give an alternative proof.

THEOREM XI. *A function that is approximately continuous is necessarily Darboux continuous.*

1.4. Non-absolutely convergent integrals. Definitions of non-absolutely convergent integrals have been given by many authors. Although continuity of the indefinite integral is required by the most familiar definitions, this is replaced in some definitions by weaker continuity properties such as approximate continuity and Cesàro continuity as in the case of Burkill's approximate Perron [**2**] and Cesàro-Perron integrals [**3**]. The most general approach to integrals of this kind is due to Ridder [**12′**]. His definitions require indefinite integrals to be [ACG] and to have an unspecified "additional property" that

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ensures the uniqueness of the definite integral. We shall show that the class of indefinite integrals corresponding to any one of the definitions that have been given for which indefinite integrals are characterized by some continuity property such as ordinary, approximate, or mean continuity is a subclass of the class of all functions that are [ACG] and Darboux continuous.

DEFINITION 6. *We call a class of functions, that are defined on* $[a, b]$ *and approximately derivable almost everywhere on* $[a, b]$, *a class of generalized primitives or indefinite integrals* (*a P-class*) *if it has the following fundamental properties*:

(1) *Linearity. Any linear sum of any two functions belonging to the class also belongs to the class.*

(2) *Uniqueness. Any two functions of the class having the same approximate derivative almost everywhere on* $[a, b]$ *differ by a constant on* $[a, b]$. This implies the uniqueness of definite integrals defined as increments of indefinite integrals over intervals.

(3) *Order. If* $F(x)$ *and* $G(x)$ *belong to the class and if* $ADF \geqslant ADG$ *almost everywhere on* $[a, b]$ *then* $F(b) - F(a) \geqslant G(b) - G(a)$.

(4) *Compatibility. Every P-class must be compatible with the class of all indefinite general Denjoy* (*D*) *integrals*, i.e. *if* $F(x)$ *belongs to an arbitrary P-class and* $G(x)$ *belongs to the class all indefinite D-integrals relative to an interval* $[a, b]$ *and if* $ADF = ADG$ *almost everywhere on* $[a, b]$ *then* $F(x)$ *and* $G(x)$ *differ by a constant on* $[a, b]$ This implies that any function that is D-integrable and integrable in some generalized sense must be integrable to the same value in both senses.

THEOREM XII. *Any linear subclass of the class of all functions that are Darboux continuous and* [ACG] *on an arbitrary interval* $[a, b]$ *is a P-class.*

A study of the properties of arbitrary linear subclasses of the class of Darboux continuous, [ACG] functions gives a general approach to the study of the properties of known general non-absolutely convergent integrals characterized by different continuity properties such as mean and approximate continuity [see §2.3]. In particular, properties (2) — (4) follow immediately from Theorem XII. Furthermore Theorem XII shows that any new definition of continuity that implies Darboux continuity and for which linear sums of continuous functions are continuous in the same sense will, along with [ACG], characterize a class of functions having the principal properties, including properties (1) — (4) of those classes characterized by mean and approximate continuity.

From the results of §1.3 we can consider the properties (2) — (4) possessed by more general linear classes. In the following table the properties listed in the first two columns determine a class of functions relative to an arbitrary interval $[a, b]$. The third column lists the properties possessed by arbitrary linear subclasses of these respective classes. When a property is not listed

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there exist linear subclasses of the corresponding general class that do not have the property.

Functions with the condition (N) that are approximately derivable	D_d	(1)
	D_*, $\mathrm{D}_k \equiv \mathrm{D}_1$ or D	(1), (2)
ACG	D_d	(1)
	D_*, D_1 or D	(1), (2)
	D′, D″	(1), (2) ,(3)
[ACG]	D_*	(1), (2), (3)
	$\mathrm{D}_d \equiv \mathrm{D}_1 \equiv \mathrm{D} \equiv \mathrm{D}' \equiv \mathrm{D}''$	(1), (2), (3), (4).

Although arbitrary P-classes cannot by definition be incompatible with the class of indefinite D-integrals there are P-classes that are not compatible with each other. It has been shown [**6**] for instance, that certain classes of approximately continuous integrals and mean continuous integrals are not compatible. This shows in particular that there can be no maximal P-class containing all other P-classes even if we consider only those P-classes that are sub-classes of all [ACG], Darboux continuous functions.

We can make correspond to each P-class a descriptive definition of integration whereby *a function $f(x)$ is integrable on $[a, b]$ if there exists a function $F(x)$ in the P-class with $ADF = f$ almost everywhere on $[a, b]$. The function F is the indefinite integral of f and $F(b') - F(a')$ is the definite integral of f over $[a', b']$, $a \leqslant a' < b' \leqslant b$.*

The paper concludes with a consideration of some further properties of general P-classes and the corresponding integrable functions.

2. Proof of the theorems and additional details.

2.1. *Proof of Theorem* A. Rademacher [**11**, p. 196] proves that the condition (N) is necessary and that, for continuous functions, it is also sufficient [**11**, p. 200]. To show that the condition is sufficient for measurable functions let $F(x)$ be a measurable function and A be an arbitrary measurable set over which F is defined and finite. Applying Lusin's Theorem [**13**, p. 72] for a sequence of values $\epsilon_n \to 0$, $A = \sum_{n=1}^{\infty} E_n + N$, where each set E_n is closed, where F is continuous over each E_n and N is null. Since a continuous function maps closed sets into closed sets $F[E_n]$ is measurable for each n and $F[A]$ is measurable as the sum of a denumerable sequence of closed sets plus a null set.

2.2. *Proof of Theorem I.* Suppose that $F(x)$ has the property D_k on $[a, b]$. This implies that if y lies between $F(l)$ and $F(m)$, $a \leqslant l < m \leqslant b$, there exists a sequence x_i of points of (l, m) with $\lim F(x_i) = y$.

Suppose that $F(l) \leqslant \alpha < \beta \leqslant F(m)$. The proof would be the same for $F(l) > F(m)$. Let $\{x_i\}$, $\{x'_i\}$ be sequences of points of $[l, m]$ with $\lim F(x_i) = \alpha$, $\lim F(x'_i) = \beta$. If there is a point $x \in [l, m]$ with $F(x) = \alpha$ we may take

$x_i = x$, $(i = 1, 2, \ldots)$ and a similar consideration applies to β.) For $\epsilon > 0$,

$$\frac{\mu^*\{F[l,m]\,.\,[\alpha,\beta]\}}{\beta-\alpha} > \frac{\mu^*\{F[l.m]\,.\,[x_i, x'_i]y\}}{|[x_i,x'_i]y|} - \epsilon$$
$$\geqslant \frac{\mu^*\{F[x_i,x'_i]\,.\,[x_i, x'_i]y\}}{|[x_i,x'_i]y|} - \epsilon$$
$$\geqslant k - \epsilon$$

for i sufficiently large. Since ϵ is arbitrary we conclude that

$$\mu^*\{F[l, m]\,.\,[\alpha, \beta] \geqslant k(\beta - \alpha)$$

for every α, β, $F(l) \leqslant \alpha < \beta \leqslant F(m)$. It follows from a theorem of Jacobsthal and Knopp [8] that

$$\mu^*\{F[l, m]\,.\,[\alpha, \beta]\} = (\beta - \alpha)$$

for all α, β, $F(l) \leqslant \alpha < \beta \leqslant F(m)$ and in particular that

$$\mu^*\{F[l, m]\,.\,[l, m]y\} = |[l, m]y|.$$

2.3. *Proof of Theorem* II. We shall use the following lemma.

LEMMA 1. *If $F(x)$ satisfies the condition (I) of Theorem* II *on an interval $[a, b]$ and if $F(x)$ is Darboux continuous on every interval $[a', b']$, $a \leqslant a' < b' \leqslant b$, then $F(x)$ is Darboux continuous on $[a, b]$.*

The lemma follows easily since (I) prevents $F(x)$ from having a discontinuity of the first kind at a or b.

Let S be the set of points of Darboux discontinuity of $F(x)$ on $[a, b]$. The set S is easily seen to be closed. Let E_n be the closed sets over which $F(x)$ is continuous and let $[\alpha, \beta]$ be any interval contained in $[a, b]$. By Baire's Theorem [13, p. 54] there exists an integer k and an interval $[\alpha', \beta']$ contained in $[\alpha, \beta]$ such that $E_k[\alpha', \beta'] \equiv [\alpha', \beta']$. It follows that $F(x)$ is continuous and therefore Darboux continuous on $[\alpha', \beta']$. Hence S is non-dense on $[a, b]$.

Lemma 1 shows that S cannot contain isolated points. By Baire's theorem there exists an interval $[l, m]$ containing points of S and an integer p such that $S[l, m] \equiv E_p[l, m]$ and therefore $F(x)$ is continuous on $S[l, m]$. Diminishing $[l, m]$ if necessary we may suppose without loss of generality that l and m are points of S with points of S on (l, m) and that there exists a value c between $F(l)$ and $F(m)$ that is not taken for x on $[l, m]$. Let (l_i, m_i) be the open intervals complementary to S on $[l, m]$. By Lemma 1 $F(x)$ has the property D on each of the closed intervals $[l_i, m_i]$.

Set $G(x) = F(x)$ for x any point of $S[l, m]$ and let $G(x)$ be linear on each interval $[l_i, m_i]$ contained in $[l, m]$. Then $G(x)$ is continuous and therefore Darboux continuous on $[l, m]$. It follows that any value c between $F(l) = G(l)$ and $F(m) = G(m)$ is taken by $G(x)$ at some point of (l, m). Now each value taken by $G(x)$ on an interval (l_i, m_i) lies between $F(l_i) = G(l_i)$ and $F(m_i)$

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$= G(m_i)$ and is therefore taken by $F(x)$ on (l_i, m_i). For the remaining points of $[l, m]$, $F(x) = G(x)$. It follows that $F(x)$ takes the value c on $[l, m]$ giving a contradiction. Hence S is empty and it follows that F is Darboux continuous on $[a, b]$.

Clearly Darboux continuity implies condition (I) of the theorem. On the other hand the function $H(x)$ of Example 4 is Darboux continuous without being [CG] showing that the [CG] condition is not necessary. Theorem II′ is a generalization of II obtained by retaining in the hypotheses only those properties of [CG] actually used in the proof of II. Since properties D_d, D_k $(k \leqslant 1)$ imply (I) of Theorem II we obtain the following corollary.

COROLLARY. *For functions that are* [CG] *properties* D_d, D_k $(k \leqslant 1)$ *and* D *are equivalent.*

2.4. *Proof of Theorem* III. We shall prove first:

THEOREM III′. *If* $F(x)$ *is Darboux continuous and* [CG] *on* $[a, b]$, *then* $F(x)$ *has the property* D″ *on* $[a, b]$.

LEMMA 2. *If* $F(x)$ *is continuous on* $[a, b]$ *then the set of values of* y *assumed a countably infinite number of times on* $[a, b]$ *for which* $F^{-1}(y)$ *is dense in the sense of order is empty. In particular* $F(x)$ *has property* D″ *on* $[a, b]$.

Consider any interval $[l, m]$, $a \leqslant l < m \leqslant b$ and suppose that $F(x)$ takes the value y_0 a countably infinite number of times on $[l, m]$. The set E of points of $[l, m]$ at which $F = y_0$ is closed and therefore contains its derived set E'. The set E' is then itself closed, non-empty, at most enumerable and must contain at least one isolated point x_0 for otherwise it would be perfect, contradicting the fact that E is countable. In the neighbourhood of x_0 the set E contains only isolated points and E is therefore not dense in the sense of order [cf. **13**, p. 281].

LEMMA 3. *If* $F(x)$ *has property* D″ *on every interval* $[a', b']$, $a < a' < b' < b$ *and if* F *satisfies* (I) *of Theorem* II *then* $F(x)$ *has property* D″ *on* $[a, b]$.

The function $F(x)$ is Darboux continuous on $[a, b]$ by Lemma 1. Consider a sequence $\{[a_n, b_n]\}$ of intervals with $a < a_n < b_n < b$, $a_n \to a$, $b_n \to b$. The points y of the set E_y of values assumed a countably infinite number of times on $[a, b]$ and for which $F^{-1}(y)$ is dense, are each assumed a countable number of times on one at least of the intervals $[a_n, b_n]$, and $F^{-1}(y) \, . \, [a_n, b_n]$ is dense. The set E_y is therefore null since it is contained in the sum of a countable number of null sets.

Proof of Theorem III. With S denoting the set of points of D″-discontinuity of F on $[a, b]$ we may repeat the steps of the proof of Theorem II and obtain an interval $[l, m]$ with l and m points of S, with F continuous on $S[l, \mathrm{m}]$ and where F has property D″ on the closures of each of the intervals (l_i, m_i) complementary to S on $[l, m]$. Let U_i and L_i be the upper and lower bounds of F on $[l_i, m_i]$. Since F is Darboux continuous on $[a, b]$ and continuous except at the end points of the interval, U_i and L_i are the maximum and minimum values attained by F on $[l_i, m_i]$.

It follows from Lemma 2 that the values of $y \in F[l, m]$ assumed only on $S[l, m]$ and for which $F^{-1}(y)$ is countable and dense form a null set since they are assumed at the same points by the continuous function $G(x)$. The set E_y of the values y taken two or more times on at least one interval $[l_i, m_i]$ and for which $F^{-1}(y)$ is countably infinite and dense is null since it is contained in the sum of a countable number of null sets. We have left to consider those values taken not more than once on any interval $[l_i, m_i]$. From the Darboux continuity of F it follows that the only values of y outside $[l_i, m_i]Fy$ not taken at least twice for x on $[l_i, m_i]$ are the values U_i, L_i and these form a countable set. Suppose that $y_0 = F(x_0) \in [l_i, m_i]Fy$ is taken only once on $[l_i\, m_i]$ and not more than once on any interval $[l_j, m_j]$, $j \neq i$. Then $G(x'_0) = y_0$ for some point x'_0, $l_i \leqslant x'_0 \leqslant m_i$. If there is no other point with $G(x) = y_0$ then $F(x)$ takes this value only at x_0. If there are other points with $G(x) = y_0$ there exists a point x'_1 nearest x'_0 with $G(x'_1) = y_0$ and with $x'_1 \in [l_j, m_j]$ for some j. Then $F(x) = y_0$ for some point $x_1 \in [l_j, m_j]$, there are no points between x_0 and x_1 at which $F(x) = y_0$ and $F^{-1}(y_0)$ is not dense . Hence F has property D$''$ on $[l, m]$ and we conclude that the set S is empty.

Since the property D$''$ always implies D$'$ Theorem III is also established. Since D$'$ implies D the corollary follows.

2.5. *Proof of Theorem* IV. By Lusin's theorem [**13**, p. 73] there exists for each n a closed set E_n with $(b-a) - |E_n| < 1/n$ and such that $F(x)$ is continuous over the set E_n. Let $F_n(x)$ denote the continuous function coinciding with $F(x)$ on E_n and linear on the intervals complementary to E_n on $[a, b]$. Then F_n fulfils the condition (N) on $[a, b]$ and therefore has the property (T_2) on $[a, b]$. Set $E = E_1 + E_2 + \ldots)$. For each n, denote by S_n the set of values of $F_n(x)$ taken a non-countable infinity of times on E. Then $|S_n| = 0$, $(n = 1, 2, \ldots)$. Every value taken a non-countable infinity of times on E must be taken a non-countable infinity of times by at least one function F_n. The measure of the set of all values taken a non-countable number of times by any one of the functions F_n does not exceed $|S_1| + |S_2| + \ldots = 0$. Finally $(b-a) - |E| = 0$, so that $F[(a, b)] - E = 0$, since F fulfils the condition (N) on $[a, b]$. We conclude that F has the property (T_2) on $[a, b]$.

2.6. *Proof of Theorem* V. Let $[a', b']$ be any interval, $a \leqslant a' < b' \leqslant b$ with $F(a') \neq F(b')$. Points of discontinuity of $F(x)$ and $H(x)$ correspond and both functions have the same saltus at each point of discontinuity. For $a' \leqslant x_0 < b'$ suppose that $U = \limsup_{h \to 0+} F(x_0 + h) \neq \liminf_{h \to 0+} F(x_0 + h) = L$. We may assume without loss of generality that $G(x_0) = 0$. Suppose that there exists an interval $[\alpha, \beta]$, $L < \alpha < \beta < U$ and a value δ such that $H(x)$ takes no values between α and β for x on $(x_0, x_0 + \delta)$. Given $\epsilon > 0$ there exists $h' \leqslant \delta$

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such that $|G(x_0 + h) - G(x_0)| < \epsilon$ for all $h < h'$. Hence on $(x_0, x_0 + h)$, $|F(x) - H(x)| < \epsilon$ and, for ϵ sufficiently small, there are values between α and β not taken by $F(x)$ on $(x_0,x_0 + h)$ contradicting the Darboux continuity of $F(x)$ in the neighbourhood of x_0. We have shown that $H[a', b']$ is everywhere dense on $[L, U]$.

The interval $[a', b']Hy$ is covered by the values corresponding to points of continuity of $H(x)$ together with the aggregate of intervals $[L, U]$ corresponding to all of the points of discontinuity (on either or both sides) of $H(x)$. We conclude that $H[a', b']$ is everywhere dense on $[a', b']Hy$.

2.7. *Proof of Theorem* VI. The function $H(x)$ is [CG], has property D_d by Theorem V, and is therefore Darboux continuous by the corollary following Theorem III.

3. Differential properties.

3.1. *Proof of Theorem* VII. Let E be the set of points of $[a, b]$ at which $ADF(x) = 0$. From the definition of E, $F(x)$ fulfils Saks's condition $(D_{\eta,1-\epsilon})$ [**13**, p. 290] at each point of E, where $0 < \epsilon < 1$ and $\eta > 0$ are arbitrary. Then [**13**, p. 290]

$$|F[E]| \leqslant 2\eta|E|/(1 - \epsilon).$$

Keeping ϵ fixed and letting $\eta \to 0$ shows that $|F[E]| = 0$. Since $|E| = b-a$ and $F(x)$ fulfils the condition (N) on $[a, b]$, $|F[a, b]| = 0$. Property D_* then implies that $F(x)$ is constant on $[a, b]$.

3.2. *Proof of Theorem* VIII. The necessity part follows from Lemma 2. We shall show that the condition is sufficient. Let $I = [a, b]$ and let S be the set of points at which $ADF \leqslant 0$. Then $|F[I]| = |F[I\text{-}S]|$ from the condition (N). Let E_y be the points of $F[I\text{-}S]$ taken at most a countable infinity of times on I. Since $F(x)$ is measurable and fulfils the condition (N) the points E_y taken more than a countable number of times form a null set by Theorem IV. Property D′ then implies that the set of values of E_y for which $F^{-1}(y)$ is dense in the ordered sense form a null set. Hence for almost every value $y \in E_y$, $F^{-1}(y)$ consists of a single point or contains two points x_0 and x_1 with no points between x_0 and x_1. The second case together with Darboux continuity implies that either $F(x) > y$ or $F(x) < y$ at all points of (x_0, x_1). This contradicts the fact that $ADF > 0$ at both x_0 and x_1. It follows that almost all points of E_y and therefore of $F[I]$ are assumed just once on I.

Suppose, if possible, that $F(a') = F(b')$ for a pair of points a', b' with $a \leqslant a' < b' \leqslant b$. Then F is constant on $(a' b')$ for, if $F(x') \neq F(a')$ at any point x', $a' < x' < b'$, all values between $F(a')$ and $F(x')$ are assumed by $F(x)$ on (a', x') and again on (x', b'). Hence a set of positive measure is assumed more than once which is a contradiction. Since $ADF \neq 0$ almost everywhere, $F(x)$ cannot be constant and we conclude that there is no value taken more than once on I. Thus since $F(x)$ is Darboux continuous it is strictly increasing or strictly decreasing and continuous on I. Since $ADF > 0$ almost everywhere, $F(x)$ is increasing, continuous and, in fact, AC [**13**, p. 227].

The corollary is established by applying the theorem to $F(x) - \epsilon x$ where $\epsilon > 0$ is arbitrary. It is necessary to replace the condition (N) by ACG in order to ensure that when a linear function is added to $F(x)$ the sum is subject to the condition (N).

3.3. *Proof of Theorem* IX. For functions that are [ACG] properties D_* and D are the only two distinct Darboux properties. If $F(x)$ is Darboux continuous Theorem IX follows immediately from Theorem III and the corollary of Theorem VIII. More generally property D_* ensures that if $F(x)$ is non-decreasing on (a, b) then it is non-decreasing on $[a, b]$. With this result replacing Theorem I_r of [**5**], the proof is the same as that of Theorem II_r of [**5**]. The example given before Theorem I shows that $F(x)$ need not then be continuous.

3.4. *Proof of Theorem* X. The function $H(x) = F(x) - G(x)$ is [ACG] on $[a, b]$ as the sum of two [ACG] functions, $H(x)$ is Darboux continuous on $[a, b]$ by Theorem VI and $ADH \geqslant 0$ almost everywhere on $[a, b]$. By Theorem IX $H(x)$ is AC on $[a, b]$. Therefore $F(x)$ is AC or continuous and [ACG] according as $G(x)$ is AC or continuous and [ACG].

To end this section we consider certain general continuity definitions implying Darboux continuity. Functions that are Cesàro [**3**] or M_1-continuous [**5**] are everywhere the derivative of their indefinite D-integrals and are therefore Darboux continuous by the classical theorem of Darboux.

3.5. *Proof of Theorem* XI. There is no difficulty in showing that every bounded approximately continuous function is everywhere the derivative of its indefinite Lebesgue integral [**4**, p. 172] and is therefore Darboux continuous. Suppose that $F(x)$ is not bounded and not Darboux continuous on $[a, b]$. There must then exist values $F(l) \neq F(m)$, $a \leqslant l < m \leqslant b$, and a number c between $F(l)$ and $F(m)$ that is not taken by $F(x)$ on (l, m). For $M > \max[|F(l)|, |F(m)|]$ define $F_M(x) = F(x)$ when $|F(x)| \leqslant M$, $F_M(x) = M$ when $F(x) > M$ and $F_M(x) = -M$ when $F(x) < -M$. The function $F_M(x)$ is then bounded and approximately continuous on (l, m); it is therefore Darboux continuous on (l, m) and so takes the value c on (l, m). It follows that $F(x)$ takes the value c on (l, m) giving a contradiction.

From Lemma 1_r and Theorem VII_r of [**5**] and Theorem II above, it follows that any [ACG], C_r- or M_r-continuous function [**3**, **5**] and in particular any indefinite C_rP- or GM_r-integral is Darboux continuous [cf. **5**, Theorem IX_r].

4. Applications to non-absolutely convergent integrals. The Riemann, Lebesgue, special Denjoy and Perron integrals are included in the general Denjoy integral, i.e., every function that is integrable in any of these senses is D-integrable to the same value. Since indefinite D-integrals are continuous and ACG they are Darboux continuous and [ACG], and the classes of indefinite integrals corresponding to each of the above definitions are subclasses of the class $[F]$ of all Darboux continuous, [ACG] functions. If infinite discontinuities in the final stage of the construction of the Young integral [**7**, p. 719] are not allowed, indefinite Young integrals are Darboux continuous [**7**, p. 720] and the methods of [**9**] (Theorems I and II) apply to show that they are [ACG].

Bosanquet's Cesàro-Lebesgue [**1**], Burkill's Cesàro-Perron [**3**], and Sargent's Cesàro-Denjoy [**14**] integrals of each order are included in the GM-integrals of the same order [**5**, Theorem VII]. Since indefinite GM_r-integrals $(r = 1, 2, \ldots)$ are Darboux continuous and [ACG], each of the classes of indefinite integrals corresponding to the above definitions is a subclass of $[F]$.

Finally, Burkill's approximate Perron integral [**2**] and Ridder's α-integrals are included in Ridder's β-integrals [**12**, p. 162]. Indefinite integrals in any of these senses are approximately continuous and therefore also Darboux continuous by Theorem XI. It follows from the descriptive definition of the β-integral [**12**, p. 148] that they are [ACG].

Each of the subclasses of $[F]$ mentioned above except the class of indefinite Young integrals is linear. For instance, the class of functions that are indefinite β-integrals on an interval $[a, b]$ coincides with the class of all functions that are [ACG] and approximately continuous on $[a, b]$. Since any linear sum of two [ACG], approximately continuous function is likewise [ACG] and approximately continuous this class is linear. On the other hand the class $[F]$ of all functions that are [ACG] and Darboux continuous is not linear since it contains, for example, the functions

$$F(x) = - G(x) = \sin \frac{1}{x-c}, \; x \neq c, a < c < b; F(c) = 1; G(c) = 0.$$

4.1. *Proof of Theorem* XII. If $F(x)$ and $G(x)$ belong to a linear subclass of $[F]$ so does $H(x) = F(x) - G(x)$, i.e., $H(x)$ is Darboux continuous and [ACG] on $[a, b]$. If $ADF(x) = ADG(x)$ almost everywhere on $[a, b]$ $ADH(x) = 0$ almost everywhere on $[a, b]$ and property (2) follows from Theorem VII. If $ADF \geqslant ADG$ almost everywhere then $ADH \geqslant 0$ almost everywhere and property (3) follows from Theorem IX. Finally, if $F(x)$ belongs to an arbitrary subclass of $[F]$ (not necessarily linear) and $G(x)$ belongs to the class of indefinite D-integrals both with respect to an interval $[a, b]$ then $H(x)$ is Darboux continuous by Theorem VI and is [ACG] as the sum of two [ACG] functions and, as before, $H(x)$ is constant if $ADF = ADG$ almost everywhere on $[a, b]$.

4.2. The theorems of §1.3 may also be used to investigate more general linear classes. A consideration of properties (1) and (2) leads us to rule out a wide class of functions that are almost everywhere approximately derivable. First suppose that $F(x)$ is a Cantor function, continuous, increasing, with $DF = 0$ almost everywhere on $[a, b]$. Then $F(x)$ and $2F(x)$ have the same derivatives almost everywhere on $[a, b]$ without differing by a constant. We consider only functions having Lusin's property (N), ruling out all Cantor functions and functions of a similar nature. Secondly, any linear class containing a step function $F(x)$ cannot have the uniqueness property (2) even though $F(x)$ fulfils the condition (N). We eliminate step functions by considering only functions having some Darboux property. (Actually all of the Darboux properties except D_* eliminate all functions having discontinuities of the first kind.) Theorem VII then shows that an arbitrary linear subclass of the class of all functions that are almost everywhere approximately derivable and have property D_* and fulfil the condition (N) on an interval $[a, b]$, has property (2), i.e., that functions of the class are uniquely determined apart from an additive constant by a knowledge of their approximate derivatives almost everywhere. The remainder of the table in §1.4 may be completed similarly from the results in §1.3.

To verify that there are linear subclasses of the class of all functions that are ACG and have property D′ on [0, 1] and do not have property (4) consider the class of functions $kH(x)$ where $H(x)$ is the function defined in Example 4 and where k runs through all real numbers. This class has properties (1)-(3) but Example 4 shows that it does not have property (4). Similar examples may be given where any other property is not listed in the table.

5. Additional properties of P-classes. We next state some additional properties of P-classes. We shall distinguish those P-classes that are subclasses of the class $[F]$ of all [ACG], Darboux continuous functions by P_*. P- and P_*-integrability (see 1.4) are taken below to refer to a single, arbitrary P- or P_*-class.

5.1. *If f and g are P-integrable on $[a, b]$ then any linear combination $\alpha f + \beta g$ is P-integrable and*

$$\int_a^b (\alpha f + \beta g)dx = \alpha\int_a^b f\,dx + \beta\int_a^b g\,dx.$$

5.2. *If f is P-integrable on $[a, b]$ then f is necessarily measurable* [**13**, p. 299].

5.3. *If f is P-integrable on $[a, b]$ and if there exists a Lebesgue integrable (D-integrable) function g such that $f \leqslant g$ almost everywhere on $[a, b]$ then f is Lebesgue (D-integrable) on $[a, b]$.*

This follows from Theorem X. In particular if f is bounded above or below and P_*-integrable on $[a, b]$ then it is necessarily Lebesgue integrable on $[a, b]$.

5.4. *Consider an arbitrary P_*-class containing the class of all* AC *functions. Given a non-decreasing sequence f_n of P_*-integrable functions whose generalized integrals over $[a, b]$ constitute a sequence bounded above, then the function $f(x) = \lim f_n(x)$ is P_*- integrable and*

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\int_a^b f_n(x)\,dx$$

[cf. **13**, p. 243].

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For a P_*-class that does not contain the class of all AC functions, the class consisting of all the members of the original class, all AC functions, and all linear combinations of members of the two classes, is a P_*-class and the theorem is true for it.

5.5. *If f is P_*-integrable and Darboux continuous on $[a, b]$ there exists a point ξ, $a < \xi < b$ such that*

$$\int_a^b f(x)\,dx = F(b) - F(a) = (b-a)f(\xi).$$

5.6. *Let f and g be P-integrable on $[a, b]$, F and G be indefinite P-integrals of f and g. Suppose that $F(x)G(x)$ belongs to the same P-class as F and G and that $F(x)g(x)$ is P-integrable on $[a, b]$. Then $f(x)G(x)$ is P-integrable on $[a, b]$ and*

$$\text{(i)} \qquad \int_a^b f(x)G(x)\,dx = F(b)G(b) - F(a)G(a) - \int_a^b F(x)g(x)\,dx.$$

Set $H(x) = F(x)G(x) - \int_a^x F(x)g(x)dx$. Since $\int_a^x F(x)g(x)dx$ belongs to the P-class, $H(x)$ belongs to the P-class by the linearity property. Now $ADG(x) = F(x)ADG(x) + ADF(x)g(x) - F(x)g(x) = f(x)G(x)$ almost everywhere on $[a, b]$ so that $F(x)G(x)$ is P-integrable on $[a, b]$ and (i) follows.

The necessity of the hypotheses that $F(x)G(x)$ belong to the P-class depends on the P-class. For instance, if the P-class is the class of all ACG and continuous, or all [ACG] and approximately continuous functions then $F(x)G(x)$ necessarily belongs to the same class. On the other hand the product of two mean continuous functions or even the product of a mean continuous and a continuous function need not be mean continuous.

The condition that $F(x)g(x)$ be P-integrable may be replaced by conditions on $g(x)$ if the P-class is one of the classes of indefinite Cesàro or GM_r-integrals. However, this condition is necessary for the class of all [ACG], approximately continuous functions since functions of this class need not be integrable in any sense.

6. Appendix (Details of examples listed in 1.2). We denote by $I_{pk} = [a_{pk}, b_{pk}]$ $(k = 1, 2, \ldots, 2^{p-1})$ the closures of the pth set of intervals deleted in the usual definition of Cantor's ternary set and let $E = [0, 1] - \sum_{p,k} I_{pk}$. Then $|E| = 0$. Each of the functions we shall define will be AC on each of the intervals I_{pk} and on the set E and therefore ACG on [0, 1]. Except for the continuous function defined in example 4 they will not be [CG] or [ACG].

Example 1. Arrange the rational numbers between 0 and 1 in the order 1, 1/2, 2/2, 1/3, ..., $1/n$, $2/n$, ... n/n, ... Let r_p denote the pth number of this sequence. Set $F(x) = 0$ on E and, on each of the intervals I_{pk} $(k = 1, 2, \ldots, 2^{p-1})$ set $F(x) = r_p$. Then $F(x)$ is ACG on $[a, b]$, $DF = 0$ almost everywhere on $[a, b]$ and F has property D_d without having any of the other Darboux properties.

Example 2. Define $F(x) = 0$ on E and on each interval I_{pk} set

$$F(x) = 1 + \frac{2(x - b_{pk})}{b_{pk} - a_{pk}}.$$

Example 3. Set $F(x) = 1, x \in E$,

$$F(x) = 1 + \frac{(2^p - 1)(x - b_{pk})}{2^p(b_{pk} - a_{pk})} \text{ on } I_{pk} \text{ for } p \text{ even},$$

$$F(x) = -1 + \frac{(2^p - 1)(x - a_{pk})}{2^p(b_{pk} - a_{pk})} \text{ on } I_{pk} \text{ for } p \text{ odd}.$$

Example 4. Let $F(x)$ be the function defined in example 1. Using as bases each of the closed intervals I_{pk} for which $r_p = i/n$ $(i = 1, 2, \ldots, n; n = 1, 2, \ldots)$, construct isosceles triangles with altitudes $1/n$. Define $G(x) = 0$ for $x \in E$ and define $G(x)$ as the corresponding point in the side of the triangle above x for the remaining points of [0, 1]. Then $G(x)$ is continuous on [0, 1] and is [ACG] on [0, 1] since it is AC on each of the closed intervals I_{pk} and on the perfect set $\overline{E}$, the closure of E. The function $H(x) = F(x) - G(x)$ is Darboux continuous, in fact has property D″, on [0, 1] while $H(x) - G(x)$ has property D_d without having any other Darboux property.

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