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OPERATORS ON LOCALLY CONVEX SPACES OF VECTOR-VALUED CONTINUOUS FUNCTIONS

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Let E and F be locally convex spaces and let K be a compact Hausdorff space. $C(K,E)$ is the space of all E-valued continuous functions defined on K, endowed with the uniform topology.

Starting from the well-known fact that every linear continuous operator T from $C(K,E)$ to F can be represented by an integral with respect to an operator-valued measure, we study, in this paper, some relationships between these operators and the properties of their representing measures. We give special treatment to the unconditionally converging operators.

As a consequence we characterise the spaces E for which an operator T defined on $C(K,E)$ is unconditionally converging if and only if (Tf_n) tends to zero for every bounded and converging pointwise to zero sequence (f_n) in $C(K,E)$.

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1. Introduction

Throughout this paper K is a compact Hausdorff topological space, Σ the Borel σ-field of K, E and F are quasicomplete Hausdorff locally convex spaces, P_E and P_F saturated families of seminorms defining the topologies of E and F respectively, $C(K,E)$ is the space of all continuous E-valued functions defined on K, with the uniform convergence topology.

We are interested in operators (= continuous linear operators) T from $C(K,E)$ to F and their operator-valued representing measures. The study of the relationship between an operator and its representing measure has been considered by many authors, see for instance [1], [2], [3], [5], [9], [11] or [12]. Some interesting characterisations for several properties of T in terms of properties of m are known when E and F are Banach spaces. In this paper we consider this class of problems in the general case where E and F are locally convex spaces.

The notation and terminology used and not defined can be found in [4] or [8].

Before proceeding further, let us give some definitions and results for reference purpose.

DEFINITION 1. [3] If $m:\Sigma \to L(E,F)$ is a *(finitely additive) operator-valued measure*, $q \in P_F$, $p \in P_E$ and $A \in \Sigma$ then define

$$\tilde{m}_{(p,q)}(A) = \sup\{q(\sum_{A_i\in\pi} m(A_i)x_i) : \pi\in\Pi(A), \quad x_i\in V_p\}$$

where $\pi(A)$ denotes the set of disjoint finite Σ-partitions of A and $V_p = \{x\in E: p(x) \le 1\}$.

We say that m has bounded semivariation if for each q in P_F there is a p in P_E with $\tilde{m}_{(p,q)}(K)$ finite, and we write $p \sim q$ to denote this correspondence.

THEOREM 2. [3]. *If* $T:C(K,E) \to F$ *is an operator, then there is a a unique representing measure* $m:\Sigma \to L(E,F'')$ *such that*

i) m *has bounded semivariation*

ii) *for* $x \in E$ *and* $z' \in F'$, $m_{xz'}(.) = \langle m(.)x, z'\rangle$ *is a finite regular Borel measure*

iii) *for* $f \in C(K,E)$

$$T(f) = \int_K f dm .$$

The reader could consult [3], [11] and [12] for more information about representing measures.

Remarks: It is easy to prove for $f \in C(K,E)$, $A \in \Sigma$ and $p \sim q$ that

$$(1) \qquad q\left(\int_A f dm\right) \le \tilde{m}_{(p,q)}(A) \sup\{p(f(t)) : t \in A\} .$$

If $x \in E$, the vector measure defined from Σ with values in F by $m_x(.) = m(.)x$ is the representing measure of the operator $T_x: C(K) \to F$, $T_x(\Psi) = T(x\Psi)$; so an easy extension of a classical theorem of Bartle, Dunford and Schwartz (VI.2.1. of [4]), proves that $m(\Sigma) \subseteq L(E,F)$ if and only if T_x is a weakly compact operator for every $x \in E$.

2. The strongly continuous at ϕ measures

In this section we introduce a new concept of semivariation for an operator-valued measure very helpful in characterising some properties of an operator T from $C(K,E)$ to F .

DEFINITION 3. For $m:\Sigma \to L(E,F)$, $q \in P_F$, B a bounded subset of E and $A \in \Sigma$, we define $\tilde{m}_{Bq}(A)$ by

$$\tilde{m}_{Bq}(A) = \sup\{q(\Sigma_{A_i \in \pi} m(A_i)x_i) : \pi \in \Pi(A), \ \{x_i\} \subset B\} .$$

We say that m is strongly continuous at ϕ (s.c.v.) if for each bounded set $B \subseteq E$ and each $q \in P_F$

$$\lim \tilde{m}_{Bq}(A_n) = 0$$

for every decreasing sequence $(A_n) \downarrow \phi$ in Σ .

When E and F are Banach spaces, the s.c.v. measures are the s-bounded measures of [3], or those with semivariation continuous at ϕ (see [2], [5] or [9]), so the representing measure of every compact, weakly compact, absolutely summing, nuclear or unconditionally converging operator possesses this property.

Now we study some properties of $\tilde{m}_{Bq}$.

For each z' in F' , let $m_{z'}$ be the vector measure, with values in the locally convex space $(E', \beta(E',E))$, defined by

$$\langle x, m_{z'}(A)\rangle = \langle m(A)x, z'\rangle \qquad x \in E,\ A \in \Sigma .$$

Whenever m is a representing measure, $m_{z'}$ has bounded variation, that is:

$$|m_{z'}|_{p'}(K) = \sup\{\sum_{i=1}^{n} p'(m_{z'}(A_i)):\quad \{A_i\} \in \Pi(K)\}<\infty$$

for every continuous seminorm p' on E' . Indeed each $|m_{z'}|_{p'}$ is a finite positive Borel regular measure on K .

It can also easily be shown that the following property holds:

If B is a bounded set in E, p_B is the seminorm defined on E' by $p_B(x') = \sup\{|\langle x,x'\rangle|: x \in B\}$ and $q \in P_F$, then

(2) $$\tilde{m}_{Bq}(A) = \sup\{|m_{z'}|_{p_B}(A): z' \in V_q^\circ\}, A \in \Sigma .$$

PROPOSITION 4. *Let $m:\Sigma \to L(E,F)$ be a representing measure, B a bounded disc (absolutely convex set) in E and $A \in \Sigma$, then:*

a) $\tilde{m}_{Bq}(A) = \sup\{q(\int_A f dm): f \in C(K,E),\ f(A) \subseteq B\}$ *for* $q \in P_F$,

b) $|m_{z'}|_{p_B}(A) = \sup\{|\langle\int_A f dm, z'\rangle|: f \in C(K,E),\ f(A) \subseteq B\}$ *for* $z' \in E'$.

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Proof. We prove a), the proof of b) is similar.

For $f \in C(K,E)$ and $f(A) \subseteq B$, there is a net (f_j) of Σ-simple functions which converges uniformly to f and $f_j(A) \subseteq B$ for every j. Then

$$q(\int_A f dm) = q(\lim \int_A f_j dm) \le \tilde{m}_{Bq}(A) \ .$$

On the other hand, for $\varepsilon > 0$ there is a partition $\pi \in \Pi(A)$ $\pi = \{A_1,\dots,A_n\}$, a finite set $\{x_1,\dots x_n\} \subseteq B$ and a $z' \in V_q^\circ$ such that

$$\tilde{m}_{Bq}(A) - \varepsilon < |\sum_{i=1}^{n} \langle m(A_i)x_i, z'\rangle| = |\sum_{i=1}^{n} m_{x_i z'}(A_i)| \ .$$

For the regularity of $m_{x_i z'}$, we can choose some compact sets $K_i \subset A_i$ and disjoint open sets $G_i \supseteq K_i$, with

$$|m_{x_i z'}|(A_i \setminus K_i) < \frac{\varepsilon}{2n} , \quad |m_{x_i z'}|(G_i \setminus K_i) < \frac{\varepsilon}{2n} \ .$$

(Here $|.|$ denotes the variation of the scalar measure).

Now there are functions $\Psi_i \in C(K)$, with $0 \le \Psi_i \le 1$, $\Psi_i(K_i) = \{1\}$ and $\Psi_i(K \setminus G_i) = \{0\}$. Let $f \in C(K,E)$ be

$$f = \sum_{i=1}^{n} x_i \Psi_i \ ,$$

then

$$\tilde{m}_{Bq}(A) - \varepsilon < |\sum_{i=1}^{n} m_{x_i z'}(A_i) - \sum_{i=1}^{n} m_{x_i z'}(K_i)| + |\sum_{i=1}^{n} m_{x_i z'}(K_i)$$

$$- \sum_{i=1}^{n} \int_A \Psi_i dm_{x_i z'}| + |\langle \int_A f dm, z'\rangle| < \varepsilon + q(\int_A f dm) \ .$$

Since ε is arbitrary, this completes the proof.

Remark: Looking at the above proof, we can deduce, when A is an open set, that:

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$$\tilde{m}_{Bq}(A) = \sup\{q(\int_A f dm) : f \in C(K,E),\ f(A) \subset B,\ \mathrm{supp}(f) \subseteq A\};$$

$$|m_{z'}|_{p_B}(A) = \sup\{|<\int_A f dm, z'>| : f \in C(K,E),\ f(A) \subseteq B,\ \mathrm{supp}(f) \subset A\}.$$

In the next theorem, the equivalence $a \Leftrightarrow d$ gives an interesting characterisation of the operators with s.c.v. representing measure.

THEOREM 5. *Let* $T:C(K,E) \to F$ *be an operator with representing measure* $m:\Sigma \to L(E,F'')$. *Then, the following assertions are equivalent:*

a) m *is s.c.v.;*

b) $m(\Sigma) \subseteq L(E,F)$ *and for every bounded disc* $B \subseteq E$ *and every* $q \in P_F$, *the set of scalar measures* $\{|m_{z'}|_{p_B} : z' \in V_q^\circ\}$ *is uniformly countably additive;*

c) *For each* B *and* q *as in b) there is a finite positive regular Borel control measure* μ *on* K *such that*

$$\lim_{\mu(A)\to 0} \tilde{m}_{Bq}(A) = 0 \ ;$$

d) (Tf_n) *tends to zero for every uniformly bounded sequence* $(f_n) \subseteq C(K,E)$ *converging pointwise to zero.*

Proof: The equivalence $a \Leftrightarrow b \Leftrightarrow c$ follows from (2) and from some classical results for sets of scalar measures (see I. 2 of [4] or IV. 9 of [6]).

$c \Rightarrow d)$ Let $(f_n) \subseteq C(K,E)$ be a uniformly bounded sequence, converging pointwise to zero, we shall prove that (Tf_n) tends to zero. Let $B \subseteq E$ be a bounded disc with $f_n(K) \subset B$ for every n . If $q \in P_F$, there is a finite positive regular Borel measure μ and a $\partial > 0$ such that

$$\tilde{m}_{Bq}(A) < \tfrac{1}{2} \ \text{when} \ \mu(A) < \partial$$

Now let $p \in P_E$ satisfy $p \sim q$, then the sequence $(p\circ f_n) \subseteq C(K)$ converges pointwise to zero, so that, by the Egoroff theorem, there is a

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$K_\circ \in \Sigma$, with $\mu(K\backslash K_\circ) < \partial$, and $n_\circ$ such that

$$p(f_n(s)) < \frac{1}{2\tilde{m}_{(p,q)}(K)}$$

for $s \in K$ and $n > n_\circ$. Then

$$q(Tf_n) \leq q(\int_{K_\circ} f_n dm) + q(\int_{K\backslash K_\circ} f_n dm) \leq \frac{\tilde{m}_{(p,q)}(K_\circ)}{2\tilde{m}_{(p,q)}(K)} + \tilde{m}_{Bq}(K\backslash K_\circ)$$

Hence $q(Tf_n) < 1$ for $n > n_\circ$ and we conclude that $(Tf_n) \to 0$.

$d \Rightarrow b)$ Since $C(K)$ has the reciprocal Dunford-Pettis property (see [7]), for each $x \in X$ the operator T_x is weakly compact, so $m(\Sigma) \subseteq L(E,F)$ and it suffices to show that for any bounded disc $B \subset E$ and any $q \in P_F$ the family of scalar measures $\{|m_{z'}|_{p_B} : z' \in V_q^\circ\}$ is uniformly countably additive. Indeed if it were not, then there is a sequence $(z_n') \subset V_q^\circ$, and another (G_n) of disjoint open sets in K , with

$$|m_{z_n'}|_{p_B}(G_n) > \epsilon .$$

Now by proposition 4 and its remark, we can choose a sequence of functions $(f_n) \subset C(K,E)$ such that for every n we have

$$f_n(K) \subseteq B,\ f_n(K\backslash G_n) = \{0\} ,\quad |\langle\int_K f_n dm, z_n'\rangle| > \epsilon$$

This sequence is uniformly bounded and converges pointwise to zero. However (Tf_n) does not converge to zero in F because

$$q(Tf_n) \geq |\langle \int_K f_n dm, z_n' \rangle| > \epsilon$$

and this contradicts d).

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3. Unconditionally converging operators

In the following, we are going to characterise the unconditionally converging operators from $C(K,E)$ to F.

Recall that an operator T between E and F is unconditionally converging if T maps weakly unconditionally Cauchy (w.u.c.) series into unconditionally convergent ones, or, what is equivalent, (Tx_n) tends to zero in F when Σx_n is a w.u.c. series in E.

The next result follows from 14.6 of [8].

LEMMA 6. *For every sequence (x_n) in E, the following assertions are equivalent:*

a) Σx_n *is w.u.c.;*

b) $\Sigma |< x_n, x'>| < \infty$ *for each* $x' \in E'$*;*

c) $\{\sum_{n \in M} x_n : M \in F(\mathbb{N})\}$ *is a bounded set in* E.

Here $F(\mathbb{N})$ *denotes the system of all finite subsets of* $\mathbb{N}$.

THEOREM 7. *Let* $T: C(K,E) \to F$ *be an unconditionally converging operator, then its representing measure* m *satisfies*

a) m *is s.c.v.;*

b) for every $A \in \Sigma$, $m(A): E \to F$ *is an unconditionally converging operator.*

Proof. The proof of a) is just like that of "$d \Rightarrow b$" in Theorem 5, since the sequence (f_n) mentioned there satisfies:

i) $\{\sum_{n \in M} f_n : M \in F(\mathbb{N})\}$ is a bounded set in $C(K,E)$. So Σf_n is a w.u.c. series;

ii) (Tf_n) does not converge to zero.

b) Suppose that T is an unconditionally converging operator, $A \in \Sigma$ and Σx_n a w.u.c. series in E. We shall prove that $(m(A)x_n)$ tends to zero in F.

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Let B be a bounded disc in E such that $\{x_n : n \in \mathbb{N}\} \subseteq B$. If $q \in P_F$, using the existence of a regular control measure for $\tilde{m}_{Bq}$, we can find a compact H and an open G in K with $H \subseteq A \subseteq G$ and $\tilde{m}_{Bq}(G\backslash K) < \frac{1}{2}$, then there is a function $\Psi \in C(K)$ such that $0 \le \Psi \le 1$, $\Psi(G\backslash H) = \{0\}$ and $\Psi(H) = \{1\}$. We define $f_n \in C(K,E)$ by $f_n = x_n\Psi$, it is clear that Σf_n is a w.u.c. series, so (Tf_n) tends to zero and we have

$$q(Tf_n - m(A)x_n) = q\left(\int_K (\Psi - \chi_A x_n)dm\right) \le \tilde{m}_{Bq}(G\backslash H) < \tfrac{1}{2} .$$

Therefore we obtain that $q(m(A)x_n) < 1$ for almost every n. Hence $(m(A)x_n)$ converges to zero and the proof is complete.

An immediate consequence of Teorems 5 and 7 is:

COROLLARY 8. *If $T:C(K,E) \to F$ is an unconditionally converging operator, then (Tf_n) tends to zero for every uniformly bounded sequence $(f_n) \subset C(K,E)$ converging pointwise to zero.*

The converse of the above result is not true in general. Now, we characterise those spaces E for which this converse holds.

DEFINITION 9. *A locally convex space E is weakly Σ-complete if every w.u.c. series in E is weakly convergent.*

All the weakly sequentially complete spaces, and so all the semi-reflexive ones, are weakly Σ-complete. An easy extension of the Bessaga-Pelczynski theorem proves that a sequentially complete locally convex space E is weakly Σ-complete if and only if it does not contain a copy of C_0.

If E is a weakly Σ-complete space, the converse of Corollary 8 is true; furthermore this property characterises the weakly Σ-complete spaces, as we prove in the next theorem.

THEOREM 10. *The following assertions are equivalent;*

a) *E is weakly Σ-complete;*

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b) *for any compact Hausdorff space* K *and any space* F, *an operator* $T:C(K,E) \to F$ *is unconditionally converging if and only if its representing measure is s.c.v.;*

c) *there is a compact* K *such that every operator* T *from* $C(K,E)$ *to* E *with representing measure s.c.v. is unconditionally converging.*

Proof. a ⇒ b) Let Σf_n be a w.u.c. series, then $\Sigma f_n(t)$ is weakly convergent for every $t \in K$, then, according to the Orlicz-Pettis theorem, $\Sigma f_n(t)$ is convergent for each t. Therefore (Tf_n) tends to zero in F, because (f_n) is a uniformly bounded sequence converging pointwise to zero in $C(K,E)$ and m is s.c.v.

b ⇒ c) Trivial.

c ⇒ a) First we fix $a \in K$ and define an operator T on $C(K,E)$ by $T(f) = f(a)$. Then, by Theorem 5, the representing measure of T is s.c.v., so T is unconditionally converging.

Now we consider a function $\Psi \in C(K)$ with $0 \le \Psi \le 1$ and $\Psi(a) = 1$. If Σx_n is a w.u.c. series in E, then Σf_n, with $f_n = x_n \Psi$ is w.u.c. in $C(K,E)$, so $\Sigma T(f_n) = \Sigma x_n$ is unconditionally convergent in E. Hence E is weakly Σ-complete.

The result "$b \Rightarrow a$" of the above theorem extends, with an easier proof, an analogous theorem proved by Saab in [9] for E and F Banach spaces.

Bombal and Cembranos show in [2] that conditions a) and b) in theorem 7 characterise the unconditionally converging operators from $C(K,E)$ to F, for E and F Banach spaces, if and only if K is a dispersed compact (that is, it does not contain any perfect set). In our case this result is also true.

THEOREM 11. *Let* K *be a dispersed compact and* T *an operator from* $C(K,E)$ *to* F, *with representing measure* m, *then the following assertions are equivalent:*

a) T *is an unconditionally converging operator*

b) m *is s.c.v. and for each* $A \in \Sigma$, $m(A):E \to F$ *is an unconditionally converging operator.*

Proof. The proof of "$b \Rightarrow a$" is similar to that of Theorem 7 of [2], but we use that for a regular Borel measure μ in a dispersed compact K there is a countable family (x_n) in K such that

$$\mu = \sum_{n \in \mathbb{N}} \mu(x_n)\, \partial_{x_n}$$

(see [10] p.338) instead, to consider a metrisable quotient of K.

Remark: It is also possible to prove an analogue of the previous theorem for compact and weakly compact operators from $C(K,E)$ to F.

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