Canad. Math. Bull. Vol. **46** (4), 2003 pp. 538–545

Subdifferentials Whose Graphs Are Not Norm×Weak* Closed

Jonathan Borwein, Simon Fitzpatrick and Roland Girgensohn

Abstract. In this note we give examples of convex functions whose subdifferentials have unpleasant properties. Particularly, we exhibit a proper lower semicontinuous convex function on a separable Hilbert space such that the graph of its subdifferential is not closed in the product of the norm and bounded weak topologies. We also exhibit a set whose sequential normal cone is not norm closed.

1 Introduction

The subgradient of a convex function provides a central example for two modern theories:

(i) non-smooth analysis ([B-L], [Cl], [R-W]) and
(ii) maximal monotone operators ([Ph], [Si]).

In both settings one constructs a multi-function that emulates a derivative and is hoped to have good closure properties as is described in detail for monotone operators below. Such closure, and semi-continuity, properties are quite essential for both analytic and algorithmic use of subdifferentials. They are less critical in the production of calculus rules and of necessary optimality conditions [Mo].

A proper lower semicontinuous convex function function f on E is a function which takes values in $(-\infty,\infty]$ with epi $f := \{(x,r) \in E \times \mathbb{R} : r \geq f(x)\}$ a closed convex nonempty set. For such a function we define the subdifferential by $x^* \in \partial f(x)$ provided $x^* \in E^*$ satisfies $f(y) - f(x) \geq \langle y - x, x^* \rangle$ for all $y \in E$. For other functions there are more complicated ways of defining a useful subdifferential.

Thus, one typically has some notion of the generalized subdifferential as a mapping from a Banach space to its dual. Then one establishes, under appropriate local boundedness or compactness hypotheses, that its graph is closed in the product of the norm and weak topologies. Now local boundedness is automatic for locally Lipschitz, and hence continuous convex, functions and so no pathology occurs in this setting. Similarly, in finite dimensions little can go wrong, because the unit ball is norm compact.

Alternatively, one takes limits of bounded weak-star convergent nets or sequences and fails, in general, to obtain a topologically closed graph. Such issues are addressed in [B-F2], [M-S] and Section 4.

It is the purpose of this note to show that such a dichotomy is intrinsic by (i) exhibiting a proper lower semicontinuous convex function on a separable Hilbert space

Received by the editors December 19, 2001; revised January 21, 2003.
AMS subject classification: Primary: 46N10; secondary: 47H05.

such that the graph of its subdifferential is not closed in the product of the norm and bounded weak topologies, and (ii) showing that the sequential limiting normal cone (or limiting proximal normal cone) need not even be norm closed.

The paper is organized as follows. In Section 2, we discuss monotone operators and provide our core examples. In Section 3, we make an extension to show that such behaviour occurs in all infinite dimensional Banach spaces. Finally, in Section 4 we record a corresponding failure of the normal cone. Our notation when not given explicitly is consistent with [B-L], [Cl], and [Ph].

2 The Graph of a Monotone Operator

Let E be a Banach space and let M be a *maximal monotone operator* on E. We recall that a multifunction $M\colon E \to 2^{E^*}$ is *monotone* if $\langle x^* - y^*, x - y\rangle \geq 0$, whenever $x^* \in M(x)$ and $y^* \in M(y)$. Also, M is maximal if its graph is maximal with respect to set inclusion among all monotone mappings ([Ph], [Si]). This means that $\langle x^* - y^*, x - y\rangle \geq 0$ for all $(y, y^*) \in \operatorname{graph} M$ allows us to conclude that $(x, x^*) \in \operatorname{graph} M$. A maximal monotone operator is very well behaved topologically on the interior of its domain [Ph], but not more generally as we now indicate.

If (x_α, x_α^*) is a net in the graph of M which converges to (x, x^*) norm×weak* then for every $y^* \in M(y)$ we have $\langle x_\alpha^* - y^*, x_\alpha - y\rangle \geq 0$. This implies

$$
\begin{aligned}
\langle x^* - y^*, x - y\rangle &= \langle x_\alpha^* - y^*, x - y\rangle + \langle x^* - x_\alpha^*, x - y\rangle \\
&= \langle x_\alpha^* - y^*, x_\alpha - y\rangle + \langle x_\alpha^* - y^*, x - x_\alpha\rangle + \langle x^* - x_\alpha^*, x - y\rangle \\
&\geq \langle x_\alpha^*, x - x_\alpha\rangle - \langle y^*, x - x_\alpha\rangle + \langle x^* - x_\alpha^*, x - y\rangle.
\end{aligned}
$$

Since $\langle y^*, x - x_\alpha\rangle \to 0$ and $\langle x^* - x_\alpha^*, x - y\rangle \to 0$, if we could show $\langle x_\alpha^*, x - x_\alpha\rangle \to 0$ we would have $\langle x^* - y^*, x - y\rangle \geq 0$ and therefore $x^* \in M(x)$ by maximality.

Clearly if the set of x_α^* is bounded then this holds. (Note, en passant, that all weak*-convergent sequences on the dual of a Banach space are necessarily bounded. This implies that graph M is norm×weak* sequentially closed.) So if we define a topology τ on E^* by declaring that a net x_α^* converges to x^* for τ if and only if x_α^* converges to x^* weak* and the x_α^* are bounded then the graph of M is norm×τ closed.

A weaker (than τ but stronger than weak*) but related topology on E^* is the bounded weak* topology, bw*, which is the polar topology generated by the compact subsets of E, see [Ho]. Some unbounded nets converge for this topology so the above argument does not work for bw* in the absence of local boundedness of M. (See [B-F], [Ph] or [Si] for some conditions under which a monotone operator is locally bounded.) However, it seems to be worthwhile to give an explicit example to show that the graph of M can fail to be norm×bw* closed.

If the Banach space E is reflexive then the bw* topology on E^* is just the bounded weak topology, bw, on E^* (as bw is then generated by the compact subsets of E^{**} which is just E) and that is the case for our example which is a maximal cyclically monotone operator (see [Ph]) on separable Hilbert space.

https://doi.org/10.4153/CMB-2003-051-5 Published online by Cambridge University Press

Example 1 A proper lower semicontinuous convex function f on a separable Hilbert space such that the graph of the maximal monotone operator ∂f is not norm $\times$ bw closed.

Let $E := \ell_2(\mathbb{N})$. To make things clearer we will keep E^* and E separate. Define

$$e_{p,m} := \frac{1}{p}(e_p + e_{p^m}), \quad e^*_{p,m} := e^*_p + (p-1)e^*_{p^m}$$

for $m, p, r, s \in \mathbb{N}$, p prime and $m \geq 2$. Here e_n and e^*_n denote the unit vectors in E and E^* respectively.

Then we have

$$\langle e^*_{p,m}, e_{p',m'} \rangle = \begin{cases} 0, & \text{if } p \neq p' \\ 1/p, & \text{if } p = p', m \neq m' \\ 1, & \text{if } p = p', m = m'. \end{cases}$$

Further, for $x \in E$ define

$$f(x) := \max(\langle e^*_1, x\rangle + 1, \sup\{\langle e^*_{p,m}, x\rangle : p \text{ prime}, m \geq 2\})$$

so f is a proper lower semicontinuous convex function on E. Then $f(0) = f(e_{p,m}) = 1$, $f(-e_1) = 0$ and $f(x) \geq \langle e^*_{p,m}, x\rangle$ for all $x \in E$ and p prime, $m \geq 2$, which implies $e^*_{p,m} \in \partial f(e_{p,m})$. In fact,

$$f(x) - f(e_{p,m}) = f(x) - 1 \geq \langle e^*_{p,m}, x\rangle - 1 = \langle e^*_{p,m}, x - e_{p,m}\rangle \quad \text{for all } x \in E.$$

We also have $0^* \notin \partial f(0)$, since $0^* \in \partial f(0)$ is equivalent to $f(x) - f(0) \geq 0$ for all $x \in E$ (immediately from the definition of ∂f), which is not true for $x = -e_1$. Thus $(0, 0^*)$ is not in the graph of ∂f.

So we may now prove that the graph of ∂f is not norm$\times$bw closed by proving that $(0, 0^*)$ is in the norm$\times$bw closure of the set

$$\{(e_{p,m}, e^*_{p,m}) : p \text{ prime}, m \geq 2\} \subseteq \operatorname{graph} \partial f.$$

Informally, this is true, since $e_{p,m}$ tends in norm to 0 for large p, and also 0^* is a bw-cluster point of the $e^*_{p,m}$. A more precise argument is the following.

Let $\varepsilon > 0$ and a compact $A \subseteq E$ be arbitrarily given. We have to prove that there exist indices p, m with $\|e_{p,m}\| \leq \varepsilon$ and $e^*_{p,m} \in A^\circ$. Pick $n_0 \in \mathbb{N}$ such that $\|e_{p,m}\| \leq \varepsilon$ and $\sup_{a\in A}\langle e^*_p, a\rangle \leq 1/2$ for all $p \geq n_0$. (This is possible since $\|e_{p,m}\| = 2/p$ and A is compact, so that $\langle e^*_p, a\rangle$ tends to 0 for $p \to \infty$ uniformly in $a \in A$.)

Then for each prime p pick $m_0 = m_0(p)$ such that

$$\sup_{a\in A}\langle e^*_{p^m}, a\rangle \leq \frac{1}{2(p-1)} \quad \text{for all } m \geq m_0,$$

once more using compactness of A. Now for all $p \geq n_0$ and $m \geq m_0(p)$ we have

$$\sup_{a\in A}\langle e^*_{p,m}, a\rangle = \sup_{a\in A} t\big(\langle e^*_p, a\rangle + (p-1)\langle e^*_{p^m}, a\rangle\big) \leq \frac{1}{2} + \frac{1}{2},$$

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thus $\|e_{p,m}\| \leq \varepsilon$ and $\langle e^*_{p,m}, a\rangle \leq 1$ for all $a \in A$ (*i.e.*, $e^*_{p,m} \in A^\circ$). ■

Remarks 2 (a) Example 1 was constructed as a separable and more illustrative version of an earlier unpublished non-separable example due to the second author. In fact, this example takes $E = \ell^2([0,1])$ and defines

$$f_1(x) := \max\bigl(\langle e_0, x\rangle + 1, \sup\{r^{-1}\langle e_r, x\rangle : 0 < r \leq 1\}\bigr).$$

The interested reader will be able to emulate the previous argument. In the previous case we relied on an unbounded sequence having a weak* cluster point. Here we rely instead on the fact that $\{r^{-1}e_r : 0 < r \leq 1\}$ has 0^* in its bounded weak* closure. Indeed the polar of a compact set in E contains all but countably many points of $\{r^{-1}e_r : 0 < r \leq 1\}$.

This was constructed in response to a letter from Isaac Namioka, who pointed out that in an early draft of [B-F-K] the authors had assumed that the bounded weak* topology is better behaved than it actually is.

(b) As we noted, the graph ∂f is sequentially norm×weak* closed. Thus, the sequential closure is in general smaller than the topological closure, even for convex subdifferentials. In particular, we cannot have a sequence in $\{(e_{p,m}, e^*_{p,m}) : p$ prime, $m \geq 2\}$ that converges norm×bw to $(0, 0^*)$. Note also that Example 1 embeds the classical fact that in the weak topology

$$e_n + ne_m \rightharpoonup_m e_n \rightharpoonup_n 0,$$

so that 0 is in the weak sequential closure of the weak sequential closure of $\{e_n + ne_m\}$. Hence, we emphasize that the weak sequential convergence is not a closure operator.

(c) These examples also show that a local boundedness hypothesis is missing in Lemma 8 (i) of [Ko].

3 A General Construction

A similar construction works for an arbitrary Banach space E, using the fact that every separable subspace Y of a Banach space E has a *normalized Markushevich basis* (M-basis). Let $\{e_n\}$ be the (densely spanning) basis and $\{e^*_n\}$ its dual coefficients (which separate points of Y), satisfying $\langle e^*_m, e_n\rangle = \delta_{n,m}$ and $\|e^*_n\|\,\|e_m\| \leq 2$, [F-H-H, page 188]. Fix an infinite dimensional subspace Y, with M-basis as above.

Define $e_{p,m}$ and $e^*_{p,m}$ as before. Then define

$$f_2(x) := \begin{cases} \max\bigl(e^*_1(x) + 1, \sup\{e^*_{p,m}(x) : p \text{ prime}, m \geq 2\}\bigr), & \text{if } x \in Y, \\ +\infty, & \text{else.} \end{cases}$$

Again, f_2 is lsc and convex. Then, as before, $f_2(0) = f_2(e_{p,m}) = 1$, $f_2(-e_1) = 0$ and $f_2(x) \geq \langle e^*_{p,m}, x\rangle$ for all $x \in E$ and p prime, $m \geq 2$, which implies that $e^*_{p,m} \in \partial f_2(e_{p,m})$.

Since $\{e_n, e^*_m\}$ is biorthogonal and bounded, e^*_m converges weak* to 0. The same arguments now show that the graph of ∂f_2 is not norm×bw* closed. This proves:

https://doi.org/10.4153/CMB-2003-051-5 Published online by Cambridge University Press

Theorem 3 *Let E be a Banach space. The following are equivalent:*

(i) *E is finite dimensional.*
(ii) *The graph of ∂f is* norm$\times$bw* *closed for each closed proper convex f on E.*
(iii) *The graph of each maximal monotone operator T on E is* norm$\times$bw* *closed.*

This result again emphasizes that all limiting constructions of generalized gradients that capture the convex subdifferential must fail to be closed for general lower semi-continuous mappings, unless they are locally bounded. It would be interesting to determine whether such examples can be constructed with f having a point of continuity.

4 The Sequential Normal Cone

Similar, quite comprehensive, related problems arise when defining *normal cones* ([B-L], [Cl], [M-S], [R-W]) outside of finite dimensions. Recall that for $\varepsilon \geq 0$, the *ε-Fréchet normal cone* to a set Ω at a point $x \in \Omega$ is

$$\hat{N}_\varepsilon(x;\Omega) := \left\{ x^* \in E^* : \limsup_{u\to x, u\in\Omega} \frac{\langle x^*, u-x\rangle}{\|u-x\|} \leq \varepsilon \right\}.$$

We set $\hat{N}(x;\Omega) := \hat{N}_0(x;\Omega)$. Thus, in the case of a convex set, $\hat{N}(x;\Omega)$ coincides with the classical normal cone from convex analysis. The sequential *(limiting-Fréchet) normal cone* to a set Ω at a point $\bar{x} \in \Omega$ is then

$$N(\bar{x};\Omega) := \limsup_{x\to\bar{x},\,\varepsilon\downarrow 0} \hat{N}_\varepsilon(x;\Omega),$$

which again coincides with the convex normal cone for a convex set. Here, for a multifunction $\Lambda\colon E \to 2^{E^*}$, 'limsup' denotes the *sequential* Kuratowski-Painlevé upper limit with respect to the norm topology in E and the (bounded) weak-star topology in E^*:

$$\limsup_{x\to\bar{x}} \Lambda(x) := \{x^* \in E^* : \exists x_n \to \bar{x},\ x_n^* \rightharpoonup x^*,\ x_n^* \in \Lambda(x_n),\ \forall n \in \mathbb{N}\}.$$

We settle here for exhibiting the possible behaviour in $\ell_2(\mathbb{N})$. The general argument is quite similar, again using Markusevich bases [F-H-H]. The argument again exploits the fact that weak* convergent sequences are bounded.

Example 4 If Ω is a closed subset of separable Hilbert space then $N(0;\Omega)$ need not be even norm closed. Indeed, let $H := \ell_2(\mathbb{N})$ and let Ω be the norm closed set

$$\{s(e_1 - je_j) + t(je_1 - e_k) : k > j > 1,\ s,t \geq 0\} \cup \{te_1 : t \geq 0\}$$

where $e_1, e_2, \ldots, e_n, \ldots,$ is the usual basis for ℓ_2. Then $N(0;\Omega)$ is not closed since

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(i) $e_1^* + j^{-1}e_j^* \in N(0;\Omega)$,
(ii) $e_1^* \notin N(0;\Omega)$ and
(iii) $e_1^* + j^{-1}e_j^* \to e_1^*$.

Proof If $e_{j,k}^* := e_1^* + j^{-1}e_j^* + je_k^*$ for $1 < j < k$ then $e_{j,k}^* \in \hat{N}\big(k^{-1}(je_1 - e_k);\Omega\big)$, as is easily computed. For each j we have $k^{-1}(je_1 - e_k) \to 0$ and $e_{j,k}^* \rightharpoonup e_1^* + j^{-1}e_j^*$ as $k \to \infty$. Thus $e_1^* + j^{-1}e_j^* \in N(0;\Omega)$ which establishes (i). It is easy to verify (iii). Also it is not hard to show that Ω is closed.

So we need to show $e_1^* \notin N(0;\Omega)$. Suppose not: then there are $x_n \to 0$, $\epsilon_n \downarrow 0$ and $x_n^* \in \hat{N}_{\epsilon_n}(x_n;\Omega)$ such that $x_n^* \rightharpoonup e_1^*$. Suppose some $x_n = t_ne_1$ for $t_n \geq 0$. Put $u := x_n + re_1$ for $r > 0$ so we have

$$\epsilon_n \geq \limsup_{\substack{\Omega\\ u\to x_n}}\left\langle x_n^*, \frac{u - x_n}{\|u - x_n\|}\right\rangle \geq \limsup_{r\to 0+}\left\langle x_n^*, \frac{re_1}{\|re_1\|}\right\rangle = \langle x_n^*, e_1\rangle.$$

On the other hand, $x_n^* \rightharpoonup e_1^*$ implies $\langle x_n^*, e_1\rangle \to 1$, so that only finitely many x_n can be of the form $x_n = t_ne_1$ for $t_n \geq 0$. So all but finitely many x_n are necessarily of the form $s(e_1 - je_j) + t(je_1 - e_k)$ where $k > j > 1$, $s, t \geq 0$.

Now let $x_n = s(e_1 - je_j) + t(je_1 - e_k)$ where $s = s(n) \geq 0$, $t = t(n) \geq 0$, $j = j(n) > 1$ and $k = k(n) > j(n)$. Hence, considering $u := x_n + r(je_1 - e_k)$, we get

$$\begin{aligned}\epsilon_n &\geq \limsup_{\substack{\Omega\\ u\to x_n}}\left\langle x_n^*, \frac{u - x_n}{\|u - x_n\|}\right\rangle \geq \limsup_{r\to 0+}\left\langle x_n^*, \frac{r(je_1 - e_k)}{\|r(je_1 - e_k)\|}\right\rangle\\ &= \left\langle x_n^*, \frac{je_1 - e_k}{\|je_1 - e_k\|}\right\rangle,\end{aligned}$$

so that

$$\langle x_n^*, e_1 - j^{-1}e_k\rangle \leq \epsilon_n\sqrt{1 + j^{-2}}, \tag{1}$$

while considering $u := x_n + r(e_1 - je_j)$, we have

$$\begin{aligned}\epsilon_n &\geq \limsup_{\substack{\Omega\\ u\to x_n}}\left\langle x_n^*, \frac{u - x_n}{\|u - x_n\|}\right\rangle \geq \limsup_{r\to 0+}\left\langle x_n^*, \frac{r(e_1 - je_j)}{\|r(e_1 - je_j)\|}\right\rangle\\ &= \left\langle x_n^*, \frac{e_1 - je_j}{\|e_1 - je_j\|}\right\rangle,\end{aligned}$$

so that

$$\langle x_n^*, e_1\rangle \leq \langle x_n^*, je_j\rangle + \epsilon_n\sqrt{1 + j^2}. \tag{2}$$

Letting $n \to \infty$ in (1) we obtain

$$1 \leq \liminf\langle x_n^*, j(n)^{-1}e_{k(n)}\rangle.$$

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If the $j(n)$ are unbounded then this shows the sequence x_n^* is unbounded, contradicting its weak convergence. We therefore have only finitely many $j(n)$. But then (2) contradicts $x_n^* \rightharpoonup e_1^*$. ■

Remarks 5 (a) Since each $e_{j,k}^*$ is a proximal normal to Ω at $k^{-1}(je_1 - e_k)$ we see that a similar result holds for the *limiting proximal normal cone*, a preferred tool for many authors (see ([B-L], [Cl], [M-S], [R-W] for the definitions and for the consequences of using this alternative normal cone).

(b) Of course, one may simply take the closure in the definition of $N(\bar{x}; \Omega)$ but this has some drawbacks, as not every member of the normal cone is then a sequential limit. An alternative is to define the limiting Fréchet normal cone more topologically, which allows one to repair the lack of closure at the expense of a more cumbersome and less intuitive limiting construction. Related issues are discussed in [B-F2] and [M-S].

Acknowledgements This work was started by the first two authors during visits in the early nineties. It was completed while the third author was visiting Simon Fraser University in 2001. He thanks the university for its hospitality and his home institution, the GSF-Forschungszentrum, for granting him the leave. The first author's research was funded by NSERC and by the Canada Research Chair Programme.

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Department of Mathematics
Simon Fraser University
Burnaby, British Columbia
V5A 1S6
e-mail: jborwein@cecm.sfu.ca

Department of Mathematics
University of Western Australia
Nedlands, Western Australia
6009 Australia
e-mail: fitzpatr@maths.uwa.edu.au

GSF-Forschungszentrum
Postfach 1129
85758 Neuherberg
Germany
e-mail: girgen@gsf.de

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