CLOSABLE DERIVATIONS OF SIMPLE C^*-ALGEBRAS

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1. Introduction. In this note we show that any derivation of a simple C^*-algebra, whose range is not dense, is closable. We also derive a necessary and sufficient condition for a *-derivation of a C^*-algebra, which is defined on the domain of a closed *-derivation, to be closed.

A linear mapping δ from a dense *-subalgebra $D(\delta)$ of a C^*-algebra $\mathscr{A}$ into $\mathscr{A}$ is called a *derivation* if $\delta(ab)=\delta(a)b+a\,\delta(b)(a, b\in D(\delta))$. If in addition $\delta(a^*)=\delta(a)^*$, then δ is called a *-derivation. For a linear mapping δ from a linear subspace D of a Banach space $\mathscr{A}$ into $\mathscr{A}$, we let $\sigma(\delta)$ denote the set $\{b\in\mathscr{A}$: there is a sequence (a_n) in D with $a_n\to 0$ and $\delta(a_n)\to b\}$, and call it the *separating space of* δ. The hypothesis on $\sigma(\delta)$ forces $\sigma(\delta)$ to be a closed linear subspace of $\mathscr{A}$, and δ is closable if and only if $\sigma(\delta)=(0)$ [**4**, p. 8]. We show that the separating space of a derivation of a C^*-algebra is a closed two sided ideal. Then we apply this result to prove the main result of this paper. In the paper $R(\delta)$ denotes the range of the derivation δ; i.e. $R(\delta)=\delta(D(\delta))$. S. Sakai [**3**] has asked: when is the range of a closed *-derivation of a simple C^*-algebra not dense? Our result implies the answer to a converse question, namely, if the range of a *-derivation δ of a simple C^*-algebra is not dense, then δ is closable.

Let δ and δ_0 be derivations of a C^*-algebra defined on the same domains D, say; then δ is called δ_0-*bounded* if there is a number $M>0$ such that $\|\delta(a)\|\leq M(\|a\|+\|\delta_0(a)\|)$ $(a\in D)$. It follows from [**1**] that if δ_0 is a closed *-derivation and δ is a *-derivation with $D(\delta)\supseteq D(\delta_0)$, then δ is δ_0-bounded. Sakai conjectured that δ should be closable [**2**]. An easy argument shows that if $D(\delta)=D(\delta_0)$, then δ is closed if and only if δ_0 is δ-bounded.

2. The results. It will now be shown that, under one restriction, a derivation of a simple C^*-algebra admits a closed extension.

THEOREM 1. *Let $\mathscr{A}$ be a simple C^*-algebra and δ be a derivation of $\mathscr{A}$. Then δ is closable if $R(\delta)$ is not dense in $\mathscr{A}$.*

Proof. It suffices to show that the separating space of δ is $\{0\}$. The separating space $\sigma(\delta)$ is obviously a linear subspace of $\mathscr{A}$. We show that it is a closed two-sided ideal in $\mathscr{A}$. Suppose (b_n) is a sequence in $\sigma(\delta)$ and $b_n\to b$; then there is a sequence $(c_n)\subseteq D(\delta)$ such that $\|c_n\|<1/n$ and $\|\delta(c_n)-b_n\|<1/n$; it therefore follows that $c_n\to 0$ and $\delta(c_n)\to b$. We conclude that $\sigma(\delta)$ is closed. Let $c\in D(\delta)$ and $b\in\sigma(\delta)$; then there exists a sequence (a_n) in $D(\delta)$ such that $a_n\to 0$ and $\delta(a_n)\to b$. Hence $ca_n\to 0$, $a_nc\to 0$, and

$$\delta(ca_n)=\delta(c)a_n+c\,\delta(a_n)\to cb,$$

$$\delta(a_nc)=\delta(a_n)c+a_n\,\delta(c)\to bc.$$

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Thus $cb, bc \in \sigma(\delta)$. Suppose now that $b \in \sigma(\delta)$ and $c \in \mathscr{A}$. The density of $D(\delta)$ in $\mathscr{A}$ implies that there is a sequence (c_n) in $D(\delta)$ such that $c_n \to c$ and hence $c_n b$, $bc_n \in \sigma(\delta)$, cb, $bc \in \sigma(\delta)$, since $\sigma(\delta)$ is closed. Thus $\sigma(\delta)$ is a closed two-sided ideal in $\mathscr{A}$. It follows now that $\sigma(\delta) = \{0\}$ or $\mathscr{A}$. If $\sigma(\delta) = \mathscr{A}$, then $R(\delta)$ would be dense in $\mathscr{A}$; however, by our assumption $R(\delta)$ is not dense. This contradiction shows that $\sigma(\delta) = \{0\}$ and therefore δ is closable.

COROLLARY 2. *Let δ be a *-derivation of a simple C*-algebra $\mathscr{A}$. Then δ is closable if one of the two sets $\{a \pm \delta(a) : a \in D(\delta)\}$ is not dense in $\mathscr{A}$.*

Proof. This follows from the proof of the theorem above.

Let δ_0 be a closed *-derivation of a C*-algebra $\mathscr{A}$ and let δ be a *-derivation of $\mathscr{A}$ with the domain $D(\delta) = D(\delta_0)$. The next result gives a necessary and sufficient condition for δ to be closed. By **[1]** δ is δ_0-bounded. Suppose moreover that δ_0 is δ-bounded; then there exists two real numbers $M, K > 0$ such that for $a \in D(\delta) = D(\delta_0)$ we have

$$\|\delta(a)\| \leq M(\|a\| + \|\delta_0(a)\|),$$
$$\|\delta_0(a)\| \leq K(\|a\| + \|\delta(a)\|).$$

If $a_n \to a$ $(a_n \in D(\delta))$ and $\delta(a_n) \to b$, then

$$\|\delta_0(a_n) - \delta_0(a_m)\| = \|\delta_0(a_n - a_m)\| \leq K(\|a_n - a_m\| + \|\delta(a_n) - \delta(a_m)\|);$$

thus $(\delta_0(a_n))$ is a Cauchy sequence in $\mathscr{A}$ and hence is convergent. Thus $a \in D(\delta_0)$ and $\delta_0(a_n) \to \delta_0(a)$ and so $\delta(a_n) \to \delta(a)$. This gives the following theorem.

THEOREM 3. *Let δ_0 and δ be *-derivations of a C*-algebra $\mathscr{A}$. Suppose δ_0 is closed and $D(\delta) = D(\delta_0)$. Then δ is closed if and only if δ_0 is δ-bounded.*

3. Comments. Let $\mathscr{A}$ be a normed space. A subset $\mathscr{A}_0$ of $\mathscr{A}$ is said to be a G_δ set if there exists a countable family $\{G_n\}$ of open sets such that $\mathscr{A}_0 = \bigcap_{n=1}^{\infty} G_n$. For a closed linear mapping δ of a normed space $\mathscr{A}$ into $\mathscr{A}$ with $\overline{R(\delta)} = \mathscr{A}$, the closedness condition of $R(\delta)$ is equivalent to $R(\delta)$ being a G_δ set. In fact if $R(\delta)$ is of second category, then $R(\delta) = \mathscr{A}$. If we can show that the range $R(\delta)$ of a closed *-derivation δ of a simple C*-algebra $\mathscr{A}$ is not closed and is a G_δ set, then it follows that $\overline{R(\delta)} \subsetneq \mathscr{A}$. The following problem poses itself: suppose δ_0 is a closed *-derivation of a simple C*-algebra $\mathscr{A}$ and $\overline{R(\delta_0)} \subsetneq \mathscr{A}$. Let δ be a *-derivation of $\mathscr{A}$ with its domain $D(\delta) = D(\delta_0)$. Then can we conclude that $\overline{R(\delta)} \subsetneq \mathscr{A}$ and thus δ is closable?

Let δ_0 be a closed *-derivation of a C*-algebra $\mathscr{A}$ and δ be a *-derivation of $\mathscr{A}$ with its domain $D(\delta) = D(\delta_0)$; then $(\delta_0 - \delta)$ is δ_0-bounded. Hence there are two positive numbers N, M such that

$$\|(\delta_0 - \delta)(a)\| \leq N\|a\| + M\|\delta_0(a)\|.$$

An easy computation shows that δ is closed if $M < 1$.

We note that the proof of Theorem 1 shows that the separating space of a derivation from a Banach algebra is a closed two-sided ideal. It also follows that if T is a densely defined operator from a Banach algebra and $D(T)\sigma(T)\subseteq\sigma(T)$, $\sigma(T)D(T)\subseteq\sigma(T)$, then $\sigma(T)$ is a closed two-sided ideal.

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