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MULTIPLE SOLUTIONS FOR A KIRCHHOFF EQUATION WITH NONLINEARITY HAVING ARBITRARY GROWTH

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Abstract

We prove the existence of infinitely many solutions $u \in W_0^{1,2}(\Omega)$ for the Kirchhoff equation

$$-\Big(\alpha + \beta \int_\Omega |\nabla u|^2\,dx\Big)\Delta u = a(x)|u|^{q-1}u + \mu f(x,u) \quad \text{in } \Omega,$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $a(x)$ is a (possibly) sign-changing potential, $0 < q < 1$, $\alpha > 0$, $\beta \geq 0$, $\mu > 0$ and the function f has arbitrary growth at infinity. In the proof, we apply variational methods together with a truncation argument.

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1. Introduction

In this paper, we consider a version of the problem

$$-\Big(\alpha + \beta \int_\Omega |\nabla u|^2\,dx\Big)\Delta u = h(x,u) \quad \text{in } \Omega, u \in W_0^{1,2}(\Omega),$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $\alpha > 0$ and $\beta \geq 0$. It is the stationary state of the hyperbolic equation

$$v_{tt} - \Big(\alpha + \beta \int_\Omega |\nabla v|^2\,dx\Big)\Delta_x v = h(x,v) \quad \text{in } \Omega \times (0,T),$$

which was proposed, for $N = 1$, by Kirchhoff [10] as an extension of the classical d'Alembert wave equation for free vibrations of elastic strings. The main point in this model is that it allows changes to the length of the string during the vibration. Such problems are called nonlocal due to presence of the term $\int_\Omega |\nabla v|^2\,dx$. After the paper of Lions [11], this kind of problem has been the subject of intensive research.

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In [2], the authors presented a variational approach to deal with the stationary equation. Since then, many authors have applied critical point theory to obtain existence and multiplicity of solutions for related problems.

Here we are interested in the case where the right-hand side of the equation presents a competition between concave and convex terms near the origin. More specifically, we shall consider

$$\begin{cases} -\Big(\alpha+\beta\displaystyle\int|\nabla u|^2\,dx\Big)\Delta u = a(x)|u|^{q-1}u+\mu f(x,u) & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{P}$$

where $\Omega\subset\mathbb{R}^N$, $N\geq 3$, is a bounded smooth domain, $0<q<1$, $\alpha>0,\beta\geq 0$ and $\mu>0$. The main assumptions on f are that:

(f_0) $f\in C(\Omega\times\mathbb{R},\mathbb{R})$ and there exists $\delta>0$ such that $f(x,s)$ is odd in s for any $x\in\Omega$ and $|s|\leq\delta$; and

(f_1) $f(x,s)=o(|s|^q)$, as $s\to 0$, uniformly in Ω.

In order to introduce the regularity condition on the potential $a(x)$, we define $2^*:=2N/(N-2)$ and consider the sequence $(p_n)\subset\mathbb{R}$ defined as $p_1:=2^*$ and

$$p_{n+1}=\begin{cases}\dfrac{Np_n}{N-2p_n} & \text{if } 2p_n<N,\\ p_n+1 & \text{if } 2p_n\geq N,\end{cases} \tag{1.1}$$

for each $n\in\mathbb{N}$. A straightforward calculation shows that (p_n) is increasing and unbounded. Hence

$$m:=\min\{n\in\mathbb{N}:2p_n>N\}$$

is well defined. The main assumption on the potential $a(x)$ is

(a_0) $a\in L^{\sigma_q}(\Omega)$, with $\sigma_q:=p_m/(1-q)$.

We denote by H the Sobolev space $W_0^{1,2}(\Omega)$ with the norm $\|u\|=(\int_\Omega|\nabla u|^2\,dx)^{1/2}$. From the variational point view, the equation in (P) is the Euler–Lagrange equation of the energy functional

$$I(u)=\frac{\alpha}{2}\|u\|^2+\frac{\beta}{4}\|u\|^4-\frac{1}{q+1}\int a(x)|u|^{q+1}\,dx-\mu\int F(x,u)\,dx, \tag{1.2}$$

where $F(x,s):=\int_0^s f(x,t)\,dt$. Since we have no control over the behaviour of f at infinity, this functional is not well defined in the entire space H. However, in view of (f_0)–(f_1), it is finite for every function $u\in H\cap L^\infty(\Omega)$ such that $\|u\|_{L^\infty(\Omega)}$ is sufficiently small.

In our first result, we consider the definite case and prove the following theorem.

THEOREM 1.1. *Suppose that* $0<q<1$, *the function* f *satisfies* (f_0)–(f_1), *the potential* a *satisfies* (a_0) *and*

(a_1) *there exists* $a_0>0$ *such that* $a(x)\geq a_0$, *for almost every (a.e.)* $x\in\Omega$.

Then, for any $\alpha > 0$, $\beta \geq 0$ *and* $\mu > 0$, *the problem* (P) *has a sequence of solutions* $(u_k) \subset W_0^{1,2}(\Omega)$ *such that* $\|u_k\|_{L^\infty(\Omega)} \to 0$ *as* $k \to \infty$. *Moreover,* $I(u_k) < 0$ *and* $I(u_k) \to 0$ *as* $k \to \infty$.

We emphasise that the theorem holds independently of the growth of f far from the origin. In order to proceed variationally, we use an argument borrowed from [12]. It consists of considering a modified functional J_θ, defined in the entire space H, whose critical points with small L^∞-norm are weak solutions of (P). After obtaining infinitely many critical points for J_θ, we use a kind of iteration regularity process to prove that these solutions go to zero in $L^\infty(\Omega)$.

In our second result, we consider potentials which can be indefinite in sign.

Theorem 1.2. *Suppose that* $0 < q < 1$, *the function* f *satisfies* $(f_0) - (f_1)$, *the potential* a *satisfies* (a_0) *and*

$(\widetilde{a_1})$ *there exists* $a_0 > 0$ *and an open set* $\widetilde{\Omega} \subset \Omega$ *such that* $a(x) \geq a_0$, *for a.e.* $x \in \widetilde{\Omega}$.

Then the problem (P) *has a sequence of solutions* $(u_k) \subset W_0^{1,2}(\Omega)$ *such that* $I(u_k) < 0$ *and* $I(u_k) \to 0$ *as* $k \to \infty$ *when:*

(i) $\alpha > 0$, $\beta \geq 0$ *and* $\mu \in (0, \mu^*)$, *for some* $\mu_* > 0$; *and*
(ii) $\beta \geq 0$, $\mu > 0$ *and* $\alpha \in (\alpha^*, \infty)$, *for some* $\alpha^* > 0$.

In our final result, we present a version of Theorem 1.2 with no restriction on the size of the parameters. In this case, we need to replace the condition (f_1) by a stronger one. This result can be stated as follows.

Theorem 1.3. *Suppose that* $0 < q < 1$, *the function* f *satisfies* (f_0) *and*

$(\widetilde{f_1})$ $f(x, s) = o(|s|)$, *as* $s \to 0$, *uniformly in* Ω,

and the potential a *satisfies* (a_0) *and* $(\widetilde{a_1})$. *Then the same conclusion as in Theorem 1.1 holds.*

We recall that, in their celebrated paper [3], Ambrosetti, Brezis and Cerami studied the problem

$$-\Delta u = \lambda |u|^{q-2}u + |u|^{p-2}u \quad \text{in } \Omega, u = 0 \text{ on } \partial\Omega,$$

with $1 < q < 2$ and $2 < p < 2^*$. Among other results, the existence of two positive solutions is obtained for $\lambda > 0$ small. After this work, many authors have considered the effect of concave–convex terms in Dirichlet problems. Since it is impossible to give a complete list of references, we just cite some results closely related to ours.

In [12], the author considered the local case $\alpha = 1$, $\beta = 0$, $\mu = 1$ and $a(x) = \lambda > 0$. Under the conditions (f_0)–(f_1), he obtained the existence of infinitely many solutions as in Theorem 1.1. The same result was proved in [8], by assuming that $a \in C(\overline{\Omega})$ has nonzero positive part and f satisfies $(\widetilde{f_1})$ instead of (f_1). For the nonlocal problem, we can cite the paper [6], where the authors considered a more general nonlocal term, $a(x) \equiv \lambda > 0$, $\mu = 1$ and $f(x, s) = |s|^{p-1}s$, with $1 < p \leq (N + 2)/(N - 2)$, and obtained

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infinitely many solutions for low dimension $N \leq 3$ and some technical conditions on the size of the parameters λ and β. We also refer to [4, 13] for some related results.

The main theorems of this paper extend and complement the aforementioned works in several ways: in contrast to [8, 12], we consider the case $\beta > 0$; our potential $a(x)$ can be nonconstant, nonsmooth and indefinite in sign; there is no restriction on the dimension; and there is no restriction on the size of β. It is worth mentioning that, even in the local case $\beta = 0$, our results seem to be new. Finally, we notice that the same results hold for $N = 1$ and $N = 2$. In this case, it is sufficient to consider $2^* = +\infty$ and choose $p_1 \in (1, +\infty)$ in a arbitrary way.

The rest of the paper is organised as follows. In the next section, after presenting some auxiliary results, we prove the first two theorems. In the final section, we prove Theorem 1.3.

2. The case $f(x, s) = o(|s|^q)$

For any $u \in L^1(\Omega)$, we write only $\int u$ to denote $\int u(x)\,dx$. If $1 \leq p \leq \infty$, $\|u\|_p$ stands for the $L^p(\Omega)$-norm of the function $u \in L^p(\Omega)$. Hereafter, we assume that conditions (a_0) and (f_0) hold.

Let H be the Sobolev space $W_0^{1,2}(\Omega)$ endowed with the norm

$$\|u\| := \left(\int |\nabla u|^2 \right)^{1/2}.$$

As stated in the introduction, the functional I given by (1.2) is not well defined in H. In order to overcome this difficulty we use a truncation argument. So we start by presenting a version of [12, Lemma 2.3].

Lemma 2.1. *Suppose that f satisfies (f_1). Then, for any given $\theta > 0$, there exist $0 < \xi < \delta/2$ and $g \in C(\Omega \times \mathbb{R}, \mathbb{R})$, odd in the second variable, such that*

(g_1) $g(x, s) = f(x, s)$, *for all* $(x, s) \in \Omega \times [-\xi, \xi]$.

Moreover, if $G(x, s) := \int_0^s g(x, t)\,dt$ *then, for any* $(x, s) \in \Omega \times \mathbb{R}$:

(g_2) $g(x, s)s - 2G(x, s) \leq \theta |s|^{q+1}$;
(g_3) $g(x, s)s - (q + 1)G(x, s) \leq \theta |s|^{q+1}$;
(g_4) $|G(x, s)| \leq \frac{1}{2}\theta |s|^{q+1}$; *and*
(g_5) $|g(x, s)| \leq \theta |s|^q$.

Proof. Given $0 < \varepsilon < \theta/14$, we obtain from (f_1) a number $0 < \xi < \delta/2$ such that

$$\max\{|F(x, s)|, |f(x, s)s|\} \leq \varepsilon |s|^{q+1} \quad \text{for all } (x, s) \in \Omega \times [-2\xi, 2\xi].$$

Let $\rho \in C^1(\mathbb{R}, [0, 1])$ be an even function satisfying, for any $s \in \mathbb{R}$,

$$\rho \equiv 1 \text{ in } [-\xi, \xi], \quad \rho \equiv 0 \text{ in } \mathbb{R} \setminus (-2\xi, 2\xi), \quad |\rho'(s)| \leq 2/\xi, \quad \rho'(s)s \leq 0.$$

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Pick $0 < \gamma < \theta/12$, consider $F_\infty(s) := \gamma|s|^{q+1}$ and define the function $g \in C(\Omega \times \mathbb{R}, \mathbb{R})$ by setting

$$g(x,s) := \rho'(s)F(x,s) + \rho(s)f(x,s) + (1-\rho(s))F'_\infty(s) - \rho'(s)F_\infty(s).$$

A straightforward calculation shows that, for any $(x,s) \in \Omega \times \mathbb{R}$,

$$G(x,s) = \rho(s)F(x,s) + (1-\rho(s))F_\infty(s).$$

Using the properties of ρ, it is easy to see that g is continuous, odd in the second variable and satisfies (g_1), (g_4) and (g_5). In order to prove (g_2), note that

$$\begin{aligned} g(x,s)s - 2G(x,s) = {} & s\rho'(s)F(x,s) + s\rho(s)f(x,s) + s(1-\rho(s))F'_\infty(s) \\ & - s\rho'(s)F_\infty(s) - 2\rho(s)F(x,s) - 2(1-\rho(s))F_\infty(s). \end{aligned}$$

Recalling that $sF'_\infty(s) = \gamma(q+1)|s|^{q+1}$, we get, for $|s| \le 2\xi$,

$$\begin{aligned} g(x,s)s - 2G(x,s) &\le 2\xi\frac{2}{\xi}|F(x,s)| + |sf(x,s)| + \gamma(q+1)|s|^{q+1} \\ &\quad + 2\xi\frac{2}{\xi}\gamma|s|^{q+1} + 2|F(x,s)| \\ &\le 6|F(x,s)| + |sf(x,s)| + 6\gamma|s|^{q+1} \\ &\le (7\varepsilon + 6\gamma)|s|^{q+1} < \theta|s|^{q+1}. \end{aligned}$$

On the other hand, for $|s| > 2\xi$,

$$g(x,s)s - 2G(x,s) = sF'_\infty(s) - 2F_\infty(s) < sF'_\infty(s) < \theta|s|^{q+1}.$$

So, we conclude that (g_2) holds. The property (g_3) can be proved by an analogous argument. □

For any $\theta > 0$, it follows from (g_4)–(g_5) that the functional

$$J_\theta(u) := \frac{\alpha}{2}\|u\|^2 + \frac{\beta}{4}\|u\|^4 - \frac{1}{q+1}\int a(x)|u|^{q+1} - \mu\int G(x,u)$$

belongs to $C^1(H, \mathbb{R})$ and, for any $u, v \in H$,

$$J'_\theta(u)v = (\alpha + \beta\|u\|^2)\int(\nabla u \cdot \nabla v) - \int a(x)|u|^{q-1}uv - \mu\int g(x,u)v.$$

Thus, if $u \in H \cap L^\infty(\Omega)$ with $\|u\|_\infty < \xi$, it follows from (g_1) that $g(x,u(x)) = f(x,u(x))$ a.e. in Ω. We then conclude that any critical point of J_θ with small L^∞-norm is a weak solution of (P).

Now we prove a technical result.

Lemma 2.2. *Suppose that the functions a and f satisfy (a_1) and (f_1), respectively. If*

$$0 < \theta < \frac{(1-q)a_0}{(1+q)\mu}, \tag{2.1}$$

then $J_\theta(u) = J'_\theta(u)u = 0$ if and only if $u = 0$.

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Proof. It is obvious that $J_\theta(0) = J'_\theta(0)0 = 0$ independently of θ. On the other hand, if $J_\theta(u) = J'_\theta(u)u = 0$, then we can use (a_1), $J'_\theta(u)u - 2J_\theta(u) = 0$ and (g_2) to get

$$\begin{aligned}\frac{(1-q)a_0}{q+1}\int |u|^{q+1} &\le \frac{\beta}{2}\|u\|^4 + \frac{1-q}{q+1}\int a(x)|u|^{q+1}\\ &= \mu\int (g(x,u)u - 2G(x,u))\\ &\le \mu\theta\int |u|^{q+1}.\end{aligned}$$

The above inequality and (2.1) imply that $u = 0$. □

Notice that the positivity of the potential a was essential in the above proof. If we are in the setting of the local condition $(\widetilde{a_1})$, we can obtain *a priori* estimates for the functions satisfying $J_\theta(u) = J'_\theta(u)u = 0$. More precisely, if we set

$$S_{q+1} := \inf_{\substack{u\in H\\ u\neq 0}} \frac{\int |\nabla u|^2}{\left(\int |u|^{q+1}\right)^{2/(q+1)}} > 0,$$

we have the following lemma.

Lemma 2.3. *Suppose that the functions a and f satisfy $(\widetilde{a_1})$ and (f_1), respectively. If*

$$0 < \theta < \frac{(1-q)}{2} S_{q+1}^{(q+1)/2}, \tag{2.2}$$

then $J_\theta(u) = J'_\theta(u)u = 0$ implies that $\|u\| \le (\mu/\alpha)^{1/(1-q)}$.

Proof. If $J_\theta(u) = J'_\theta(u)u = 0$, then the equality $J'_\theta(u)u - (q+1)J_\theta(u) = 0$ implies that

$$\frac{\alpha(1-q)}{2}\|u\|^2 + \frac{\beta(3-q)}{4}\|u\|^4 = \mu\int (g(x,u)u - (q+1)G(x,u)). \tag{2.3}$$

It follows from (g_3), the embedding $H \hookrightarrow L^{q+1}(\Omega)$ and (2.2) that

$$\frac{\alpha(1-q)}{2}\|u\|^2 \le \mu\theta S_{q+1}^{-(q+1)/2}\|u\|^{q+1} \le \frac{\mu(1-q)}{2}\|u\|^{q+1}$$

and the result follows. □

Lemma 2.4. *For any $\theta > 0$ the functional J_θ is coercive and satisfies the Palais–Smale condition.*

Proof. Since the sequence (p_n), defined in the introduction, is increasing,

$$\sigma_q = \frac{p_m}{1-q} \ge \frac{2^*}{1-q} > \frac{2^*}{2^* - (q+1)} = \left(\frac{2^*}{q+1}\right)', \tag{2.4}$$

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and therefore $1 < \sigma'_q(q+1) < 2^*$. Hence, we can use Hölder's inequality, (g_4) and the Sobolev embeddings to get

$$\begin{aligned} J_\theta(u) &\geq \frac{\alpha}{2}\|u\|^2 + \frac{\beta}{4}\|u\|^4 - \frac{1}{q+1}\|a\|_{\sigma_q}\|u\|^{q+1}_{\sigma'_q(q+1)} - \frac{\mu\theta}{2}\|u\|^{q+1}_{q+1} \\ &\geq \frac{\alpha}{2}\|u\|^2 + \frac{\beta}{4}\|u\|^4 - C\|u\|^{q+1}, \end{aligned}$$

for some constant $C > 0$. Recalling that $(q+1) < 2$, we conclude that $J_\theta(u) \to \infty$ if $\|u\| \to +\infty$, that is, J_θ is coercive.

Suppose now that $(u_n) \subset H$ is such that $J_\theta(u_n) \to c$ and $J'_\theta(u_n) \to 0$. By the above considerations, (u_n) is bounded. Hence, up to a subsequence, for some $A \geq 0$ and $u \in H$,

$$\|u_n\| \to A, \quad u_n \rightharpoonup u \text{ weakly in } H, \quad u_n \to u \text{ strongly in } L^p(\Omega),$$

for any $p \in [1, 2^*)$. By (2.4), there exists $p_0 \in (q+1, 2^*)$ such that $\sigma_q = (p_0/(q+1))'$. Hölder's inequality and the above convergence results imply that

$$\left|\int a(x)|u_n|^{q-1}u_n(u_n - u)\right| \leq \|a\|_{\sigma_q}\|u_n\|^q_{p_0}\|u_n - u\|_{p_0} \to 0$$

as $n \to \infty$. Moreover, by (g_5) and Hölder's inequality again,

$$\left|\int g(x, u_n)(u_n - u)\right| \leq \theta\|u_n\|^q_{q+1}\|u_n - u\|_{q+1} \to 0.$$

Thus

$$o_n(1) = J'_\theta(u_n)(u_n - u) = (\alpha + \beta\|u_n\|^2)\left(\|u_n\|^2 - \int \nabla u_n \cdot \nabla u\right) + o_n(1).$$

Taking the limit, we obtain $(\alpha + \beta A^2)(A^2 - \|u\|^2) = 0$, which implies that $\|u\| = A$. It follows from the weak convergence that $u_n \to u$ strongly in H. □

In order to prove our result, we shall apply the following variant of a result due to Clark [5] (see also [9, Theorem 2.1, Proposition 2.2]).

Theorem 2.5. *Let X be a Banach space and let $J \in C^1(X, \mathbb{R})$ be an even functional bounded from below that satisfies the Palais–Smale condition and $J(0) = 0$. If, for each $k \in \mathbb{N}$, there exists a k-dimensional subspace $X^k \subset X$ and $\rho_k > 0$ such that*

$$\sup_{u \in X^k \cap S_{\rho_k}} J(u) < 0, \tag{2.5}$$

where $S_\rho := \{u \in X : \|u\|_X = \rho\}$, then J has a sequence of critical values $(c_k) \subset (-\infty, 0)$ such that $c_k \to 0$ as $k \to +\infty$.

We are now ready to prove our first result.

Proof of Theorem 1.1. Let $\xi > 0$ be given by Lemma 2.1 with $\theta > 0$ satisfying (2.1). As noted before, any critical point of J_θ such that $\|u\|_\infty < \xi$ is a weak solution of (P). We are going to apply Theorem 2.5 with this modified functional and show that the solutions we obtain have small L^∞-norm.

It is clear that the even functional J_θ satisfies $J_\theta(0) = 0$. Moreover, by Lemma 2.4, it also satisfies the Palais–Smale condition. Since J_θ is bounded on bounded sets of H, we also conclude from the same lemma that J_θ is bounded from below.

For any given $k \in \mathbb{N}$, we set $X^k := \operatorname{span}\{\phi_1, \ldots, \phi_k\}$, where $\{\phi_n\}_{n\in\mathbb{N}}$ is a Hilbertian basis of H. Since all the norms in X^k are equivalent, we can use (a_1), (g_4) and $0 < \theta < (1-q)a_0/(1+q)\mu < a_0/(1+q)\mu$ to get, for any $u \in X^k$,

$$\begin{aligned} J_\theta(u) &\le \frac{\alpha}{2}\|u\|^2 + \frac{\beta}{4}\|u\|^4 - \frac{a_0}{q+1}\|u\|_{q+1}^{q+1} + \frac{\mu}{2}\theta\|u\|_{q+1}^{q+1} \\ &\le \frac{\alpha}{2}\|u\|^2 + \frac{\beta}{4}\|u\|^4 - \frac{a_0}{2(q+1)}C\|u\|^{q+1}, \end{aligned} \tag{2.6}$$

for some constant $C > 0$ independent of u. Recalling that $(q+1) < 2$, we can choose $\rho_k > 0$ to be small in such a way that J_θ satisfies (2.5).

From Theorem 2.5, there is a sequence of critical points $(u_k) \subset H$ which satisfies $J_\theta(u_k) = c_k \to 0$ as $k \to \infty$. Since (u_k) is a Palais–Smale sequence at level $c = 0$, by Lemma 2.4, we may suppose that $u_k \to u$ strongly in H. Hence $J_\theta(u) = J'_\theta(u)u = 0$ and we infer from Lemma 2.2 that $u = 0$, that is, $u_k \to 0$ strongly in H.

Notice that each function u_k is a weak solution of

$$-\Delta u = \frac{a(x)|u_k(x)|^{q-1}u_k(x) + \mu g(x, u_k(x))}{\alpha + \beta\|u_k\|^2} \quad \text{in } \Omega, \qquad u = 0 \text{ on } \partial\Omega.$$

If we denote by h_k the right-hand side of the first equation above, by (g_5), we see that $|h_k(x)| \le \alpha^{-1}(|a(x)||u_k(x)|^q + \mu\theta|u_k(x)|^q)$ a.e. in Ω. Hence

$$\int |h_k(x)|^{2^*} \le C_1\alpha^{-2^*}\Big(\int |a(x)|^{2^*}|u_k|^{q2^*} + (\mu\theta)^{2^*}\int |u_k|^{q2^*}\Big),$$

with $C_1 := 2^{2^*-1}$. Since $\sigma_q \ge 2^*/(1-q) > 2^*$,

$$\tau_q := q\Big(\frac{\sigma_q}{2^*}\Big)' = \frac{q\sigma_q}{\sigma_q - 2^*} \le 1.$$

Thus Hölder's inequality implies that

$$\int |a(x)|^{2^*}|u_k|^{q2^*} \le \|a\|_{\sigma_q}^{2^*}\Big(\int |u_k|^{\tau_q 2^*}\Big)^{q/\tau_q} \le C_2\|a\|_{\sigma_q}^{2^*}\|u_k\|_{2^*}^{q2^*}$$

and

$$\int |u_k(x)|^{q2^*} \le C_3\|u_k\|_{2^*}^{q2^*},$$

with $C_2 := |\Omega|^{q(1-\tau_q)/\tau_q}$ and $C_3 := |\Omega|^{1-q}$. We conclude that $h_k \in L^{2^*}(\Omega)$ and therefore, by the Agmon–Douglis–Nirenberg result [1] (see also [7, Lemma 9.17]), $u_k \in W^{2,2^*}(\Omega)$ and there exists $C_4 = C_4(\Omega)$ such that

$$\|u_k\|_{W^{2,2^*}} \le C_4\|h_k\|_{2^*} \le C_5\|u_k\|_{2^*}^q \le \widehat{C}_1\|u_k\|^q, \tag{2.7}$$

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for each $k \in \mathbb{N}$, where $C_5 = C_4\alpha^{-1}\{C_1(C_2\|a\|_{\sigma_q}^{2^*} + C_3(\mu\theta)^{2^*})\}^{1/2^*}$ and $\widehat{C_1} = C_5 S^{-q/2}$. Since $u_k \to 0$ strongly in H, we see that $\|u_k\|_{W^{2,2^*}} \to 0$. If $m = 1$, that is, $2p_1 = 2 \cdot 2^* > N$ (see the definition of m and of the sequence (p_n) in (1.1)), the continuous embedding $W^{2,2^*}(\Omega) \hookrightarrow C(\overline{\Omega})$ implies that $\|u_k\|_\infty \to 0$.

On the other hand, if $2 \cdot 2^* \le N$ (that is, equivalently, $m > 1$), the embedding $W^{2,2^*}(\Omega) \hookrightarrow L^{p_2}(\Omega)$ and (2.7) imply that, for some $C_6 = C_6(\Omega) > 0$,

$$\|u_k\|_{p_2} \le C_6\|u_k\|_{W^{2,2^*}} \le C_6\widehat{C_1}\|u_k\|^q,$$

where p_2 is the second term of the sequence defined in (1.1). Furthermore, since, in this case, $\sigma_q \ge p_2/(1-q) > p_2$, we can argue as above to conclude that $h_k \in L^{p_2}(\Omega)$. Hence $u_k \in W^{2,p_2}(\Omega)$ and

$$\|u_k\|_{W^{2,p_2}} \le C_7\|u_k\|_{p_2}^q \le \widehat{C_2}\|u_k\|^{q^2},$$

where $C_7 > 0$ is independent of k and $\widehat{C_2} = C_7(C_6\widehat{C_1})^q$.

Since $\sigma_q \ge p_n/(1-q) > p_n$ for $n = 1, \ldots, m$, we can repeat this argument until we get $u_k \in W^{2,p_m}(\Omega)$ and

$$\|u_k\|_{W^{2,p_m}} \le \widehat{C_m}\|u_k\|^{q^m}, \tag{2.8}$$

with $\widehat{C_m} > 0$ independent of k. Then, $\|u_k\|_{W^{2,p_m}} \to 0$ as $k \to \infty$. But $2p_m > N$ provides $W^{2,p_m}(\Omega) \hookrightarrow C(\overline{\Omega})$, and therefore we conclude that $\|u_k\|_\infty \to 0$. So there exists $k_0 \in \mathbb{N}$ such that

$$\|u_k\|_\infty \le \frac{\xi}{2} \quad \text{for all } k \ge k_0,$$

and the theorem is proved. □

Proof of Theorem 1.2. In the setting of item (i), that is, $\alpha > 0$ and $\beta \ge 0$ fixed, we suppose that $\mu \le 1$ and choose

$$0 < \theta < \min\left\{\frac{1-q}{2}S_{q+1}^{(q+1)/2}, \frac{a_0}{1+q}\right\}.$$

In the setting of item (ii) ($\beta \ge 0$ and $\mu > 0$ fixed) we choose

$$0 < \theta < \min\left\{\frac{1-q}{2}S_{q+1}^{(q+1)/2}, \frac{a_0}{\mu(1+q)}\right\}.$$

In order to obtain a sequence of critical points for J_θ we argue as in Theorem 1.1. The first difference appears when we try to prove (2.5). Indeed, since a is no longer positive, we need an alternative construction for the finite-dimensional subspace. For any given $k \in \mathbb{N}$, we choose k linearly independent functions $\phi_1, \ldots, \phi_k \in C_0^\infty(\widetilde{\Omega})$, where the open set $\widetilde{\Omega} \subset \Omega$ comes from the condition $(\widetilde{a_1})$. Since $a(x) \ge a_0$ a.e. in $\widetilde{\Omega}$, the inequality (2.6) still holds for $u \in X^k$. Hence, there is a sequence of critical points $(u_k) \subset H$ such that $J_\theta(u_k) = c_k \to 0$ as $k \to \infty$. Again, there exists $u \in H$ such that $u_k \to u$ strongly in H. Although we cannot guarantee that $u = 0$, it follows from Lemma 2.3 that

$$\|u\| \le \left(\frac{\mu}{\alpha}\right)^{1/(1-q)}. \tag{2.9}$$

As in the proof of Theorem 1.1, we see that $u_k \in W^{2,p_m}(\Omega)$ and inequality (2.8) holds. Hence, we can use the embedding $W^{2,p_m}(\Omega) \hookrightarrow C(\overline{\Omega})$ to get

$$\|u_k\|_\infty \le C_0\|u_k\|_{W^{2,p_m}} \le C_0\widehat{C}_m\|u_k\|^{q^m},$$

for some constant $C_0 = C_0(\Omega) > 0$. Since $\|u_k\| \to \|u\|$, it follows from (2.9) that there exists $k_0 \in \mathbb{N}$ such that

$$\|u_k\|_\infty \le C_0\widehat{C}_m 2^{q^m}\Big(\frac{\mu}{\alpha}\Big)^{q^m/(1-q)} \quad \text{for all } k \ge k_0.$$

A simple inspection of the proof of Theorem 1.1 shows that the constant $\widehat{C}_m = \widehat{C}_m(\mu, \alpha)$ is directly proportional to both μ and α^{-1}. Thus, if $\alpha > 0$ is fixed (item (i)), the L^∞-norm of the function u_k becomes small if μ is close to zero. On the other hand, if $\mu > 0$ is fixed (item (ii)), the same occurs if $\alpha > 0$ is large. In both cases, the approximated solutions are weak solutions of (P). □

3. The case $f(x, s) = o(|s|)$

In this section, we prove Theorem 1.3. The ideas are analogous to those used in the previous section. We only need to adapt the auxiliary results.

Lemma 3.1. *Suppose that f satisfies $(\widetilde{f_1})$. Then, for any given $\theta > 0$, there exist $0 < \xi < \delta/2$ and $g \in C(\Omega \times \mathbb{R}, \mathbb{R})$, odd in the second variable, such that (g_1) (see Lemma 2.1) holds. Moreover, if $G(x, s) := \int_0^s g(x, t)\, dt$, then, for any $(x, s) \in \Omega \times \mathbb{R}$:*

- $(\widetilde{g_3})$ $g(x, s)s - (q + 1)G(x, s) \le \theta|s|^2$;
- $(\widetilde{g_4})$ $|G(x, s)| \le \frac{1}{2}\theta|s|^2$; *and*
- $(\widetilde{g_5})$ $|g(x, s)| \le \theta|s|$.

Proof. It follows from $(\widetilde{f_1})$ that, for any given $0 < \varepsilon < \theta/14$, there exists $0 < \xi < \delta/2$ such that

$$\max\{|F(x, s)|, |f(x, s)s|\} \le \varepsilon|s|^2 \quad \text{for all } (x, s) \in \Omega \times [-2\xi, 2\xi].$$

The argument now is analogous to that presented in the proof of Lemma 2.1. We omit the details. □

As in the previous section, for any $\theta > 0$, we consider the function g given by the above lemma and define $J_\theta : H \to \mathbb{R}$ by setting

$$J_\theta(u) := \frac{\alpha}{2}\|u\|^2 + \frac{\beta}{4}\|u\|^4 - \frac{1}{q + 1}\int a(x)|u|^{q+1} - \mu\int G(x, u).$$

In the next result, we denote by $\lambda_1 > 0$ the first eigenvalue of $(-\Delta, W_0^{1,2}(\Omega))$.

Lemma 3.2. *Suppose that the functions a and f satisfy $(\widetilde{a_1})$ and $(\widetilde{f_1})$, respectively. If*

$$0 < \theta < \frac{(1 - q)\alpha\lambda_1}{2\mu}, \tag{3.1}$$

then $J_\theta(u) = J'_\theta(u)u = 0$ if and only if $u = 0$.

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Proof. If $J_\theta(u) = J'_\theta(u)u = 0$, it follows from (2.3), $(\widetilde{g_3})$ and Poincaré's inequality that

$$\frac{\alpha(1-q)}{2}\|u\|^2 \le \mu \int_\Omega (g(x,u)u - (q+1)G(x,u)) \le \mu\theta \int |u|^2 \le \frac{\mu\theta}{\lambda_1}\|u\|^2.$$

The result follows from (3.1). □

Lemma 3.3. *For any $\theta > 0$ satisfying (3.1), the functional J_θ is coercive and satisfies the Palais–Smale condition.*

Proof. As in the proof of Lemma 2.4, $1 < \sigma'_q(q+1) < 2^*$. Hence we can use Hölder's inequality and $(\widetilde{g_4})$ to obtain

$$\begin{aligned} J_\theta(u) &\ge \frac{\alpha}{2}\|u\|^2 + \frac{\beta}{4}\|u\|^4 - \frac{1}{q+1}\|a\|_{\sigma_q}\|u\|^{q+1}_{\sigma'_q(q+1)} - \frac{\mu\theta}{2}\|u\|_2^2 \\ &\ge \frac{\alpha}{2}\|u\|^2 + \frac{\beta}{4}\|u\|^4 - \frac{1}{q+1}\|a\|_{\sigma_q}\|u\|^{q+1}_{\sigma'_q(q+1)} - \frac{(1-q)\alpha\lambda_1}{4}\|u\|_2^2 \\ &\ge \frac{(1+q)\alpha}{4}\|u\|^2 + \frac{\beta}{4}\|u\|^4 - C\|u\|^{q+1}, \end{aligned}$$

for some constant $C > 0$. It is sufficient now to argue as in the proof of Lemma 2.4. □

The proof of our last result is now straightforward.

Proof of Theorem 1.3. The proof is a consequence of the above lemmas and the same argument used in the proofs of Theorems 1.1 and 1.2. □

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