# ON THE PROJECTIVE CENTRES OF CONVEX CURVES

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**1. Introduction.** We consider a closed curve $C$ in the projective plane and the projective involutions which map $C$ into itself. Any such mapping $\gamma$, other than the identity, is a harmonic homology whose axis $\eta$ we call a projective axis of $C$ and whose centre $p$ we call an interior or exterior projective centre according as it is inside or outside $C$.[1] The involutions are the generators of a group $\Gamma$, and the set of centres and the set of axes are invariant under $\Gamma$. The present paper is concerned with the type of centre sets which can exist and with the relationship between the nature of $C$ and its centre set.

If $C$ is a conic, then every point which is not on $C$ is a projective centre. Conversely, it was shown by Kojima **(4)** that if $C$ has a chord of interior centres, or a full line of exterior centres, then $C$ is a conic. Kojima's results, and in fact all the problems considered here, have interpretations in Hilbert geometries. If $C$ is convex and $x$ and $y$ are points interior to $C$, then the line $\eta = x \times y$ cuts $C$ in points $a$ and $b$, and the Hilbert distance defined by $h(x, y) = |\log R(a, b; x, y)|$ induces a metric on the interior of $C$. An involution $\phi$ leaving $C$ invariant preserves this distance and hence is a motion of the Hilbert plane onto itself. If the centre $p$ of $\phi$ is inside $C$, then the motion is a reflection in the point $p$. If $p$ is outside $C$, then the motion is a reflection in the Hilbert line carried by the axis $\eta$. When $C$ is a conic, then the Hilbert geometry is the Klein representation of hyperbolic geometry. Thus Kojima's results imply that a Hilbert geometry is hyperbolic if it possesses reflections in every point of a line, or reflections in every line of a pencil.

In §2 we determine the convex closed curves which admit a continuous group of projective transformations and hence the plane Hilbert metrics with a continuous group of isometries. In §3 we apply these results in order to sharpen and extend Kojima's characterizations. In §4 we consider curves with discontinuous transformation groups generated by projective involutions. Finally, in §5, we extend our results to higher dimensional projective spaces.

**2. Curves which are invariant under a continuous group of projective transformations.** In this section we wish to characterize the closed convex curves in the projective plane which permit infinitesimal projective

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[1]These terms are justified by the fact that each interior projective centre is an affine centre if the corresponding axis is taken as the line at infinity, while the axis corresponding to an exterior centre becomes an affine axis (the locus of midpoints of parallel chords) if the exterior centre is at infinity. For an application of this concept see **(2)**.

transformations onto themselves, and hence the Hilbert geometries which permit connected continuous groups of isometries.

For this purpose we first determine the orbits of points under one-dimensional continuous subgroups of the projective group. Every such subgroup can be represented by $G = \{\exp (tA) | -\infty < t < \infty\}$ where $A$ is a $3 \times 3$ matrix.

By suitable choice of co-ordinates we can reduce $A$ to one of the following forms (over the complex field)

$$\text{(i)}\quad A = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}, \qquad \exp (tA) = \begin{pmatrix} e^{at} & 0 & 0 \\ 0 & e^{bt} & 0 \\ 0 & 0 & e^{ct} \end{pmatrix};$$

$$\text{(ii)}\quad A = \begin{pmatrix} a+ib & 0 & 0 \\ 0 & a-ib & 0 \\ 0 & 0 & c \end{pmatrix}, \qquad \exp (tA) = \begin{pmatrix} e^{at}e^{ibt} & 0 & 0 \\ 0 & e^{at}e^{-ibt} & 0 \\ 0 & 0 & e^{ct} \end{pmatrix};$$

$$\text{(iii)}\quad A = \begin{pmatrix} a & 1 & 0 \\ 0 & a & 0 \\ 0 & 0 & c \end{pmatrix}, \qquad \exp (tA) = \begin{pmatrix} e^{at} & te^{at} & 0 \\ 0 & e^{at} & 0 \\ 0 & 0 & e^{ct} \end{pmatrix};$$

$$\text{(iv)}\quad A = \begin{pmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & a \end{pmatrix}, \qquad \exp (tA) = \begin{pmatrix} e^{at} & te^{at} & \frac{t^2}{2} e^{at} \\ 0 & e^{at} & te^{at} \\ 0 & 0 & e^{at} \end{pmatrix};$$

where $a$, $b$, $c$ are real.

*Case* (i). The orbit of $(x_0, y_0)$ is $x = x_0\, e^{(a-c)\,t}$, $y = y_0\, e^{(b-c)\,t}$ which is affine equivalent to one of the convex arcs

$$y = x^m,\ 0 < x < \infty,\ -1 \leqslant m < 1$$

or a single point.

*Case* (ii). A suitable complex affine transformation yields the real orbit $x = e^{(a-c)\,t}[x_0 \cos bt - y_0 \sin bt]$, $y = e^{(a-c)\,t}\ [x_0 \sin bt + y_0 \cos bt]$. This is the affine equivalent of a circle if $a = c$ and of a spiral $r = e^{\kappa\theta}$ if $a \neq c$. The degenerate orbit is a single point.

*Case* (iii). The orbit is

$$x = e^{(a-c)\,t}(x_0 + ty_0),\ y = e^{(a-c)\,t}\, y_0.$$

This is affine equivalent to

$$x = y \log y, \quad 0 < y < \infty;$$

with the full line, the half-line and the point as degenerate orbits.

*Case* (iv). The orbit is

$$x = x_0 + ty_0 + \frac{t^2}{2}, \qquad y = y_0 + t$$

which is the affine equivalent of the parabola $x = y^2$.

A closed curve which admits a one-dimensional group of projective transformations must consist entirely of orbits of its points under that group. We thus obtain the following.

THEOREM 1. *The convex closed curves in the projective plane which are invariant under a connected continuous group of projective transformations are projective equivalents of one of the following:*

1. *Two straight lines.*

2. *A curve consisting of the two arcs* $y = x^m$, $x > 0$ *and* $y = -ax^m$, $x > 0$, $a \geqslant 0$ *and their common endpoints, where* $0 < m < 1$. (*For* $m = \frac{1}{2}$ *this case includes the differentiable union of two conical arcs; and, in particular, the conic.*)

3. *A curve consisting of the arc* $y = x^m$, $x < 0$ $(0 > m \leqslant 1)$, *the line segment* $x = 0$, $y \leqslant 0$; *and the segment on the line at infinity which corresponds to non-positive slopes.* (*For* $m = 1$ *this includes the triangle.*)

4. *A curve consisting of the arc* $y = x \log x$, $x > 0$ *and the segment* $x = 0$, $0 \leqslant y \leqslant \infty$.

**3. Curves whose projective centres have limit points which are not on the curve.** If the projective centres $\{p_i\}$ of $C$ have a limit point $p \notin C$ then the corresponding projective involutions $\{\gamma_i\}$ of $C$ have a limit involution $\gamma$ whose centre is $p$. Hence $C$ admits the projective transformations $\{\gamma_i\gamma\}$ which approach the identity. In other words $C$ admits a connected continuous group of projective transformations.

THEOREM 2. *If the projective centres of a closed convex curve $C$ have a limit point not on $C$ then one of the following cases holds:*

1. *$C$ is a conic and all points not on $C$ are projective centres of $C$.* (*This includes two lines as a degenerate case which admits all points of the plane as projective centres.*)

2. *$C$ is the union of arcs of two different conics with common endpoints which are points of differentiability of $C$* (*This includes the case in which one of these arcs degenerates to two tangent line segments from a point to the other arc*). *Here the projective centres consist of the points exterior to $C$ on the line of the common chord of the two conical arcs.*

3. *$C$ is the union of a conical arc and the chord joining its endpoints* (*this includes the triangle for a degernerate conical arc*). *Here the projective centres consist of the points on the line of the chord exterior to the chord. In the case of the triangle all points on the three lines which are not on the three sides are projective centres.*

In order to prove this theorem we first establish the following extension of Kojima's results.

LEMMA 1. *Let $a$ and $b$ be two points on the* (*convex*) *closed curve $C$ so that all*

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*points on the line $a \times b$ which are exterior to $C$ are projective centres of $C$. Then either*

(i) *$C$ is differentiable at $a$ and $b$ and the two arcs of $C$ with endpoints $a, b$ are conical arcs (one of which may be degenerate), or*

(ii) *the segment $a, b$ is on $C$ and the other arc of $C$ with endpoints $a, b$ is conical (possibly degenerate).*

*Proof.* For any exterior point $x$ on $\sigma = a \times b$ the involution $\gamma_x$ interchanges $a$ and $b$, so both are regular points or both are corner points. If they are corner points then the two one-sided tangents at $a$ and the two at $b$ cannot be four distinct lines for then they would determine a quadrilateral, two of whose vertices would have to be invariant under every $\gamma_x$, which would imply that all $\phi_x$ had a common axis. Thus, in this case, one of the one-sided tangents at $a$ coincides with one of the one-sided tangents at $b$. In other words,[2] the segment $(a, b)$ lies on $C$. Next, let $\omega$ be a one-sided tangent at an arbitrary point $p$ of $C$, $p \notin (a, b)$. If $x = \sigma \times \omega$ is exterior to $C$ then $\gamma_x$ leaves $p$ and $\omega$ fixed. Because $\gamma_x$ interchanges two arcs at $p$, it also interchanges the half tangents at $p$, so both must coincide with $\omega$. The only case in which $\sigma \times \omega$ is not exterior to $C$ is the case in which it is one of the points $a, b$ and therefore one arc of $C$ with endpoints $a, b$ consists of the line segments $(a, p)$, $(b, p)$.

Thus we have the following alternatives. Either $C$ is everywhere differentiable; or $C$ has a single corner point $p$ $(\neq a, b)$ in which case it contains the line segments $(p, a)$ and $(p, b)$, or $C$ has two corner points, in which case these corner points are the points $a, b$ and $C$ contains the segment $(a, b)$;[2] or $C$ has three corner points, in which case it is a triangle.

In case $C$ is not a triangle, let $y$ be a point on a differentiable arc $C_{a,b}$ of $C$ with endpoints $a, b$ and $C_{a,b} \neq (a, b)$. There exists a unique conic $K_y$ which passes through $y$ and is tangent to $\omega_a$ at $a$ and $\omega_b$ at $b$, where $\omega_a$, $\omega_b$ are the one-sided tangents to $C_{a,b}$ at its endpoints. Let $x = \sigma \times \omega_y$ where $\omega_y$ is the tangent to $C$ at $y$, then $\gamma_x$ leaves $y$, $C$ and $K_y$ invariant and hence $\omega_y$ is tangent to $K_y$. Now the family of conics tangent to $\omega_a$ and $\omega_b$ at $a$ and $b$ has no proper envelope. Hence $C_{a,b}$ must be an arc of one of the conics of that family.

*Proof of Theorem* 2. Since our hypothesis implies that $C$ admits a continous group of projective transformations we need only consider the cases enumerated in Theorem 1:

*Case* 1. Obvious.

*Case* 2. If $m \neq \frac{1}{2}$ then the origin and the point at infinity are distinguished by the fact that they either are not points of analyticity or that they are points of zero curvature. That is to say, any involution must permute the origin and the point at infinity between themselves. Thus all interior centres are on the

[2] If $C$ is not assumed convex, the points $a, b$ may be cusps formed by arcs with common one-sided tangents at $a, b$.

(positive) $x$-axis; and the exterior centres are either on the (negative) $x$-axis or the point at infinity of the $y$-axis.

Now the curve permits the affine transformations $x \to tx, y \to t^m y, t > 0$. Thus if $(x_0, 0)$ is a projective centre then so is $(tx_0, 0)$ for all $t > 0$. If $x_0 > 0$, then, by Kojima's theorem, $C$ would be a conic and if $x_0 < 0$ then, by Lemma 1, $C$ would consist of conical arcs.

The point at infinity on the $y$-axis is a projective centre if and only if the $x$-axis is a Euclidean axis of symmetry. In other words, if and only if $a = 1$. To sum up: If a curve in this case is not the union of two conical arcs, then it has at most one (exterior) projective centre. If $m = \frac{1}{2}$ and $a = 1$, then $C$ is a conic. If $m = \frac{1}{2}$ and $a \neq 1$, then $C$ is the union of two conical arcs and all points on the negative $x$-axis are projective centres of $C$.

*Case* 3. If $m = 1$ and $C$ is a triangle then the situation is obvious. If $m \neq 1$ then every projective mapping of $C$ onto itself must permute the origin and the point at infinity on the $x$-axis among themselves and must leave the point at infinity on the $y$-axis fixed. Thus all projective centres must be exterior (since an involution of C which corresponds to an interior centre can have no fixed points) and lie on the negative $x$-axis. As in Case 2 we see that if there is one centre then all points on the negative $x$-axis are centres and by Lemma 1 we have $m = \frac{1}{2}$ so that $C$ is the union of a conical arc and a degenerate conical arc which is differentiable at the common endpoints.

*Case* 4. Any involution must preserve the straight line segment on this curve. Thus there can be no interior projective centre and an exterior projective centre would have to lie on the (negative) $y$-axis.

Now the curve admits the affine transformations $x \to tx, y \to ty + (t \log t)x$, $t > 0$; so that if $(0, y_0)$ is a projective centre then so is $(0, ty_0)$ for all $t > 0$. In other words, if there is a projective centre then all points on a supporting line which are not on the curve are projective centres. By Lemma 1 this would imply that the arc $y = x \log x, x > 0$ is a conical arc.

To sum up: A curve of Case 4 has no projective centres.

It is easy to see that the only non-convex closed curves whose projective centres have a limit point not on $C$ are the unions of two conical arcs with common one-sided tangents at their juncture.

## 4. Projective centres of general convex plane curves.

THEOREM 3. *If there are two centres $p_0$ and $p_1$ interior to $C$, then there is an infinite sequence $\{p_n\}$, $n = 0, \pm 1, \pm 2, \ldots$, of interior centres on the line $p_0 \times p_1$. The points of intersection of $C$ and $p_0 \times p_1$ are the two limit points of the sequence,*

$$p_{-\infty} = \lim_{n \to -\infty} p_n$$

*and*

$$p_{\infty} = \lim_{n \to \infty} p_n,$$

*and C is differentiable at these points. If C is not a conic then its curvature has a singularity of the second kind at $p_{-\infty}$ and $p_\infty$.*

*Proof.* Let the involutions corresponding to $p_0$ and $p_1$ be $\gamma_0$ and $\gamma_1$. Then they generate the *centre sequence* $\{p_n\}$ and the corresponding sequence of involutions $\{\gamma_n\}$; $n = 0, \pm 1, \pm 2, \ldots$, defined recursively by

$$\left.\begin{aligned}\gamma_n &= \gamma_{n-1}\gamma_{n-2}\gamma_{n-1}\\ p_n &= p_{n-2}\gamma_{n-1}\end{aligned}\right\} n = 2, 3, 4, \ldots; \qquad \left.\begin{aligned}\gamma_n &= \gamma_{n+1}\gamma_{n+2}\gamma_{n+1}\\ p_n &= p_{n+2}\gamma_{n+1}\end{aligned}\right\} n = -1, -2 \ldots.$$

Because the involutions $\gamma_i$ are motions of the Hilbert plane defined by $C$, the Hilbert distance between any two successive centres in the $p_i$ or in the $p_{-i}$ subsequence is the same. Thus, in the Hilbert sense, the $p_i$ and $p_{-i}$ sequences correspond to the points obtained by starting with $p_0$ and $p_1$ and then repeatedly stepping off the distance $h(p_0, p_1)$ in the two directions along the line respectively. Hence one sequence converges (in the topology of the projective plane) to $a$ and the other to $b$, and these points are in some order the points $p_{-\infty}$ and $p_\infty$.

From the collinearity of the centres $p_n$, it follows that the axes $\eta_i$ belong to a pencil together with their limit lines $\eta_{-\infty}$ and $\eta_\infty$, which are lines of support to $C$ at $p_{-\infty}$ and $p_\infty$ respectively. Each conic which is tangent to $\eta_{-\infty}$ and $\eta_\infty$ at $p_{-\infty}$ and $p_\infty$ respectively in an invariant of all $\gamma_n$. Let $\mathscr{F}$ be the family of these conics. Then for each $q$ on $C$, $q \neq a, b$, the sequence of points $q_n = q\phi_n$ lies on the (unique) conic $K_q$, which is in $\mathscr{F}$ and passes through $q$, and the sequence has $p_{-\infty}$ and $p_\infty$ for its only limit points. The arc $A$ of $C$, with ends $q$ and $q\gamma_0\gamma_1$ and which does not contain $p_\infty$, determines $C$ completely. Let $K_1$ be the maximal conic of $\mathscr{F}$ whose interior does not intersect $A$, and let $K_2$ be the minimal conic of $\mathscr{F}$ which contains $A$. Then $C$ lies entirely exterior to $K_1$ and interior to $K_2$. Since $K_1$ and $K_2$ have common tangents at $p_{-\infty}$ and $p_\infty$, these tangents must also be tangents to $C$.

Finally, if $C$ is not a conic then $K_1 \neq K_2$ and $C$ intersects every conic of $\mathscr{F}$ between $K_1$ and $K_2$ infinitely often in every neighbourhood of $p_{-\infty}$ and of $p_\infty$. Thus, in every such neighbourhood the curvature of $C$ oscillates between that of $K_1$ and that of $K_2$.

In a completely analogous manner we can prove the following.

THEOREM 4. *If there are two exterior centres $p_0$, $p_1$ of $C$ so that $p_0 \times p_1$ is a secant of $C$, then there is an infinite sequence $\{p_n\}$ $n = 0, \pm 1, \pm 2, \ldots$, of exterior centres on $p_0 \times p_1$. The points of intersection of $C$ and $p_0 \times p_1$ are the two limit points of the sequence,*

$$p_{-\infty} = \lim_{n\to-\infty} p_n$$

*and*

$$p_\infty = \lim_{n\to\infty} p_n$$

*and C is differentiable at these points. If the arc of C on either side of* $p_0 \times p_1$ *is not conic (possibly degenerate) then its curvature has a singularity of the* 2nd *kind at* $p_{-\infty}$ *and* $p_\infty$.

For the sake of completeness we prove the following theorem which is well known, though possibly not in this formulation (**1**, p. 190).

THEOREM 5. *If the set of centres of C is finite, then the number of centres is odd. There is at most one interior centre and the exterior centres are collinear on a line which does not intersect C. In other words, the finite subgroups, of the plane projective group, which are generated by involutions are isomorphic to the groups of symmetry of the regular polygons.*

*Proof.* If the number of centres is finite, then by Theorems 3 and 4 there cannot be two interior centres nor two exterior centres whose line intersects $C$ or is a line of support to $C$. In any case, there cannot be exactly two centres, for then each would have to lie on the axis of the other. But that, in turn, is the condition for the product of their involutions to be an involution, or, what is the same thing, for the two axes to intersect in a third centre.

Next suppose that $p_1$ and $p_2$ are the only exterior centres on the line $p_1 \times p_2$. This line cannot intersect $C$, so the axes $\eta_1$ and $\eta_2$ intersect at an interior centre $p_0$. Any third exterior centre $p_3$ is not on $p_1 \times p_2$ and hence its axis $\eta_3$ does not pass through $p_0$. Then $\gamma_3$ carries $p_0$ to a second interior centre, contrary to hypothesis.

We have thus proved that there is exactly one (interior or exterior) centre, or there are exactly one interior and two exterior centres, or every line through two exterior centres also passes through a third exterior centre. But the last condition applied to a finite set (here the exterior centres) is known to imply that the set is collinear.

To see that the number of centres is odd, we consider first the case in which an interior centre $p_0$ exists, and $p_1, p_2, \ldots, p_j$; $j \geqslant 3$, are the exterior centres which obviously lie on $\eta_0$. Then $p_0$ lies on $\eta_i$, $i = 1, 2, \ldots, j$. Since $p_1$ is on $\eta_0$ and $p_0$ is on $\eta_1$, the intersection $\eta_0 \times n_1$ is an exterior centre, say $p_2$. Under $\gamma_1$, the points $p_1$ and $p_2$ are fixed and the remaining $j - 2$ exterior centres are interchanged in pairs, hence $j - 2$ is even and the total number of centres $j + 1$ is odd.

Next, suppose there is no interior centre. Then the point in which $\eta_1$ intersects the line of exterior centres cannot be itself a centre. Under $\gamma_1$ the point $p_1$ is fixed and the remaining $j - 1$ centres are interchanged in pairs, hence $j - 1$ is even and $j$ is odd.

The centre sets thus described are, up to projective transformations, the centre sets of the regular polygons.

A subset of a centre set is *independent* if, in the corresponding subset of involutions, no one of the transformations can be generated by the remaining involutions. We now consider the extent to which independent reflections can exist in non-hyperbolic Hilbert geometries.

THEOREM 6. *There exist closed convex curves which are not conics and which have infinitely many independent projective centres. In fact, for every closed convex curve $S$, there exists a closed convex curve $C$ with infinitely many independent projective centres and which coincides with $S$ except for a portion of arbitrarily small linear measure.*

*Proof.* First, to illustrate our method, we give an example of such a curve. On the $x$-axis, we pick a sequence of disjoint intervals $[u_i, v_i]$, $i = 1, 2, \ldots$, so that $u_1 < v_1 < u_2 < v_2 < \ldots$. On the parabola $y = x^2$, let the points $U_i$ and $V_i$ have the co-ordinates $(u_i, u_i^2)$ and $(v_i, v_i^2)$ respectively. Let $\gamma_1$ be an involution of the parabola which interchanges the inside and the outside of the arc $(U_i, V_i)$, and let $\Gamma$ be the group which such involutions generate. The centre of $\gamma_i$ may be any interior point of the chord $U_iV_i$ of the parabola, or it may be the intersection point of the tangents to the parabola at $U_i$ and $V_i$.

We now construct a curve $C$ to consist of two parts $C'$ and $C''$. The first part $C'$ is the union of arbitrary convex arcs $C_i$ with end points $V_{i-1}$, $U_i$, $i = 1, 2, \ldots$,

$$V_0 = \lim_{i\to\infty} V_i,$$

subject to the restriction that if the arcs $(V_{i-1}, U_i)$ of the parabola are replaced by the arcs $C_i$, then the resulting curve is still closed and convex in the projective plane (an especially simple example is that in which $C_i$ is chosen as the straight line segment from $V_{i-1}$ to $U_i$). The remainder of $C$, namely $C''$, is defined by the images of $C'$ under $\Gamma$.

To show that $C$ is a connected convex curve, we project the parabola onto the $x$-axis from the point at infinity on the $y$-axis. To the projectivity $\gamma$ in $\Gamma$ there then corresponds a one-dimensional projectivity $\gamma'$ of the $x$-axis onto itself. The mappings $\gamma'$ form a group $\Gamma'$ which is generated by $\gamma_i'$, $i = 1, 2 \ldots$. It is easily verified that $\gamma_i'$ is either an inversion in a one-dimensional circle, or is such an inversion followed by a reflection on a point. In either case, $\gamma_i'$ interchanges the interior and exterior of the interval $[u_i, v_i]$.

Let $R$ denote the points outside all the intervals $[u_i, v_i]$. If $x$ is interior to $R$, then the image of $x$ under any mapping in $\Gamma'$, other than the identity, lies inside one of the open intervals $\langle u_i, v_i\rangle$, hence $\Gamma'$ is a properly discontinuous group. The set $R$ is a fundamental region. From the theory of discontinuous groups it follows that the images of $R$ under $\Gamma'$ fill the intervals $[u_i, v_i]$ without gaps or overlap. Since $R$ is the projection of $C'$ and the $\Gamma$ images of $C'$ project to the $\Gamma'$ images of $R$, it follows that $C$ is a connected curve. The convexity of the arcs of $C'$ is preserved by $\Gamma$. Since the parabola remains convex when its arcs $(V_{i-1}, U_i)$ are replaced by those of $C'$, it will still remain convex when the arcs $(V_{i-1}\gamma, U_i\gamma)$ are replaced by those of $C'\gamma$, for any $\gamma$ in $\Gamma$. Repeating this argument, we find that the curve $C(\gamma^{(1)}, \gamma^{(2)}, \ldots, \gamma^{(n)})$, which consists of $C' \cup C'\gamma^{(1)} \cup \ldots \cup C'\gamma^{(n)}$ on the

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arcs $(V_{i-1}\gamma^{(j)}, U_i\gamma^{(j)})$, and of the parabola elsewhere, is convex. Thus it follows that the curve

$$C = \lim_{n\to\infty} C(\gamma^{(1)}, \gamma^{(2)}, \ldots, \gamma^{(n)})$$

is convex. But if $\Gamma = (\gamma^{(1)}, \gamma^{(2)}, \ldots, \gamma^{(n)}, \ldots)$, then this curve is precisely $C = C' \cup C''$.

We now proceed with the proof of the general case. Let $S$ be a closed convex curve which is not a polygon. Then there exists an infinite sequence of disjoint arcs $\{(U_i, V_i)\}$ of $S$ which are either not straight line segments or there are lines of support to $S$ at $U_i$ and $V_i$ whose union does not contain the arc $(U_i, V_i)$ and so that the total length of the arcs $(U_i, V_i)$ is arbitrarily small. If $S$ is a polygon then we first replace it by a curve $\tilde{S}$ which is not a polygon and coincides with $S$ except on arcs of arbitrarily small total length.

To each $(U_i, V_i)$ we associate a projective involution $\gamma_i$ such that $\gamma_i$ either has the line $\sigma_i = U_i \times V_i$ as its axis and its centre is $\lambda_i \times \mu_i$, where $\lambda_i$ and $\mu_i$ are lines of support to $S$ at $U_i$ and $V_i$ respectively, or else $\gamma_i$ has its centre interior to $S$ on the interval $\langle U_i, V_i\rangle$ and interchanges $\lambda_i$ and $\mu_i$. The triangle $\Delta_i$ formed by $\sigma_i, \lambda_i, \mu_i$, which contains the arc $(U_i, V_i)$, has the property that $\Delta_i \cap \Delta_j$ is empty, for $i \neq j$. The $\gamma_i$ are therefore independent, since $\gamma_i$ maps a point of $S$ not in the union of the arcs $(U_j, V_j)$ into $\Delta_i$. Finally, let $\Gamma$ be the group generated by the mappings $\gamma_i$.

We now construct the curve $C$ by successive steps. Let $S_1$ be the curve obtained from $S$ by replacing each arc $(U_i, V_i)$ by the image under $\phi_i$ of the complementary arc $(U_i, V_i)'$ of $S$. We can now repeat this construction starting with $S_1$ and the arcs $\{(U_i, V_i)\gamma_j\}$, $i, j = 1, 2, \ldots$, with the corresponding involutions $\gamma_j\gamma_i\gamma_j$ leading to a curve $S_2$, etc. All the curves $S_n$ are convex, and they agree at more and more points so that the length of the complement of $S_n \cap S_{n+1}$ in $S_{n+1}$ converges to zero. Thus

$$C = \lim_{n\to\infty} S_n$$

exists and is a convex closed curve. Since $S_n \cap S_{n+1} \subset (S_{n-1} \cap S_n)\gamma_i$, and the

$$\lim_{n\to\infty} S_n \cap S_{n+1}$$

is dense in $C$, we have

$$C = C_1 (\lim_{n\to\infty} S_n \cap S_{n+1}) = C_1[\lim_{n\to\infty} (S_{n-1} \cap S_n)\gamma_i] = C\gamma_i$$

so $C$ has the desired involutions.

**5. Projective involutions in higher dimensional projective spaces.** The projective involutions of (real) $n$-dimensional projective space $P^n$ need not have "centres." Since the eigenvalues are all $\pm 1$ the dimensions of the characteristic spaces are determined by the signature. If that signature is

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$s = n + 1 - 2k$ then the characteristic spaces are $k - 1$ and $n - k$ dimensional, leading to "*projective* $(k - 1)$*-planes of symmetry*." In case $s = \pm (n-1)$ we again have a projective centre and a projective hyperplane of symmetry.

While it would not be difficult to obtain a characterization of the orbits of points under one-dimensional groups of projective transformations, there is no hope of obtaining a simple characterization of the convex closed surfaces which admit a continuous group of projective transformations. So, if we wish to obtain results analogous to those of § 2 we have to proceed somewhat differently.

Even the concept of a *convex surface* in $P^n$ needs some elaboration which did not arise in $P^2$. In $P^2$ a closed curve $C$ which separates the space into two open regions which are segmentwise connected is the projective equivalent of an affine convex curve (if we include two parallel lines in this description). Thus we may use as our definition of *projective convexity* either the property

(i) $C$ is simple closed so that the components of its complement are segmentwise connected, or

(ii) $C$ is projectively equivalent to a convex closed curve.

Clearly (ii) implies (i).

In $P^3$ these two definitions no longer coincide and Kneser **(3)** has shown that the only surface which satisfies (i) but not (ii) is a quadric surface. For higher dimensions the situation is not known. We shall use Kneser's definition (i) when we speak of a closed convex surface.

THEOREM 7. *A closed convex surface $S$ in $P^n$ has a projective centre $p \in S$ if and only if either*

(i) *$S$ is a convex cone with vertex $p$; or*

(ii) *$S$ consists of two hyperplanes and $p$ is any point on $S$.*

*Proof.* We first prove the theorem for $n = 2$. If we take the axis of $p$ to be the line at infinity then there is an arc of $S$ which contains $p$ and has $p$ as affine centre. Since a convex arc can be symmetric about one of its points only if it is a straight line segment, we are left with only two possible cases. Either $S$ consists of two straight lines through $p$, or of one straight line through $p$ and the line at infinity.

For general $n$ we now have that every two-plane through $p$ intersects $S$ either in $p$ alone or in a single straight line or in two straight lines or lies entirely in $S$. The 2-planes which contain points on both sides of $S$, therefore always intersect it in two straight lines. Either both of these lines go through $p$, or one goes through $p$ and the other lies in the hyperplane of symmetry, $\pi$, corresponding to $p$. For reasons of continuity we see that only one of these possibilities occurs. In the first case $S$ is a cone with vertex $p$. In the second case $S$ contains the entire plane $\pi$ and, since the only convex closed surface which contains a hyperplane consists of two hyperplanes, it follows that $S$ consists of $\pi$ and a hyperplane through $p$.

THEOREM 8. *The projective centres of a convex closed surface $S$ which lie in $P^n - S$ form the union of a discrete, relatively closed set of linear manifolds* (that is, all the points of a $k$-plane, $0 \leqslant k \leqslant n$, which lie on one side of $S$).

*If the projective centres have a limit point $p$ in $P^n - S$ then all centres which do not belong to the $k$-dimensional component $C_p^k$ of $p$ must lie either in that part of the plane $P^k$ of $C_p^k$ which lies on the other side of $S$ than $p$, or in the $(n - k - 1)$-dimensional plane of intersection of the hyperplanes of symmetry which correspond to the centres of $C_p^k$. If there are any centres in $P^k - C_p^k - S$ then all points of $P^k - S$ are centres.*

*Proof*. If the projective centres are discrete in $P^n - S$ then the theorem is true. Assume now that there is a point $p \in S$ which is the limit point of a sequence of projective centres $\{p_i\}$.

We first prove relative closure by showing that $p$ itself is a projective centre. To any point $q$ not on $S$ we can associate the locus $\Sigma_q$ as follows. Let $\lambda$ be any line through $q$ which intersects $S$ in the points $a, b$ and let $q_\lambda$ be the harmonic conjugate of $q$ with respect to $a, b$. Then $\Sigma_q$ is the locus of all $q_\lambda$.

Now $q$ is a projective centre if and only if $\Sigma_q$ is planar. Since $\Sigma_q$ is clearly a continuous function of $q$ for all $q \notin S$ it follows that

$$\Sigma_p = \lim \Sigma_{pi}$$

is planar.

Let $\gamma_i$ and $\gamma$ be the involutions with centres $p_i$ and $p$ respectively. The centres of the central involutions in $\{\gamma_i, \gamma\}$ all lie on the line $p \times p_i$, and are the images of $p$ and $p_i$ under the transformations $\{\gamma\gamma_i\}$. Since $\gamma\gamma_i \to 1$, the identity map, as $p_i \to p$ we see that the projective centres cover the line $p \times p_i$ "more and more densely" in a neighbourhood of $p$ as $p_i \to p$, and that therefore $p$ is an interior point of a segment of centres on any limit line of the lines $p \times p_i$. The endpoints of a segment of centres on such a limit line are themselves limit centres and therefore interior to a segment of centres on the line unless they lie on $S$. Thus a limit line $\lambda$ of the lines $p \times p_i$ consists entirely of projective centres if $\lambda$ does not intersect $S$; if $\lambda$ meets $S$ in a single point then all other points of $\lambda$ are projective centres; if $\lambda$ meets $S$ in two points then all points in the open component $\mu$ of $\lambda - S$ which contains $p$ are centres. The other component $\mu'$ (if any) of $\lambda - S$ may contain no projective centre, but if it contains one centre $p'$ then $p'$ can be invariant under at most one of the involutions defined by the centres in $\mu$. Hence $p'$ is itself a limit point of centres and all points of $\lambda$ except $\lambda \cap S$ are projective centres.

Now let $p' \notin \lambda$ be a projective centre in $P^n - S$. The discussion in the proof of Theorem 2 shows that the orbit $0_{p'}$ of $p'$ under the group generated by the centres of $\lambda$ is either a full conic, or a conic with one point removed, or an open conical arc with endpoints $a, b$ ($\in \lambda \cap S$), or a conic with the points $a, b$ removed, or just $p'$. The last possibility occurs only if $p'$ lies in the intersection of the planes of symmetry of all the centres on $\lambda$. In all other cases the segments of the tangent lines of $0_{p'}$ which lie on the same side of $S$ as $0_{p'}$ consist of centres. Thus the set of centres has interior points in the two-plane $\pi = p' \times \lambda$ and all points of $\pi$ which are on the same side of $S$ as $p'$ are centres. If $p$ and $p'$ are on different sides of $S$ then all points of $\pi - S$ are projective centres.

This completes the proof of our theorem.

It is easy to see that in case the centres have a limit point in $P^n - S$ most of the cases described in Theorem 8 can occur. However there are the following exceptions.

THEOREM 9. *The closed surface $S$ is quadric in each of the following cases.*

1. *The projective centres fill one component of $p^n - S$.*
2. *The projective centres fill a hyperplane $p^{n-1} - S$.*
3. *The projective centres in $p^n - S$ have two non-coplanar components with sum of dimensions $n - 1$.*

*Proof.* By induction on $n$, if $n = 2$ then Cases 1 and 2 follow immediately from Theorem 2. In Case 3 the two components must be one- and zero-dimensional respectively. In other words there is a line segment $(a, b)$ of centres and a centre $p$ not on $a \times b$. By Theorem 8 we know that $p$ is the point of intersection of the axes of symmetry corresponding to the centres of $(a, b)$ and therefore $a \times b$ is the axis of $p$. By Theorem 2, the curve $S$ consists either of two conical arcs with endpoints $a$, $b$ and differentiable at $a$, $b$, or of one conical arc and the segment $(a, b)'$ complementary to $(a, b)$. Since $S$ has the line $a \times b$ as axis of symmetry the second case is excluded and the first case possible only if $S$ is a conic.

Now consider any hyperplane $p^{n-1}$ which intersects $S$ in a convex closed surface $S'$. The projective centres which lie in $p^{n-1}$ (and hence are centres of $S'$) fill one of the components of $p^{n-1} - S'$ in Case 1, and in Case 2 they fill an $(n - 2)$-plane $p^{n-2} - S'$. Thus in these cases every plane section of $S$ is quadric and hence $S$ is quadric.

In order to prove Case 3 we first need a lemma.

LEMMA 2. *If the centres of $S$ have a $k$-dimensional component lying in the $k$-plane $P^k$, then every $(k+1)$-plane $P^{k+1}$ through $P^k$ intersects $S$ in a surface $S'$, so that each component of $S' - P^k$ is quadric (that is lies in a quadric surface). If $S' - P^k$ has more than one component, then $S'$ is differentiable on $S' \cap P^k$.*

*Proof.* For $n = 2$ the case $k = 0$ is trivial (as it is for all $n$) and the case $k = 1$ was treated in Theorem 2. We now see by induction that the intersection of each component of $S' - P^k$ with any $k$-plane $Q^k \subset P^{k+1}$ is quadric and hence each component is quadric. If $S' - P^k$ has more than one component then $S' \cap Q^k$ is differentiable on $S' \cap Q^k \cap P^k$. But a convex $(k - 1)$-surface is differentiable at a point if it is differentiable in $k - 1$ independent directions at that point.

We now return to the proof of Theorem 9. In Case 3 let one component of the set of centres be $k$-dimensional and the other $(n - k - 1)$-dimensional. Let $P^k$ be the plane of the $k$-dimensional component. By Theorem 8 we know that every hyperplane through $P^k$ is a projective plane of symmetry of $S$. Now by Lemma 2 every $(k + 1)$-plane $P^{k+1}$ through $P^k$ intersects $S$ in a surface $S'$ so that $S' - P^k$ has quadric components. Since $P^k$ is a plane of symmetry of $S'$ it follows that either $S' \cap P^k = 0$ or $S' \cap P^k$ is degenerate or $S' - P^k$ is not connected. In the last case $S'$ is differentiable on $P^k$ and hence in all three cases $S'$ is quadric. By the same argument the intersection $S''$ of $S$ with any $(n - k)$-plane through the $(n - k - 1)$-dimensional component of the set of centres is quadric.

Now one of the surfaces $S'$ together with one of the surfaces $S''$ determines a unique quadric surface $S^*$ whose intersections with the planes $P^{k+1} \supset P^k$ coincide with $P^{k+1} \cap S$. Hence $S^* = S$ and the proof is complete.

For the sake of completeness we state the following simple consequence of Theorems 3 and 4.

THEOREM 10. *If a point $p$ on the convex closed surface $S$ is a limit point of centres of $S$ which are isolated in $P^n - S$, then the curvature of $S$ has discontinuities in every neighbourhood of $p$.*

We conclude this discussion with a few comments on surfaces with a finite number of projective centres (not on the surface). The following is an extension of Theorems 3 and 4.

THEOREM 11. *If the centres $p_0, \ldots, p_k$ on one side of the surface $S$ span a $k$-plane $P^k$ which contains points on the other side of $S$, then the group generated by the corresponding involutions $\gamma_0, \ldots, \gamma_k$ contains infinitely many centres lying in $P^k$ on the same side of $S$ as $p_i$ $(i = 0, \ldots, k)$.*

*Proof.* For $k = 1$ the proof is that given in the proof of Theorem 3. Now assume the theorem true for dimensions less than $k$, so that, if $P^k$ contains only a finite number of centres of $S$, then none of the $(k - 1)$-planes determined by $k$ of the $p_i$ contains a point on the other side of $S$. Thus $S \cap P^k$ lies in a closed simplex with the vertices $p_i$ $(i = 0, \ldots, k)$.

If the number of centres in $P^k$ on the side of $p_0$ were finite then there would exist a minimal simplex $\Sigma$ in $P^k$ whose vertices are centres of $S$ and which contains $S \cap P^k$ but contains no other centres of $S$. Now the $(k - 1)$-plane of symmetry of $S \cap P^k$ which corresponds to a vertex $p$ of $\Sigma$ must intersect the side of $S \cap P^k$ which is interior to $\Sigma$. Hence that plane must intersect at least one of the edges $pq$ of $\Sigma$ at an interior point. Thus the image $q\gamma$ of $q$ under the involution on $p$ lies interior to the edge $pq$, in contradiction to the definition of $\Sigma$.

THEOREM 12. *If the (convex) closed surface $S$ has a finite number of projective centres not on $S$, then the centres on one side span a $k$-plane $P_1{}^k$, and the corre-*

*sponding planes of symmetry intersect in a* $(n - k - 1)$*-plane* $P_1^{n-k-1}$. *In a similar manner the centres on the other side determine the planes* $P_2^l$ *and* $P_2^{n-l-1}$.

*The planes* $P_1^k$ *and* $P_2^l$ *lie entirely in the (closed) opposite sides of* $S$ *and* $P_1^k \subset P_2^{n-l-1}$, $P_2^l \subset P_1^{n-k-1}$. *Thus the group* $\Gamma$ *contains the two (not necessarily proper) subgroups* $\Gamma_1$, $\Gamma_2$ *which are generated by the centres on one of the sides of* $S$ *and every element of* $\Gamma_1$ *commutes with every element of* $\Gamma_2$.

*Proof.* Let $p_1 \in P_1^k$ be a centre and let $\gamma_1$ be the corresponding involution. Since $\gamma_1$ maps centres into centres we must have $P_2^l\gamma_1 = P_2^l$. Now the only planes invariant under $\gamma_1$ are the planes through $p_1$ and those contained in the hyperplane of symmetry of $\gamma_1$. Since $P_2^l$ contains no point on the side of $S$ which contains $p_1$ it must lie in the plane of symmetry.

Hence each involution $\gamma_1$ with centre in $P_1^k$—and hence every element of $\Gamma_1$—leaves all centres in $P_2^l$ fixed. This means that every $\gamma_1 \in \Gamma_1$ commutes with the involutions which correspond to these centres, and hence that $\gamma_1\gamma_2 = \gamma_2\gamma_1$ for all $\gamma_1 \in \Gamma_1$, $\gamma_2 \in \Gamma_2$.

## References


1. H. S. M. Coxeter, *Regular polytopes* (New York, 1947).
2. Paul Kelly and E. G. Straus, *Curvature in Hilbert geometries*, Pacific J. Math., *8* (1958), 119–125.
3. H. Kneser, *Eine Erweiterung des Begriffes "konvexer Körper"*, Math. Ann., *82* (1921), 287–296.
4. T. Kojima, *On characteristic properties of the conic and the quadric*, Sci. Rep. Tohoku Univ., *8* (1919), 67–68.


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