Note on Suspension Bridge Catenaries.

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1. Most lecturers on Analytical Statics only consider the following catenaries: the common catenary, the parabolic catenary of the suspension bridge, and, less frequently, the catenary of uniform strength. Of these the second only represents the form of the suspension bridge chain when the weight of the chain is neglected in comparison with that of the roadway, while the first represents the opposite extreme when the weight of the chain alone is taken into account.

The object of this note is to show that the equations of equilibrium admit of integration in cases in which the weights of the chain and roadway, and in some cases the weights of the upright ties, are taken into account.

When the links of the chains and the distance between the uprights are regarded as infinitesimal, we can see that the weight of a small element of the chain is of the form $w\,ds$, that of an element of the roadway is $w_1 dx$, and that of the upright ties is represented by $w_2 y\,dx$, where w, w_1 and w_2 are constants. The equation of equilibrium of an element thus assumes the form

$$T_0\, d\left(\frac{dy}{dx}\right) = w\,ds + (w_1 + w_2 y)\,dx. \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

or in intrinsic coordinates

$$T_0 \sec^2\psi\, d\psi = \{w + (w_1 + w_2 y)\cos\psi\}\, ds. \quad\ldots\ldots\ldots\ldots\ldots(2)$$

If the weight of the upright ties be not neglected, we may get rid of the term involving w_1 by transferring the origin to a depth w_1/w_2 below the level of the bridge. In this case we write $w_2 y$ for $w_1 + w_2 y$ in the equations. This is the same thing as supposing the upright ties to be all increased in length by an amount of which the weight is equal to the previous weight of the corresponding element of roadway.

2. *A common catenary is a possible form of equilibrium*, for if we write the equation

$$T_0 \frac{d^2y}{dx^2} = w\sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}} + w_2 y \qquad \text{.....................(3)}$$

and substitute $y = c \cosh x/c$, we get

$$T_0 = c\,w + c^2 w_2 \qquad \text{..............................(4)}$$

If the roadway is on a level with the vertex of the catenary, we have $w_1 = c\,w_2$, so that the weight of roadway supported by each upright must be equal to the weight of a length c of that upright. The tension at any point (x, y) of the chain is given by

$$T = y\,(w + c\,w_2) \qquad \text{..............................(5)}$$

and is thus equal to the weight of a length y of *chain and roadway*, while the maximum tension of an upright at the same point is equal to the weight of a length y of the upright alone, so that in this catenary the strength of the uprights is considerably greater than that of the chains, and is thus greater than is necessary. It will also be observed that the chain hangs in the same catenary as it did before the rest of the bridge was added.

This, however, is only a particular solution of the differential equation (3). For the more general case it will be found on putting

$$q = \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}} = \sec\psi, \quad \text{that} \quad T_0\, q\, \frac{dq}{dy} = w\,q + w_2\, y. \qquad \text{......(6)}$$

This is a homogeneous equation which may be integrated by the substitution $q = vy$, the integral being

$$y = A\,(v - \alpha)^{-\frac{\alpha}{\alpha - \beta}}\,(v - \beta)^{\frac{\beta}{\alpha - \beta}} \qquad \text{....................(7)}$$

where A is a constant and α, β are the roots of the quadratic

$$T_0 v^2 - wv - w_2 = 0.$$

Here v is the reciprocal of the projection of the ordinate on the normal, which becomes $1/c$ in the case of the catenary. But if we try to express x and the arc s in terms of the parameter v by means of the substitutions

$$dx = \frac{dy}{\sqrt{(q^2 - 1)}} = \frac{dy}{\sqrt{(v^2 y^2 - 1)}}, \quad ds = \frac{q\,dy}{\sqrt{q^2 - 1}}$$

the expressions do not appear to be integrable.

3. *If the weight of the uprights is neglected*, the equation (2) gives

$$ds = \frac{T_0 \, d\psi}{\cos^2\psi \, (w + w_1 \cos\psi)} \quad \text{.......................(8)}$$

and since $dx = ds \cos\psi$, $dy = ds \sin\psi$, it follows that x, y and s are readily expressible in terms of the parameter ψ by resolving the right hand side into partial fractions and applying the formulae for the integration of $d\psi \,/\, (a + b \cos\psi)$. The results need not be given here.

4. *If the weight of the chains be neglected* and the weight of the uprights taken into account, the solution is $y = A \operatorname{ch} x/c$ where $c^2 = T_0 / w_2$; this case, however, has no practical application to suspension bridges, but it would give the form of an arched bridge in which every element of the arch supported the weight of the superincumbent column of earth, supposed pressing vertically downwards on it.

5. *For the corresponding catenaries of uniform strength* we must replace w by $w \sec\psi$ since the tension is everywhere proportional to $\sec\psi$. The general equation now becomes

$$T_0 \frac{d^2y}{dx^2} = w \left\{1 + \left(\frac{dy}{dx}\right)^2\right\} + w_1 + w_2 y. \quad \text{............ (9)}$$

6. *If w_2 be neglected* and we write

$$x = x_1 \sqrt{\frac{w}{w_1 + w}}, \quad \text{.........................(10)}$$

this reduces to the form

$$T_0 \frac{d^2y}{dx_1^2} = w \left\{1 + \left(\frac{dy}{dx_1}\right)^2\right\} \quad \text{..................... (11)}$$

i.e. the catenary of uniform strength when the weight of the roadway is negligible. It follows that when the weight of the roadway is taken into account, the form of the chain is a projection of the ordinary catenary of uniform strength in which the horizontal dimensions are shortened in the ratio of $\sqrt{w} : \sqrt{(w_1 + w)}$.

7. *If w_2 be taken into account* and we write $p = dy/dx$, the equation becomes

$$T_0 \, p \frac{dp}{dy} - w p^2 = w + w_1 + w_2 y \quad \text{...................(12)}$$

which is a linear equation in p^2 and has an integrating factor e^{-2wy/T_0}; but when we try to express x in terms of y, we obtain an integral which cannot be evaluated in a simple form.

8. An interesting case, however, occurs when both the chain and the uprights satisfy the condition for "uniform strength" throughout, the ratio of the tension to the "weight per unit length" being the same at every point both of the chain and uprights and equal to c.

In this case the vertical tension due to the upright at any point is of the form $w_2 e^{y/c}$. The integrating factor is, however, $e^{-2y/c}$, and it will be found that when the integration is performed and $e^{-y/c}$ put equal to z, the expression for x assumes the form

$$\int \frac{dz}{\sqrt{(A - 2Bz - Cz^2)}} \qquad \text{.......................(13)}$$

where A, B, C are certain constants.