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l^p-NORMS OF SOME GENERALIZED HAUSDORFF MATRICES

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ABSTRACT. In a recent paper D. Borwein [Math. Z. 183(1983), 483–487] obtained an upper bound for the l^p-norms of some generalized Hausdorff matrices, where the sequence (λ_n) satisfies the condition $\lambda_{n+1} \leqq \lambda_n + c$, for some positive c. In this paper we obtained the l^p-norms of these generalized Hausdorff matrices for which the mass functions are totally monotone.

In his dissertation, Cartlidge [4] established a number of sufficient conditions for weighted mean operators to be bounded operators on $l^p, p > 1$, and obtained the spectra of such operators in some cases. Borewein [1] used the fact that weighted mean operators are special cases of generalized Hausdorff matrices to generalize two of the results of Cartlidge, dealing with weighted mean operators with increasing weights. Borwein's result also includes Theorem 1 of [6].

In this paper we obtain the l^p norms of certain generalized Hausdorff matrices:

Let $\{\lambda_n\}$ be a sequence of real numbers satisfying $\lambda_0 \geqq 0, \lambda_n > 0$ for $n > 0$. For $0 \leqq k \leqq n$, define

$$(1) \qquad \lambda_{nk}(t) = -\lambda_{k+1}\cdots\lambda_n \frac{1}{2\pi i}\int_C \frac{t^z\,dz}{(\lambda_k - z)\cdots(\lambda_n - z)}, \quad 0 < t \leqq 1,$$
$$\lambda_{nk}(0+) = \lambda_{nk}(0),$$

where C is a positively oriented closed Jordan curve enclosing $\lambda_k, \lambda_{k+1}, \ldots, \lambda_n$. We shall accept the convention that $\lambda_{k+1}\cdots\lambda_n = 1$ when$k = n$.

A generalized Hausdorff matrix $H(\lambda, \alpha)$ is a lower triangular matrix with entries defined by

$$(2) \qquad \lambda_{nk} = \int_0^1 \lambda_{nk}(t)\,d\alpha(t),$$

where $\alpha(t) \in BV[0,1]$, continuous from the right at $t = 0$, and satisfying $\alpha(t) = [\alpha(t+) + \alpha(t-)]/2$ for each $0 < t < 1$. Borwein, Jakimowski, and Russell have investigated these matrices as bounded operators on l^p for $p > 1$ [2] and [3], and have used them to solve generalized interpolation problems [7].

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The choice $\alpha(t) = t$ yields a generalized weighted mean matrix D generated by a positive sequence $\{d_n\}$ defined by

(3)
$$d_0 > 0, D_0 = (1 + \lambda_0)d_0, D_n = (1 + \lambda_n)d_n$$
$$= \prod_{k=1}^{n}(1 + 1/\lambda_k), \quad n > 0.$$

With $\lambda_n = n, d_n = 1$ for all n, D becomes the Cesaro matrix of order 1.

We shall also assume that the $\{\lambda_n\}$ are strictly increasing, which allows (1) to be represented in the form

(4)
$$\lambda_{nk}(t) = \lambda_{k+1} \cdots \lambda_n[t^{\lambda_n}, \dots, t^{\lambda_n}], \quad 0 < t \leqq 1,$$

where $[t^{\lambda_k}, \dots, t^{\lambda_n}]$ is the divided difference defined by

$$[t^{\lambda_k}, t^{\lambda_{k+1}}] = (t^{\lambda_k} - t^{\lambda_{k+1}})/(\lambda_{k+1} - \lambda_k),$$

and

$$\frac{[t^{\lambda_k}, \dots, t^{\lambda_n}] = [t^{\lambda_k}, \dots, t^{\lambda_{n-1}}] - [t^{\lambda_{k+1}}, \dots, t^{\lambda_n}]}{\lambda_n - \lambda_k}.$$

Theorem A [1, Theorem 1]. If $p \geqq 1, c > 0$ and

(5)
$$\lambda_{n+1} \leqq \lambda_n + c, \quad n \geqq 0,$$

and if

(6)
$$\int_0^1 t^{-c/p}|d\alpha(t)| < \infty,$$

then $H(\lambda, \alpha) \in B(l^p)$ and

$$||H(\lambda, \alpha)||_p \leqq \int_0^1 t^{-c/p}|d\alpha(t)|.$$

If $\alpha(t)$ is nonnegative and nondecreasing over [0, 1], then it shall be called totally monotone.

THEOREM 1. *Suppose $H(\lambda, \alpha) \in B(l^p)$, $\alpha(t)$ totally monotone. Then (6) exists, where*

(7)
$$\mu_n = \int_0^1 t^{\lambda_n} d\alpha(t).$$

We shall first establish the following lemma.

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LEMMA. *Suppose* $\{\mu_n\}$ *has the representation* (7), *with* $\alpha(t) \in \mathrm{BV}[0,1]$ *and normalized. If*

(8) $$\sup_n \sum_{k=0}^{n} (\lambda_{k+1} + c/p) \cdots (\lambda_n + c/p) \int_0^1 [t^{\lambda_k + c/p}, \ldots, t^{\lambda_n + c/p}] d\alpha(t) < \infty, \quad c > 0,$$

then (6) *exists.*

The uniform boundedness of (8) implies, by [8], the existence of a sequence $\{\xi_n\}$ satisfying

$$\xi_n = \int_0^1 t^{\lambda_n + c/q} d\gamma(t)$$

for some $\gamma(t) \in \mathrm{BV}[0,1]$ and normalized.

We may write

$$\xi_n = \int_0^1 t^{\lambda_n} d\alpha(t),$$

where

$$\alpha(t) = \int_0^t u^{c/p} d\gamma(u), \quad 0 \leqq t \leqq 1.$$

For any $0 < \epsilon < 1$, and any subdivision $\epsilon = x_0 < x_1 < \cdots < x_n = 1$,

$$\begin{aligned} \int_\epsilon^1 t^{-c/p} |d\alpha(t)| &= \lim_{\max(x_{i+1} - x_i) \to 0} \sum_k x_{k+1}^{-c/p} \bigvee_{x_k}^{x_{k+1}} \alpha(t) \\ &= \lim_{\max(x_{i+1} - x_i) \to 0} \sum_k x_{k+1}^{-c/p} \bigvee_{x_k}^{x_{k+1}} \left(\int_0^t u^{c/p} d\gamma(u) \right) \\ &\leqq \overline{\lim_n}_{\max(x_{i+1} - x_i) \to 0} \sum_k \int_{x_k}^{x_{k+1}} u^{-c/p} u^{c/p} |d\gamma(u)| \\ &= \int_\epsilon^1 |d\gamma(u)| \leqq \int_0^1 |d\gamma(u)|. \end{aligned}$$

To prove Theorem 1, assume (6) does not exist. Let $0 < \eta < 1$ and arbitrary. Then, from the Lemma, given any $M > 0$ there exists an integer N such that $n \geqq N$ implies

$$\sum_{k=0}^{n} (\lambda_{k+1} + c/p) \cdots (\lambda_n + c/p) \int_0^1 [t^{\lambda_k + c/p}, \ldots, t^{\lambda_n + c/p}] d\alpha(t) > M(1 - \eta).$$

Choose $\omega = \epsilon + c/p, 0 < \epsilon < 0$, and define $\{a_n\}$ by

$$a_0 = 1, a_n = \lambda_1 \cdots \lambda_n / (\lambda_1 + \omega) \cdots (\lambda_n + \omega), n > 0.$$

Then $\{a_n\} \in l^p$ for each $\epsilon > 0$, and with $v_n = \lambda_{nk} a_k$, for $n \geqq N$,

$$v_n = \sum_{k=0}^{n} \lambda_{k+1} \cdots \lambda_n \int_0^1 [t^{\lambda_k}, \ldots, t^{\lambda_n}] d\alpha(t) a_k$$
$$= a_n \sum_{k=0}^{n} (\lambda_{k+1} + \omega) \cdots (\lambda_n + \omega) \int_0^1 [t^{\lambda_k}, \ldots, t^{\lambda_n}] d\alpha(t)$$
$$\geqq a_n \sum_{k=0}^{n} (\lambda_{k+1} + c/p) \cdots (\lambda_n + c/p) \int_0^1 [t^{\lambda_k}, \ldots, t^{\lambda_n}] d\alpha(t).$$

Since $\alpha(t)$ is totally monotone,

$$v_n \geqq a_n \sum_{k=0}^{n} (\lambda_{k+1} + c/p) \cdots (\lambda_n + c/p) \int_0^1 [t^{\lambda_k + c/p}, \ldots, t^{\lambda_n + c/p}] d\alpha(t)$$

Now choose ϵ so that $\sum_{n=N}^{\infty} a_n^p > (1-\eta) \sum_{n=0}^{\infty} a_n^p$. Then

$$\sum_{n=0}^{\infty} v_n^p \geqq \sum_{n=N}^{\infty} v_n^p \geqq M^p (1-\eta)^{-1} \sum_{n=N}^{\infty} a_n^p > M^p \sum_{n=0}^{\infty} a_n^p,$$

and $||H(\lambda, \alpha)||_p > M$, a contradiction.

THEOREM 2. *Let $H(\lambda, \alpha)$ be a generalized Hausdorff matrix with $\alpha(t)$ totally monotone and $\{\lambda_n\}$ satisfying* (5) *for some $c > 0$. Then $H(\lambda, \alpha) \in B(l^p)$ if and only if* (6) *exists. Moreover,*

$$||H(\lambda, \alpha)||_p = \int_0^1 t^{-c/p} d\alpha(t).$$

The first part of the Theorem merely combines Theorems A and 1. Since the integral exists, using [5, p. 179].

$$\lim_n \sum_{k=0}^{n} (\lambda_{k+1} + c/p) \cdots (\lambda_n + c/p) \int_0^1 [t^{\lambda_k + c/p}, \ldots, t^{\lambda_n + c/p}] d\alpha(t)$$
$$= \int_0^1 t^{-c/p} d\alpha(t).$$

With $0 < \eta < 1$ and arbitrary, and $\{a_n\}$ as in Theorem 1, choose N so that, for $n \geqq N$,

$$\sum_{k=0}^{n} (\lambda_{k+1} + c/p) \cdots (\lambda_n + c/p) \int_0^1 [t^{\lambda_k + c/p}, \ldots, t^{\lambda_n + c/p}] d\alpha(t)$$
$$> (1-\eta) \int_0^1 t^{-c/p} d\alpha(t).$$

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Choose ϵ as in Theorem 1. Then

$$\left(\sum_{n=0}^{\infty} v_n^p\right)^{1/p} \geqq \left(\sum_{n=N}^{\infty} v_n^p\right)^{1/p} > (1-\eta)\int_0^1 t^{-c/p} d\alpha(t) \left(\sum_{n=N}^{\infty} v_n^p\right)^{1/p}$$
$$> (1-\eta)^{1+1/p} \int_0^1 t^{-c/p} d\alpha(t) ||a||_p.$$

Since η is arbitrary

$$||H(\lambda,\alpha)||_p \geqq \int_0^1 t^{-c/p} d\alpha(t).$$

The opposite inequality follows from Theorem A.

Corollary 2 of [6] is the special case of Theorem 2 with $\lambda_n = n + \alpha$.

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